OSCILLATION AND NONOSCILLATION FOR IMPULSIVE DYNAMIC EQUATIONS ON CERTAIN TIME SCALES

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We discuss the existence of oscillatory and nonoscillatory solutions for first-order impulsive dynamic equations on time scales with certain restrictions on the points of impulse. We will rely on the nonlinear alternative of Leray-Schauder type combined with a lower and upper solutions method.

1. Introduction

This paper is concerned with the existence of oscillatory and nonoscillatory solutions of first-order impulsive dynamic equations on certain time scales. We consider the problem

$$y^\Delta(t) = f(t, y(t)), \quad t \in J_\mathbb{T} := [0, \infty) \cap \mathbb{T}, \, t \neq t_k, \, k = 1, \ldots,$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \ldots, \tag{1.1}$$

where $\mathbb{T}$ is an unbounded-above time scale with $0 \in \mathbb{T}$ $f : J_\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k \in C(\mathbb{R}, \mathbb{R})$, $t_k \in \mathbb{T}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots < \infty$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$ in the sense of the time scale; that is, in terms of $h > 0$ for which $t_k + h$, $t_k - h \in [t_0, \infty) \cap \mathbb{T}$, whereas if t_k is left-scattered (resp., right-scattered), we interpret $y(t_k^-) = y(t_k)$ (resp., $y(t_k^+) = y(t_k)$).

Impulsive differential equations have become important in recent years in mathematical models of real processes and they rise in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There have been significant developments in impulse theory also in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [5], Lakshmikantham et al. [22], Samoĭlenko and Perestyuk [25], and the references therein. In recent years, dynamic equations on times scales have received much attention.

We refer the reader to the books by Bohner and Peterson [10, 11], Lakshmikantham et al. [23], and the references therein. The time scale calculus has tremendous potential for applications in mathematical models of real processes, for example, in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences; see the monographs of Aulbach and Hilger [4], Bohner and Peterson [10, 11], Lakshmikantham et al. [23], and the references therein. The existence of solutions of boundary value problem on a measure chain (i.e., time scale) was recently studied by Henderson [20] and Henderson and Tisdell [21]. The question of existence of solutions to some classes of impulsive dynamic equations on time scales was treated very recently by Henderson [19] and Benchohra et al. in [1, 7, 8]. The aim of this paper is to initiate the study of oscillatory and nonoscillatory solutions to impulsive dynamic equations on time scales. For oscillation and nonoscillation of impulsive differential equations, see, for instance, the monograph of Bainov and Simonov [5] and the papers of Graef et al. [16, 17]. The purpose of this paper is to give some sufficient conditions for existence of oscillatory and nonoscillatory solutions of the first-order dynamic impulsive problem (1.1) on time scales. There has been, in fact, a good deal of research already devoted to oscillation questions for dynamic equations on time scales; see, for example, [2, 9, 12, 14, 15, 24]. For the purposes of this paper, we will rely on the nonlinear alternative of Leray-Schauder type combined with a lower and upper solutions method. Our results can be considered as contributions to this emerging field.

2. Preliminaries

We will briefly recall some basic definitions and facts from time scale calculus that we will use in the sequel.

A time scale $\mathbb{T}$ is an closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \tag{2.1}$$

(supplemented by $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The notations $[a,b], [a,b)$, and so on will denote time scales intervals

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}, \tag{2.2}$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

Definition 2.1. Let X be a Banach space. The function $g : \mathbb{T} \to X$ will be called $rd-$ continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, and write $g \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, X)$. For $t \in \mathbb{T}^k$, the Δ derivative of g at t, denoted by $g^{\Delta}(t)$, is the number (provided it exists) such that for all $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s| \tag{2.3}$$

for all $s \in U$. A function F is called an antiderivative of $g : \mathbb{T} \to X$ provided

$$F^\Delta(t) = g(t) \quad \text{for each } t \in \mathbb{T}^k. \tag{2.4}$$

A function $g : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)g(t) \neq 0 \quad \forall t \in \mathbb{T}, \tag{2.5}$$

where $\mu(t) = \sigma(t) - t$ which is called the graininess function. The set of all $rd-$continuous functions g that satisfy $1 + \mu(t)g(t) > 0$ for all $t \in \mathbb{T}$ will be denoted by $\mathscr{R}^+$.

The generalized exponential function e_p is defined as the unique solution $y(t) = e_p(t, a)$ of the initial value problem $y^\Delta = p(t)y$, $y(a) = 1$, where p is a regressive function. An explicit formula for $e_p(t, a)$ is given by

$$e_p(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right\} \quad \text{with } \xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases} \tag{2.6}$$

For more details, see [10]. Clearly, $e_p(t,s)$ never vanishes. $C([0,b], \mathbb{R})$ is the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$, where $[a,b] \subset \mathbb{T}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in [a,b]\}. \tag{2.7}$$

Remark 2.2. (i) If f is continuous, then f $rd-$continuous.
(ii) If f is delta differentiable at t, then f is continuous at t.

3. Main result

We will assume for the remainder of the paper that, for each $k = 1, \ldots$, the points of impulse t_k are right-dense. In order to define the solution of (1.1), we will consider the space

$$PC = \{y : J_\mathbb{T} \longrightarrow \mathbb{R} : y_k \in C(J_k, \mathbb{R}), \ k = 0, 1, \ldots, \text{ and there exist} \\ y(t_k^-) \text{ and } y(t_k^+), \ k = 1, \ldots, \text{ with } y(t_k^-) = y(t_k)\}, \tag{3.1}$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$.

Remark 3.1. In light of the right-density assumption on each impulse point, we observe that this restriction precludes certain time scales. For example, time scales that are excluded from this work include discrete time scales, time scales associated with q-differences, harmonic numbers time scales, and so forth. We observe further that, in the context of impulsive problems on time scales, such restrictions on impulse points are not uncommon; see, for example, [13, 19].

Let us start by defining what we mean by a solution of problem (1.1).

Definition 3.2. A function $y \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \ldots$, is said to be a solution of (1.1) if y satisfies the equation $y^{\Delta}(t) = f(t, y(t))$ on $J \setminus \{t_1, \ldots\}$ and the condition $y(t_k^+) = I_k(y(t_k^-))$, $k = 1, \ldots$.

Definition 3.3. A function $\alpha \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \ldots$, is said to be a lower solution of (1.1) if $\alpha^{\Delta}(t) \le f(t, \alpha(t))$ on $J_{\mathbb{T}} \setminus \{t_1, \ldots\}$ and $\alpha(t_k^+) \le I_k(\alpha(t_k))$, $k = 1, \ldots$. Similarly, a function $\beta \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \ldots$, is said to be an upper solution of (1.1) if $\beta^{\Delta}(t) \ge f(t, \beta(t))$ on $J_{\mathbb{T}} \setminus \{t_1, \ldots\}$ and $\beta(t_k^+) \ge I_k(\beta(t_k))$, $k = 1, \ldots$.

For the study of this problem, we first list the following hypotheses:
(H1) the function $f : J_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous;
(H2) for all $r > 0$, there exists a nonnegative function $h_r \in C(J_{\mathbb{T}}, \mathbb{R}^+)$ with

$$|f(t, y)| \le h_r(t) \quad \forall t \in J_{\mathbb{T}} \text{ and all } |y| \le r; \tag{3.2}$$

(H3) there exist α and $\beta \in PC \cap C^1((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \ldots$, lower and upper solutions for the problem (1.1) such that $\alpha \le \beta$;
(H4)

$$\alpha(t_k^+) \le \min_{y \in [\alpha(t_k^-), \beta(t_k^-)]} I_k(y) \le \max_{y \in [\alpha(t_k^-), \beta(t_k^-)]} I_k(y) \le \beta(t_k^+), \quad k = 1, \ldots. \tag{3.3}$$

THEOREM 3.4. *Assume that hypotheses (H1)–(H4) hold. Then the problem (1.1) has at least one solution y such that*

$$\alpha(t) \le y(t) \le \beta(t) \quad \forall t \in J. \tag{3.4}$$

Proof. The proof will be given in several steps.
Step 1. Consider the problem

$$y^{\Delta}(t) = f(t, y(t)), \quad t \in J_1 := [t_0, t_1]. \tag{3.5}$$

Transform the problem (3.5) into a fixed point problem. Consider the following modified problem:

$$y^{\Delta}(t) + y(t) = f_1(t, y(t)), \quad t \in J_1, \tag{3.6}$$

where

$$f_1(t, y) = f(t, \tau(t, y)) + \tau(t, y),$$

$$\tau(t, y) = \max\{\alpha(t), \min(y, \beta(t))\}, \tag{3.7}$$

$$\overline{y}(t) = \tau(t, y).$$

A solution to (3.6) is a fixed point of the operator $N : C([t_0, t_1], \mathbb{R}) \to C([t_0, t_1], \mathbb{R})$ defined by

$$N(y)(t) = \int_{t_0}^{t} [f_1(s, \overline{y}(s)) + \overline{y}(s) - y(s)] \Delta s. \tag{3.8}$$

Remark 3.5. (i) Notice that f_1 is a continuous function, and from (H2) there exists $M^* > 0$ such that

$$\|f_1(t,y)\| \leq M^* + \max\left\{\sup_{t\in J_1}|\alpha(t)|, \sup_{t\in J_1}|\beta(t)|\right\} := M. \tag{3.9}$$

(ii) By the definition of τ it is clear that

$$\alpha(t_k^+) \leq I_k(\tau(t_k, y(t_k))) \leq \beta(t_k^+), \quad k = 1,\ldots. \tag{3.10}$$

In order to apply the nonlinear alternative of Leray-Schauder type, we first show that N is continuous and completely continuous.

Claim 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([t_0, t_1], \mathbb{R})$. Then

$$|N(y_n)(t) - N(y)(t)| \leq \int_{t_0}^{t}[|f_1(s,\overline{y}_n(s)) - f_1(s,\overline{y}(s))\Delta s|$$

$$+ |\overline{y}_n(s) - \overline{y}(s)| + |y_n(s) - y(s)|]\Delta s$$

$$\leq \int_{t_0}^{t}|f_1(s,\overline{y}_n(s)) - f_1(s,\overline{y}(s))\Delta s|$$

$$+ (t_1 - t_0)\|\overline{y}_n(s) - \overline{y}(s)\|_\infty + (t_1 - t_0)\|y_n(s) - y(s)\|_\infty\Delta s. \tag{3.11}$$

Since f_1 is a continuous function, then we have

$$\|N(y_n) - N(y)\|_\infty \leq \|f_1(\cdot,\overline{y}_n) - f_1(\cdot,\overline{y})\|_\infty + (t_1 - t_0)\|\overline{y}_n - \overline{y}\|_\infty + (t_1 - t_0)\|y_n - y\|_\infty. \tag{3.12}$$

Thus

$$\|N(y_n) - N(y)\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Claim 2. N maps bounded sets into bounded sets in $C([t_0, t_1], \mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C([t_0, t_1], \mathbb{R}) : \|y\|_\infty \leq q\}$ one has $\|Ny\|_\infty \leq \ell$. Let $y \in B_q$. Then for each $t \in J_1$ we have

$$N(y)(t) = \int_{t_0}^{t}[f_1(s,\overline{y}(s)) + \overline{y}(s) - y(s)]\Delta s. \tag{3.14}$$

By (H1) and Remark 3.5 we have, for each $t \in J_1$,

$$
|Ny(t)| \leq \int_{t_0}^{t} [\,|f_1(t,y)| + |\overline{y}(s)| + |y(s)|\,]\Delta s +
$$

$$
\leq (t - t_0)M + (t - t_0) \max\left\{ q, \sup_{t \in J_1} |\alpha(t)|, \sup_{t \in J_1} |\beta(t)| \right\} \tag{3.15}
$$

$$
+ (t - t_0)q := \ell.
$$

Thus $\|N(y)\|_\infty \leq \ell$.

Claim 3. N maps bounded set into equicontinuous sets of PC.

Let $u_1, u_2 \in J_1$, $u_1 < u_2$ and B_q be a bounded set of PC as in **Claim 2**. Let $y \in B_q$. Then

$$
|N(u_2) - N(u_1)|
$$

$$
\leq (u_2 - u_1)M + (u_2 - u_1) \max\left\{ q, \sup_{t \in J_1} |\alpha(t)|, \sup_{t \in J_1} |\beta(t)| \right\} + (u_2 - u_1)q. \tag{3.16}
$$

As $u_2 \to u_1$, the right-hand side of the above inequality tends to zero.

As a consequence of Claims 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N : C([t_0, t_1], \mathbb{R}) \to C([t_0, t_1], \mathbb{R})$ is continuous and completely continuous.

Claim 4. A priori bounds on solutions.

Let y be a possible solution of $y = \lambda N(y)$ with $\lambda \in [0,1]$. Then we have

$$
y(t) = \lambda \int_{t_0}^{t} [f_1(t,y) + \overline{y}(s) - y(s)]\Delta s. \tag{3.17}
$$

This implies by Remark 3.5 that for each $t \in J_1$ we have

$$
|y(t)| = \left| \int_{t_0}^{t} [f_1(t,y) + \overline{y}(s) + y(s)]\Delta s \right| \leq \int_{t_0}^{t} [\,|f_1(t,y)| + |\overline{y}(s)| + |y(s)|\,]\Delta s
$$

$$
\leq (t - t_0)M + (t - t_0) \max\left(\sup_{t \in J_1} |\alpha(t)|, \sup_{t \in J_1} |\beta(t)| \right) + \int_{t_0}^{t} |y(s)|\Delta s. \tag{3.18}
$$

Now $1 \in \mathcal{R}^+$. Hence, let $e_1(t,0)$ be the unique solution of the problem

$$
y^\Delta(t) = y(t), \qquad y(0) = 1. \tag{3.19}
$$

Then from Gronwall's inequality we have

$$
|y(t)| \leq f_* + f_* \int_{t_0}^{t} e_1(t, \sigma(s))\Delta s, \tag{3.20}
$$

where

$$f_* = M(t - t_0) + (t - t_0) \max \left\{ \sup_{t \in J_1} |\alpha(t)|, \sup_{t \in J_1} |\beta(t)| \right\}. \tag{3.21}$$

Thus

$$\|y\|_\infty \le f_* + f_* \sup_{t \in J_1} \int_{t_0}^t e_1(t, \sigma(s)) \Delta s := M_1. \tag{3.22}$$

Set

$$U = \{y \in C([t_0, t_1], \mathbb{R}) : \|y\|_\infty < M_1 + 1\}. \tag{3.23}$$

From the choice of U there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [18], we deduce that N has a fixed point y in U which is a solution of the problem (3.6).

Claim 5. The solution y of (3.6) satisfies

$$\alpha(t) \le y(t) \le \beta(t) \quad \forall t \in J_1. \tag{3.24}$$

Let y be the above solution to (3.6). We prove that

$$\alpha(t) \le y(t) \quad \forall t \in J_1. \tag{3.25}$$

Suppose not. Then there exist $e_1, e_2 \in J_1$, $e_1 < e_2$ such that $\alpha(e_1) = y(e_1)$ and

$$y(t) < \alpha(t) \quad \forall t \in (e_1, e_2]. \tag{3.26}$$

In view of the definition of τ one has

$$y(t) - y(e_1) = \int_{e_1}^t [f(s, \alpha(s)) - (y(s) - \alpha(s))] \Delta s. \tag{3.27}$$

Using the fact that α is a lower solution to (3.6), the above inequality yields

$$\alpha(t) - \alpha(e_1) \le \int_{e_1}^t f(s, \alpha(s)) \Delta s < \int_{e_1}^t [f(s, \alpha(s)) - (y(s) - \alpha(s))] \Delta s$$
$$= y(t) - y(e_1) < \alpha(t) - \alpha(e_1), \tag{3.28}$$

which is a contradiction. Analogously, we can prove that

$$y(t) \le \beta(t) \quad \forall t \in [t_0, t_1]. \tag{3.29}$$

This shows that the problem (3.6) has a solution in the interval $[\alpha, \beta]$ which is solution of (3.5). Denote this solution by y_0.

Step 2. Consider the following problem:

$$y^\Delta(t) = f(t, y(t)), \quad t \in J_2 := [t_1, t_2],$$
$$y(t_1^+) = I_1(y_0(t_1^-)). \tag{3.30}$$

Consider the following modified problem:

$$y^\Delta(t) + y(t) = f_1(t, y(t)), \quad t \in J_2, \tag{3.31}$$

$$y(t_1^+) = I_1(y_0(t_1^-)). \tag{3.32}$$

A solution to (3.31)-(3.32) is a fixed point of the operator $N_1 : C([t_1, t_2], \mathbb{R}) \to C([t_1, t_2], \mathbb{R})$ defined by

$$N_1(y)(t) = \int_{t_1}^{t} [f_1(s, \overline{y}(s)) + \overline{y}(s) - y(s)] \Delta s + I_1(y_0(t_1^-)). \tag{3.33}$$

Since $y_0(t_1) \in [\alpha(t_1^-), \beta(t_1^-)]$, then (H4) implies that

$$\alpha(t_1^+) \le I_1(y_0(t_1^-)) \le \beta(t_1^+), \tag{3.34}$$

that is,

$$\alpha(t_1^+) \le y(t_1^+) \le \beta(t_1^+). \tag{3.35}$$

Using the same reasoning as that used for problem (3.5), we can conclude the existence of at least one solution y to (3.32)-(3.41). We now show that this solution satisfies

$$\alpha(t) \le y(t) \le \beta(t) \quad \forall t \in J_2. \tag{3.36}$$

Let y be the above solution to (3.32)-(3.41). We show that

$$\alpha(t) \le y(t) \quad \forall t \in J_2. \tag{3.37}$$

Assume this is false. Then since $y(t_1^+) \ge \alpha(t_1^+)$, there exist $e_3, e_4 \in J_2$ with $e_3 < e_4$ such that $\alpha(e_3) = y(e_3)$ and

$$y(t) < \alpha(t) \quad \forall t \in (e_1, e_2]. \tag{3.38}$$

In view of the definition of τ one has

$$\alpha(t) - \alpha(e_3) \le \int_{e_3}^{t} f(s, \alpha(s)) \Delta s < \int_{e_3}^{t} [f(s, \alpha(s)) - (y(s) - \alpha(s))] \Delta s$$
$$= y(t) - y(e_3) < \alpha(t) - \alpha(e_3), \tag{3.39}$$

which is a contradiction. Analogously, we can prove that

$$y(t) \le \beta(t) \quad \forall t \in [t_1, t_2]. \tag{3.40}$$

This shows that the problem (3.32)–(3.41) has a solution in the interval $[\alpha, \beta]$ which is a solution of (3.30). Denote this solution by y_1.

Step 3. We continue this process and take into account that $y_m := y|_{[t_{m-1}, t_m]}$ is a solution to the problem

$$y^\Delta(t) = f(t, y(t)), \quad t \in J_m := [t_{m-1}, t_m], \tag{3.41}$$

$$y(t_m^+) = I_m(y_{m-1}(t_{m-1}^-)). \tag{3.42}$$

Consider the following modified problem:

$$y^\Delta(t) + y(t) = f_1(t, y(t)), \quad t \in J_m,$$
$$y(t_m^+) = I_m(y_{m-1}(t_{m-1}^-)). \tag{3.43}$$

A solution to (3.43) is a fixed point of the operator $N_m : C([t_{m-1}, t_m], \mathbb{R}) \to C([t_{m-1}, t_m], \mathbb{R})$ defined by

$$N_m(y)(t) = \int_{t_m}^t [f_1(s, \overline{y}(s)) + \overline{y}(s) - y(s)] \Delta s + I_m(y(t_{m-1}^-)). \tag{3.44}$$

Using the same reasoning as that used for problems (3.5) and (3.6)-(3.30), we can conclude the existence of at least one solution y to (3.41)–(3.42). Denote this solution by y_{m-1}.

The solution y of the problem (1.1) is then defined by

$$y(t) = \begin{cases} y_0(t), & t \in [t_0, t_1], \\ y_2(t), & t \in (t_1, t_2], \\ \vdots \\ y_{m-1}(t), & t \in (t_{m-1}, t_m], \\ \vdots \end{cases} \tag{3.45}$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem gives sufficient conditions to ensure the nonoscillation of solutions of problem (1.1).

THEOREM 3.6. *Let α and β be lower and upper solutions, respectively, of (1.1) with $\alpha \le \beta$ and assume that*

(H5) *α is eventually positive nondecreasing, or β is eventually negative nonincreasing. Then every solution y of (1.1) such that $y \in [\alpha, \beta]$ is nonoscillatory.*

Proof. Assume α to be eventually positive. Thus there exists $T_\alpha > t_0$ such that

$$\alpha(t) > 0 \quad \forall t > T_\alpha. \tag{3.46}$$

Hence, $y(t) > 0$ for all $t > T_\alpha$, and $t \neq t_k, k = 1, \ldots$. For some $k \in \mathbb{N}$ and $t > t_\alpha$, we have $y(t_k^+) = I_k(y(t_k))$. From (H4) we get $y(t_k^+) > \alpha(t_k^+)$. Since for each $h > 0, \alpha(t_k + h) \geq \alpha(t_k) > 0$, then $I_k(y(t_k)) > 0$ for all $t_k > T_\alpha, k = 1, \ldots$, which means that y is nonoscillatory. Analogously, if β is eventually negative, then there exists $T_\beta > t_0$ such that

$$y(t) < 0 \quad \forall t > T_\beta, \tag{3.47}$$

which means that y is nonoscillatory. This completes the proof. $\square$

The following theorem discusses the oscillation of solutions of problem (1.1).

THEOREM 3.7. *Let α and β be lower and upper solutions, respectively, of (1.1), and assume that the sequences $\alpha(t_k)$ and $\beta(t_k)$, $k = 1, \ldots$, are oscillatory. Then every solution y of (1.1) such that $y \in [\alpha, \beta]$ is oscillatory.*

Proof. Suppose on the contrary that y is a nonoscillatory solution of (1.1). Then there exists $T_y > 0$ such that $y(t) > 0$ for all $t > T_y$, or $y(t) < 0$ for all $t > T_y$. In the case $y(t) > 0$ for all $t > T_y$, we have $\beta(t_k) > 0$ for all $t_k > T_y$, $k = 1, \ldots$, which is a contradiction, since $\beta(t_k)$ is an oscillatory upper solution. Analogously, in the case $y(t) < 0$ for all $t > T_y$, we have $\alpha(t_k) < 0$ for all $t_k > T_y$, $k = 1, \ldots$, which is also a contradiction, since $\alpha(t_k)$ is an oscillatory lower solution. $\square$

4. An example

As an application of our results, we consider the following impulsive dynamic equation

$$y^\Delta(t) = f(t, y), \quad \text{for each } t \in J_\mathbb{T} := [0, \infty) \cap \mathbb{T}, \; t \neq t_k, \; k = 1, \ldots,$$
$$y(t_k^+) = I_k(y(t_k^-)), \quad k \in \mathbb{N}, \tag{4.1}$$

where $f : J_\mathbb{T} \times \mathbb{R} \to \mathbb{R}$. Assume that there exist $g_1(\cdot), g_2(\cdot) \in C(J_\mathbb{T}, \mathbb{R})$ such that

$$g_1(t) \leq f(t, y) \leq g_2(t) \quad \forall t \in J_\mathbb{T}, \; y \in \mathbb{R}, \tag{4.2}$$

and, for each $t \in J_\mathbb{T}$,

$$\int_0^t g_1(s)\Delta s \leq I_k \left(\int_0^t g_1(s)\Delta s \right), \quad k \in \mathbb{N},$$
$$\int_0^t g_2(s)\Delta s \geq I_k \left(\int_0^t g_2(s)\Delta s \right), \quad k \in \mathbb{N}. \tag{4.3}$$

Consider the functions $\alpha(t) := \int_0^t g_1(s)\Delta s$ and $\beta(t) := \int_0^t g_2(s)\Delta s$. Clearly, α and β are lower and upper solutions of the problem (4.1), respectively; that is,

$$\alpha^\Delta(t) \leq f(t, y) \quad \forall t \in J_\mathbb{T} \quad \text{and all } y \in \mathbb{R},$$
$$\beta^\Delta(t) \geq f(t, y) \quad \forall t \in J_\mathbb{T} \quad \text{and all } y \in \mathbb{R}. \tag{4.4}$$

Since all the conditions of Theorem 3.4 are satisfied, the problem (4.1) has at least one solution y on $J_{\mathbb{T}}$ with $\alpha \leq y \leq \beta$. If $g_1(t) > 0$, then α is positive and nondecreasing, thus $y(t)$ is nonoscillatory. If $g_2(t) < 0$, then β is negative and nonincreasing, thus $y(t)$ is nonoscillatory. If the sequences $\alpha(t_k)$ and $\beta(t_k)$ are both oscillatory, then $y(t)$ is oscillatory.

Acknowledgments

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