EXPONENTIAL DICHOTOMY OF DIFFERENCE EQUATIONS IN l_p-PHASE SPACES ON THE HALF-LINE

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For a sequence of bounded linear operators $\{A_n\}_{n=0}^{\infty}$ on a Banach space X, we investigate the characterization of exponential dichotomy of the difference equations $v_{n+1} = A_n v_n$. We characterize the exponential dichotomy of difference equations in terms of the existence of solutions to the equations $v_{n+1} = A_n v_n + f_n$ in l_p spaces ($1 \le p < \infty$). Then we apply the results to study the robustness of exponential dichotomy of difference equations.

1. Introduction and preliminaries

We consider the difference equation

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{N}, \tag{1.1}$$

where A_n, $n = 0, 1, 2, \ldots$, is a sequence of bounded linear operators on a given Banach space X, $x_n \in X$ for $n \in \mathbb{N}$.

One of the central interests in the asymptotic behavior of solutions to (1.1) is to find conditions for solutions to (1.1) to be stable, unstable, and especially to have an exponential dichotomy (see, e.g., [1, 5, 7, 12, 16–20] and the references therein for more details on the history of this problem). One can also use the results on exponential dichotomy of difference equations to obtain characterization of exponential dichotomy of evolution equations through the discretizing processes (see, e.g., [4, 7, 9, 18]).

One can easily see that if $A_n = A$ for all $n \in \mathbb{N}$, then the asymptotic behavior of solutions to (1.1) can be determined by the spectra of the operator A. However, the situation becomes more complicated if $\{A_n\}_{n\in\mathbb{N}}$ is not a constant sequence because, in this case, the spectra of each operator A_n cannot determine the asymptotic behavior of the solutions to (1.1). Therefore, in order to find the conditions for (1.1) to have an exponential dichotomy, one tries to relate the exponential dichotomy of (1.1) to the solvability of the

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following inhomogeneous equation:

$$x_{n+1} = A_n x_n + f_n, \quad n \in \mathbb{N}, \tag{1.2}$$

in some certain sequence spaces for each given $f = \{f_n\}$. In other words, one wants to relate the exponential dichotomy of (1.1) to the surjectiveness of the operator T defined by

$$(Tx)_n := x_{n+1} - A_n x_n \quad \text{for } x = \{x_n\} \text{ belonging to a relevant sequence space.} \tag{1.3}$$

In the infinite-dimensional case, in order to characterize the exponential dichotomy of (1.1) defined on $\mathbb{N}$, beside the surjectiveness of the operator T, one needs a priori condition that the stable space is complemented (see, e.g., [5]). In our recent paper, we have replaced this condition by the spectral conditions of related operators (see [9, Corollary 3.3]).

At this point, we would like to note that if one considers the difference equation (1.1) defined on $\mathbb{Z}$, then the existence of exponential dichotomy of (1.1) is equivalent to the existence and uniqueness of the solution of (1.2) for a given $f = \{f_n\}_{n\in\mathbb{Z}}$, or, in other words, to the invertibility of the operator T on suitable sequence spaces defined on $\mathbb{Z}$. This means that one can drop the above priori condition in the case that the difference equations are defined on $\mathbb{Z}$ (see [7, Theorem 3.3] for the original result and see also [2, 3, 11, 15] for recent results on the exponential dichotomy of difference equations defined on $\mathbb{Z}$).

However, if one considers the difference equation (1.1) defined only on $\mathbb{N}$, then the situation becomes more complicated, because for a given $f = \{f_n\}_{n\in\mathbb{N}}$, the solutions of the difference equation (1.2) on $\mathbb{N}$ are not unique even in the case that the difference equation (1.1) has an exponential dichotomy. Moreover, one does not have any information on the negative half-line $\mathbb{Z}_- := \{z \in \mathbb{Z} : z \leq 0\}$ of the difference equations (1.1) and (1.2) (we refer the readers to [8] for more details on the differences between the exponential dichotomy of the differential equations defined on the half-line and on the whole line). Therefore, one needs new ideas and new techniques to handle the exponential dichotomy of difference equations defined only on $\mathbb{N}$. For differential equations defined on the half-line, such ideas and techniques have appeared in [14] (see also [8, 13]). Those ideas and techniques have been exploited to obtain the characterization of exponential dichotomy of difference equations defined on $\mathbb{N}$ with l_∞-phase space of sequences defined on $\mathbb{N}$ (see [9]). As a result, we have obtained a necessary and sufficient condition for difference equations to have an exponential dichotomy. This conditions related to the solvability of (1.2) in l_∞ spaces of sequences defined on $\mathbb{N}$. In the present paper, we will characterize the exponential dichotomy of (1.1) by the solvability of (1.2) in l_p spaces ($1 \leq p < \infty$) of sequences defined on $\mathbb{N}$. Moreover, we also characterize the exponential dichotomy by invertibility of a certain appropriate difference operator derived from the operator T. Consequently, we will use this characterization to prove the robustness of exponential dichotomy under small perturbations. Our results are contained in Theorems 3.2, 3.6, 3.7, and Corollary 3.3.

To describe more detailedly our construction, we will use the following notation: in this paper X denotes a given complex Banach space endowed with the norm $\|\cdot\|$. As usual, we denote by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{C}$ the sets of natural, real, nonnegative real, and complex numbers, respectively. Throughout this paper, for $1 \leq p < \infty$ we will consider the following sequence spaces:

$$\begin{aligned} l_p(\mathbb{N},X) &:= \left\{ v = \{v_n\}_{n\in\mathbb{N}} : v_n \in X : \sum_{n=0}^{\infty} \|v_n\|^p < \infty \right\} := l_p, \\ l_p^0(\mathbb{N},X) &:= \{v = \{v_n\} : v \in l_p;\ v_0 = 0\} := l_p^0 \end{aligned} \tag{1.4}$$

endowed with the norm $\|v\|_p := (\sum_{n=0}^{\infty} \|v_n\|^p)^{1/p}$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of bounded linear operators from X to X which is uniformly bounded. This means that there exists $M > 0$ such that $\|A_n x\| \leq M\|x\|$ for all $n \in \mathbb{N}$ and $x \in X$. Next we define a discrete evolution family $\mathcal{U} = (U_{n,m})_{n\geq m\geq 0}$ associated with the sequence $\{A_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} U_{m,m} &= \mathrm{Id} \quad \text{(the identity operator in } X\text{)} \\ U_{n,m} &= A_{n-1}A_{n-2}\cdots A_m \quad \text{for } n > m. \end{aligned} \tag{1.5}$$

The uniform boundedness of $\{A_n\}$ yields the exponential boundedness of the evolution family $(U_{n,m})_{n\geq m\geq 0}$. That is, there exist positive constants K, α such that $\|U_{n,m}x\| \leq Ke^{\alpha(n-m)}\|x\|$; $x \in X$; $n \geq m \geq 0$.

Definition 1.1. Equation (1.1) is said to have an *exponential dichotomy* if there exist a family of projections $(P_n)_{n\in\mathbb{N}}$ on X and positive constants N, ν such that

(1) $A_nP_n = P_{n+1}A_n$;
(2) $A_n : \ker P_n \to \ker P_{n+1}$ is an isomorphism and its inverse is denoted by $A_{|n}^{-1}$;
(3) $\|U_{n,m}x\| \leq Ne^{-\nu(n-m)}\|x\|$; $n \geq m \geq 0$; $x \in P_mX$;
(4) denote $U_{|m,n} = A_{|m}^{-1}A_{|m+1}^{-1}\cdots A_{|n-1}^{-1}$; $n > m$, and $U_{|m,m} = \mathrm{Id}$, then

$$\left\|U_{|m,n}x\right\| \leq Ne^{-\nu(n-m)}\|x\|, \quad n \geq m \geq 0;\ x \in \ker P_n. \tag{1.6}$$

The above family of projections $(P_n)_{n\in\mathbb{N}}$ is called the family of dichotomy projections.

We define a linear operator T as follows:

$$\text{If } u = \{u_n\} \in l_p \quad \text{set } (Tu)_n = u_{n+1} - A_nu_n \quad \text{for } n \in \mathbb{N}. \tag{1.7}$$

For $u = \{u_n\} \in l_p$, we have $\|(Tu)_n\| = \|u_{n+1} - A_nu_n\| \leq \|u_{n+1}\| + M\|u_n\|$, hence $Tu \in l_p$ and $\|Tu\|_p \leq (1+M)\|u\|_p$. This means that T is a bounded linear operator from l_p into l_p. We denote the restriction of T on l_p^0 by T_0, this means that $D(T_0) = l_p^0$ and $T_0u = Tu$ for $u \in l_p^0$. From the definition of T, the following are obvious.

Remark 1.2. (i) $\ker T = \{u = \{u_n\} \in l_p : u_n = U_{n,0}u_0,\ n \in \mathbb{N}\}$.

(ii) It is easy to verify that T_0 is injective. Indeed, let $u = \{u_n\}$, $v = \{v_n\} \in l_p^0$ and $T_0u = T_0v$. Then we have $u_0 = v_0 = 0$, $u_1 = (T_0v)_0 = v_1$, $u_2 = A_1u_1 + (T_0u)_1 = A_1v_1 + (T_0v)_1 = v_2, \ldots$, $u_{n+1} = A_nu_n + (T_0u)_n = A_nv_n + (T_0v)_n = v_{n+1}$, for all $n \in \mathbb{N}$. Hence, $u = v$.

Recall that for an operator B on a Banach space Y, the approximate point spectrum $A\sigma(B)$ of B is the set of all complex numbers λ such that for every $\epsilon > 0$, there exists $y \in D(B)$ with $\|y\| = 1$ and $\|(\lambda - B)y\| \leq \epsilon$.

In order to characterize the exponential stability and dichotomy of an evolution family, we need the concept of l_p-stable spaces defined as follows.

Definition 1.3. For a discrete evolution family $\mathcal{U} = (U_{m,n})_{m\geq n\geq 0}$, $m,n \in \mathbb{N}$, on Banach space X and $n_0 \in \mathbb{N}$, define the *l_p-stable space* $X_0(n_0)$ by

$$X_0(n_0) := \left\{ x \in X : \sum_{n=n_0}^{\infty} ||U_{n,n_0}x||^p < \infty \right\}. \tag{1.8}$$

An orbit $U_{m,n_0}x$ for $m \geq n_0 \geq 0$ and $x \in X_0(n_0)$ is called an l_p-stable orbit.

2. Exponential stability

In this section we will give a sufficient condition for stability of l_p-stable orbits of a discrete evolution family $\mathcal{U}$. The obtained results will be used in the next section to characterize the exponential dichotomy of (1.1).

Theorem 2.1. *Let the operator T_0 defined as above satisfy the condition $0 \notin A\sigma(T_0)$. Then every l_p-stable orbit of $\mathcal{U}$ is exponentially stable. Precisely, there exist positive constants N, ν such that for any $n_0 \in \mathbb{N}$ and $x \in X_0(n_0)$,*

$$||U_{n,n_0}x|| \leq Ne^{-\nu(n-s)}||U_{s,n_0}x||, \quad n \geq s \geq n_0. \tag{2.1}$$

Proof. Since $0 \notin A\sigma(T_0)$, we have that there exists a constant $\eta > 0$ such that

$$\eta||T_0v||_p \geq \|v\|_p \quad \text{for } v \in l_p^0. \tag{2.2}$$

To prove (2.1), we first prove that there is a positive constant l such that for any $n_0 \in \mathbb{N}$ and $x \in X_0(n_0)$,

$$||U_{n,n_0}x|| \leq l||U_{s,n_0}x||, \quad n \geq s \geq n_0 \geq 0. \tag{2.3}$$

Fix $n_0 \in \mathbb{N}$, $x \in X_0(n_0)$, and $s \geq n_0$. Taking

$$v = \{v_n\} \quad \text{with } v_n := \begin{cases} U_{n,n_0}x & \text{for } n > s, \\ 0 & \text{for } 0 \leq n \leq s, \end{cases} \tag{2.4}$$

we have $v \in l_p^0$. By definition of T_0, we have $(T_0v)_n = v_{n+1} - A_nv_n$. This yields

$$(T_0v)_n = \begin{cases} 0 & \text{for } n \leq s-1, \\ U_{s+1,n_0}x & \text{for } n = s, \\ 0 & \text{for } n > s. \end{cases} \tag{2.5}$$

By inequality (2.2), we have

$$\eta||U_{s+1,n_0}x|| \geq \left(\sum_{k=s}^{\infty}||U_{k,n_0}x||^p\right)^{1/p} \geq ||U_{n,n_0}x|| \quad \text{for } n > s \geq n_0. \tag{2.6}$$

Hence,

$$||U_{n,n_0}x|| \leq \eta||U_{s+1,n_0}x|| = \eta||U_{s+1,s}U_{s,n_0}x|| \leq \eta M||U_{s,n_0}x|| \quad \text{for } n > s \geq n_0. \tag{2.7}$$

Putting $l = \max\{1, \eta M\}$, we obtain (2.3).

We now show that there is a number $K = K(\eta, l) > 0$ such that for any $n_0 \in \mathbb{N}$ and $x \in X_0(n_0)$,

$$||U_{s+n,n_0}x|| \leq \frac{1}{2}||U_{s,n_0}x|| \quad \text{for } n \geq K,\ s \geq n_0. \tag{2.8}$$

To prove (2.8), put $u_n := U_{n,n_0}x$, $n \geq n_0$, and let $a < b$ be two natural numbers with $a \geq n_0$ such that $\|u_b\| > 1/2\|u_a\|$. From (2.3), we obtain that

$$l||u_a|| \geq ||u_n|| > \frac{1}{2l}||u_a|| \quad \text{for } a \leq n \leq b. \tag{2.9}$$

Put now

$$v = \{v_n\} \quad \text{with } v_n = \begin{cases} 0 & \text{for } 0 \leq n \leq a, \\ u_n \displaystyle\sum_{k=a+1}^{n} \frac{1}{||u_k||} & \text{for } a+1 \leq n \leq b, \\ u_n \displaystyle\sum_{k=a+1}^{b+1} \frac{1}{||u_k||} & \text{for } n \geq b+1. \end{cases} \tag{2.10}$$

Then $v \in l_p^0$. By definition of T_0, we have

$$T_0 v = \left\{(T_0 v)_n\right\} \quad \text{with } (T_0 v)_n = \begin{cases} 0, & \text{for } 0 \leq n < a, \\ \dfrac{u_{n+1}}{||u_{n+1}||} & \text{for } a \leq n \leq b-1, \\ 0 & \text{for } n \geq b. \end{cases} \tag{2.11}$$

By inequality (2.2), we obtain

$$\eta(b-a)^{1/p} \geq \|v\|_p. \tag{2.12}$$

Using Hölder inequality for v and $\chi_{[a+1,b]}$, where

$$\left(\chi_{[a+1,b]}\right)_n = \begin{cases} 1 & \text{for } a+1 \leq n \leq b, \\ 0 & \text{otherwise}, \end{cases} \tag{2.13}$$

we have that

$$\sum_{n=a+1}^{b} ||v_n|| \leq (b-a)^{1-1/p}\|v\|_p. \tag{2.14}$$

Substituting this into inequality (2.12), we obtain that

$$\eta(b-a) \geq \sum_{n=a+1}^{b} ||v_n||. \tag{2.15}$$

Using now the estimates (2.9), we have

$$\begin{aligned} \eta(b-a) &\geq \sum_{n=a+1}^{b} ||v_n|| = \sum_{n=a+1}^{b} \sum_{k=a+1}^{n} \frac{||u_n||}{||u_k||} \\ &\geq \sum_{n=a+1}^{b} \frac{1}{2l}||u_a|| \sum_{k=a+1}^{n} \frac{1}{l||u_a||} = \frac{(b-a)(b-a+1)}{4l^2} > \frac{(b-a)^2}{4l^2}. \end{aligned} \tag{2.16}$$

This yields $b-a < 4\eta l^2$. Putting $K := 4\eta l^2$, the inequality (2.1) follows.

We finish by proving (2.1). Indeed, if $n \geq s \geq n_0 \in \mathbb{N}$ writing $n-s = n_1K + r$ for $0 \leq r < K$, and $n_1 \in \mathbb{N}$, we have

$$\begin{aligned} ||U_{n,n_0}x|| &= ||U_{n-s+s,n_0}x|| = ||U_{n_1K+r+s,n_0}x|| \\ &\overset{\text{by (2.8)}}{\leq} \frac{1}{2^{n_1}}||U_{r+s,n_0}x|| \overset{\text{by (2.3)}}{\leq} \frac{l}{2^{n_1}}||U_{s,n_0}x|| \leq 2le^{-((n-s)/K)\ln 2}||U_{s,n_0}x||. \end{aligned} \tag{2.17}$$

Taking $N := 2l$ and $\nu := \ln 2/K$, the inequality (2.1) follows. □

From this theorem, we obtain the following corollary.

Corollary 2.2. *Under the conditions of Theorem 2.1, the space $X_0(n_0)$ can be expressed as*

$$X_0(n_0) = \{x \in X : ||U_{n,n_0}x|| \leq Ne^{-\nu(n-n_0)}\|x\|;\ n \geq n_0 \geq 0\}, \tag{2.18}$$

for certain positive constants N, ν. Hence, $X_0(n_0)$ is a closed linear subspace of X.

3. Exponential dichotomy and perturbations

We will characterize the exponential dichotomy of (1.1) by using the operators T_0, T. In particular, we will also get necessary and sufficient conditions for exponential dichotomy in Hilbert spaces and finite-dimensional spaces. Moreover, using our characterization of the exponential dichotomy, we can prove the robustness of the exponential dichotomy of (1.1) under small perturbations. Then we start with the following lemma which has a history that can be traced back to [14, Lemma 4.2] and to [6] and beyond.

Lemma 3.1. *Assume that (1.1) has an exponential dichotomy with corresponding family of projections P_n, $n \geq 0$, and constants $N > 0$, $\nu > 0$, then $M := \sup_{n\geq 0}\|P_n\| < \infty$.*

Proof. The proof is done in [9, Lemma 3.1]. We present it here for sake of completeness. Fix $n_0 > 0$, and set $P^0 := P_{n_0}$; $P^1 := \mathrm{Id} - P_{n_0}$, $X_k = P^k X$, $k = 0, 1$. Set $\gamma_0 := \inf\{\|x^0 + x^1\| : x^k \in X_k,\ \|x^0\| = \|x^1\| = 1\}$. If $x \in X$ and $P^k x \neq 0$, $k = 0, 1$, then

$$\begin{aligned}\gamma_{n_0} &\leq \left\| \frac{P^0 x}{\|P^0 x\|} + \frac{P^1 x}{\|P^1 x\|} \right\| \leq \frac{1}{\|P^0 x\|} \left\| P^0 x + \frac{\|P^0 x\|}{\|P^1 x\|} P^1 x \right\| \\ &\leq \frac{1}{\|P^0 x\|} \left\| x + \frac{\|P^0 x\| - \|P^1 x\|}{\|P^1 x\|} P^1 x \right\| \leq \frac{2\|x\|}{\|P^0(x)\|}.\end{aligned} \tag{3.1}$$

Hence, $\|P^0\| \leq 2/\gamma_{n_0}$. It remains to show that there is a constant $c > 0$ (independent of n_0) such that $\gamma_{n_0} \geq c$. For this, fix $x^k \in X_k$, $k = 0, 1$, with $\|x^k\| = 1$. By the exponential boundedness of $\mathcal{U}$, we have $\|U_{n,n_0}(x^0 + x^1)\| \leq K e^{\alpha(n-n_0)} \|x^0 + x^1\|$ for $n \geq n_0$ and constants K, $\alpha \geq 0$. Thus,

$$\begin{aligned}\|x^0 + x^1\| &\geq K^{-1} e^{-\alpha(n-n_0)} \|U_{n,n_0} x^0 + U_{n,n_0} x^1\| \\ &\geq K^{-1} e^{-\alpha(n-n_0)} \left(N^{-1} e^{\nu(n-n_0)} - N e^{-\nu(n-n_0)}\right) =: c_{n-n_0},\end{aligned} \tag{3.2}$$

and hence $\gamma_{n_0} \geq c_{n-n_0}$. Obviously $c_m > 0$ for m sufficiently large. Thus $0 < c_m \leq \gamma_{n_0}$. □

Now we come to our first main result. It characterizes the exponential dichotomy of (1.1) by properties of the operator T.

Theorem 3.2. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of bounded linear and uniformly bounded operators on the Banach space X. Then the following assertions are equivalent.*

(i) *Equation (1.1) has an exponential dichotomy.*

(ii) *T is surjective and $X_0(0)$ is complemented in X.*

Proof. (i)⇒(ii). Let $(P_n)_{n\geq 0}$ be the family of dichotomy projections. Then $X_0(0) = P_0 X$, and hence $X_0(0)$ is complemented. If $f \in l_p$, define $v = \{v_n\}_{n\in\mathbb{N}}$ by

$$v_n = \begin{cases} \displaystyle\sum_{k=1}^{n} U_{n,k} P_k f_{k-1} - \sum_{k=n+1}^{\infty} U_{|n,k}(\mathrm{Id} - P_k) f_{k-1} & \text{for } n \geq 1, \\ \displaystyle -\sum_{k=1}^{\infty} U_{|0,k}(\mathrm{Id} - P_k) f_{k-1} & \text{for } n = 0, \end{cases} \tag{3.3}$$

then $v_{n+1} = A_n v_n + f_n$. Moreover, since

$$\left\| \sum_{k=1}^{n} U_{n,k} P_k f_{k-1} - \sum_{k=n+1}^{\infty} U_{|n,k}(\mathrm{Id} - P_k) f_{k-1} \right\| \leq N \sum_{k=1}^{\infty} e^{-\nu|n-k|} \|f_{k-1}\| \tag{3.4}$$

and $f \in l_p$, we can easily derive that $v \in l_p$. By the definition of T, we have $Tv = f$. Therefore $T : l_p \to l_p$ is surjective.

(ii)⇒(i). We prove this in several steps.

(A) Let $Z \subseteq X$ be a complement of $X_0(0)$ in X, that is, $X = X_0(0) \oplus Z$. Set $X_1(n) = U_{n,0}Z$. Then

$$U_{n,s}X_0(s) \subseteq X_0(n), \quad U_{n,s}X_1(s) = X_1(n), \quad n \geq s \geq 0. \tag{3.5}$$

(B) There are constants $N, \nu > 0$ such that

$$||U_{n,0}x|| \geq Ne^{\nu(n-s)}||U_{s,0}x|| \quad \text{for } x \in X_1(0),\ n \geq s \geq 0. \tag{3.6}$$

In fact, let $Y := \{(v_n)_{n\in\mathbb{N}} \in l_p : v_0 \in X_1(0)\}$ endowed with l_p-norm. Then Y is a closed subspace of the Banach space l_p, and hence Y is complete. By Remark 1.2, we have $\ker T := \{v \in l_p : v_n = U_{n,0}x \text{ for some } x \in X_0(0)\}$. Since $X = X_0(0) \oplus X_1(0)$ and T is surjective, we obtain that

$$T : Y \longrightarrow l_p \tag{3.7}$$

is bijective and hence an isomorphism. Thus, by Banach isomorphism theorem, there is a constant $\eta > 0$ such that

$$\eta\|Tv\|_p \geq \|v\|_p, \quad \text{for } v \in Y. \tag{3.8}$$

To prove (3.6), we first prove that there is a positive constant l such that

$$||U_{n,0}x|| \geq l||U_{s,0}x|| \quad \text{for } x \in X_1(0),\ n \geq s \geq 0,\ n,s \in \mathbb{N}. \tag{3.9}$$

Indeed, fix $x \in X_1(0)$, $x \neq 0$, and $n \geq s \geq 0$. If $n = 0$, there is nothing to do. So, assume that $n \geq 1$. Now taking

$$v := \{v_m\} \quad \text{with } v_m := \begin{cases} U_{m,0}x & \text{for } 0 \leq m \leq n-1, \\ 0 & \text{for } m > n-1, \end{cases} \tag{3.10}$$

we have that $v \in Y$. Then, by definition of T, we obtain that

$$(Tv)_m := \begin{cases} 0 & \text{for } m > n-1, \\ -U_{n,0}x & \text{for } m = n-1, \\ 0 & \text{for } m < n-1. \end{cases} \tag{3.11}$$

Inequality (3.8) yields

$$\eta||U_{n,0}x|| \geq \left(\sum_{k=0}^{n-1}||U_{k,0}x||^p\right)^{1/p} \geq ||U_{s,0}x|| \quad \forall\ 0 \leq s \leq n-1. \tag{3.12}$$

Putting now $l := \min\{1/\eta, 1\}$, inequality (3.9) follows.

We now show that there is a number $K = K(\eta, l) > 0$ such that

$$||U_{s+n,0}x|| \geq 2||U_{s,0}x|| \quad \text{for } n \geq K,\ s \geq 0;\ x \in X_1(0). \tag{3.13}$$

Let $0 \neq x \in X_1(0)$, set $u_n := U_{n,0}x$, $n \geq 0$. By Remark 1.2 we have $u_n \neq 0$ for all $n \geq 0$. To prove (3.13), let $a < b$ be two natural numbers such that $\|u_b\| < 2\|u_a\|$. From (3.9), we obtain that

$$\frac{2}{l}||u_a|| > ||u_n|| \geq l||u_a|| \quad \forall\ a \leq n \leq b. \tag{3.14}$$

Take now $v = \{v_n\}$, where

$$v_n = \begin{cases} -u_n \displaystyle\sum_{k=a+1}^{b} \frac{1}{||u_k||} & \text{for } 0 \leq n < a, \\ -u_n \displaystyle\sum_{k=n+1}^{b} \frac{1}{||u_k||} & \text{for } a \leq n < b, \\ 0 & \text{for } n \geq b. \end{cases} \tag{3.15}$$

Then, $v \in Y$. By definition of T, we have that

$$(Tv)_n = \begin{cases} 0 & \text{for } 0 \leq n < a, \\ \dfrac{u_{n+1}}{||u_{n+1}||} & \text{for } a \leq n < b, \\ 0 & \text{for } n \geq b. \end{cases} \tag{3.16}$$

By inequality (3.8), we obtain

$$\eta(b-a)^{1/p} \geq \|v\|_p. \tag{3.17}$$

Using Hölder inequality for v and $\chi_{[a,b-1]}$, where

$$(\chi_{[a,b-1]})_n = \begin{cases} 1 & \text{for } a \leq n \leq b-1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.18}$$

we have that

$$\sum_{n=a}^{b-1} ||v_n|| \leq (b-a)^{1-1/p} \|v\|_p. \tag{3.19}$$

Substituting this into inequality (3.17), we obtain that

$$\eta(b-a) \geq \sum_{n=a}^{b-1} ||v_n||. \tag{3.20}$$

Using now the estimates (3.14), we have

$$\begin{aligned}\eta(b-a) &\geq \sum_{n=a}^{b-1}||v_n|| = \sum_{n=a}^{b-1}\sum_{k=n+1}^{b}\frac{||u_n||}{||u_k||}\\ &\geq \sum_{n=a}^{b-1} l||u_a|| \sum_{k=n+1}^{b}\frac{l}{2||u_a||} = \frac{l^2(b-a)(b-a+1)}{4} > \frac{l^2(b-a)^2}{4}.\end{aligned} \tag{3.21}$$

This yields $b-a < 4\eta/l^2$. Putting $K := 4\eta/l^2$, the inequality (3.13) follows.

We finish this step by proving inequality (3.6). Indeed, if $n \geq s \in \mathbb{N}$, writing $n-s = n_0K + r$ for $0 \leq r < K$, and $n_0 \in \mathbb{N}$, we have

$$\begin{aligned}||U_{n,0}x|| &= ||U_{n-s+s,0}x|| = ||U_{n_0K+r+s,0}x||\\ &\overset{\text{by (3.13)}}{\geq} 2^{n_0}||U_{r+s,0}x|| \overset{\text{by (3.9)}}{\geq} l2^{n_0}||U_{s,0}x|| \geq \frac{l}{2}e^{((n-s)/K)\ln 2}||U_{s,0}x||.\end{aligned} \tag{3.22}$$

Taking $N := l/2$ and $\nu := \ln 2/K$, inequality (3.6) follows.

(C) $X = X_0(n) \oplus X_1(n)$, $n \in \mathbb{N}$.

Let $Y \subset l_p$ be as in (B). Then by Remark 1.2, we have that $l_p^0 \subset Y$. From this fact and (3.8), we obtain that $\eta\|T_0v\|_{l_p} \geq \|v\|_{l_p}$, for $v \in l_p^0$. Thus,

$$0 \notin A\sigma(T_0). \tag{3.23}$$

The relation (3.23) and Corollary 2.2 imply that $X_0(n)$ is closed. From (3.5), (3.6), and the closedness of $X_1(0)$, we can easily derive that $X_1(n)$ is closed and $X_1(n) \cap X_0(n) = \{0\}$ for $n \geq 0$.

Finally, fix $n_0 > 0$, and $x \in X$ (note that we already have $X = X_0(0) \oplus X_1(0)$). For a natural number $n_1 > n_0 + 1$, set

$$\begin{gathered} v = \{v_n\} \quad \text{with } v_n = \begin{cases} 0 & \text{for } 0 \leq n < n_0,\\ (n-n_0+1)U_{n,n_0}x & \text{for } n_0 \leq n \leq n_1,\\ 0 & \text{for } n > n_1,\end{cases}\\ f = \{f_n\} \quad \text{with } f_n = \begin{cases} 0 & \text{for } 0 \leq n < n_0,\\ U_{n+1,n_0}x & \text{for } n_0 \leq n < n_1,\\ -(n_1-n_0+1)U_{n+1,n_0}x & \text{for } n = n_1,\\ 0 & \text{for } n > n_1.\end{cases}\end{gathered} \tag{3.24}$$

Then $v, f \in l_p$ and satisfy (1.2) for all $n \geq n_0 > 0$. By assumption, there exists $w \in l_p$ such that $Tw = f$. By the definition of T, w_n is a solution of (1.2). Thus,

$$v_n - w_n = U_{n,n_0}(v_{n_0} - w_{n_0}) = U_{n,n_0}(x - w_{n_0}), \quad n \geq n_0. \tag{3.25}$$

Since $v - w \in l_p$, we have that $x - w_{n_0} \in X_0(n_0)$. On the other hand, since $w_0 = w^0 + w^1$ with $w^k \in X_k(0)$, $w_{n_0} = U_{n_0,0}w^0 + U_{n_0,0}w^1$, and by (3.5), we have $U_{n_0,0}w^k \in X_k(n_0)$, $k = 0,1$. Hence $x = x - w_{n_0} + w_{n_0} = x - w_{n_0} + U_{n_0,0}w^0 + U_{n_0,0}w^1 \in X_0(n_0) + X_1(n_0)$. This proves (C).

(D) Let P_n be the projections from X onto $X_0(n)$ with kernel $X_1(n)$, $n \geq 0$. Then (3.5) implies that $P_{n+1}U_{n+1,n} = U_{n+1,n}P_n$, or $A_nP_n = P_{n+1}A_n$ for $n \geq 0$. From (3.5), (3.6), and $A_n = U_{n+1,n}$, we obtain that $A_n : \ker P_n \to \ker P_{n+1}$, $n \geq 0$ is an isomorphism. Finally, by (3.6), Theorem 2.1, and the assumption $0 \notin A\sigma(T_0)$, there exist constants $N, \nu > 0$ such that

$$\begin{aligned} ||U_{n,m}x|| &\leq Ne^{-\nu(n-m)}\|x\| \quad \text{for } x \in P_mX,\ n \geq m \geq 0, \\ ||U_{|m,n}x|| &\leq Ne^{-\nu(n-m)}\|x\| \quad \text{for } x \in \ker P_n,\ n \geq m \geq 0. \end{aligned} \tag{3.26}$$

Thus (1.1) has an exponential dichotomy. □

If X is a Hilbert space, we need only the closedness of $X_0(0)$. Therefore, we obtain the following corollary.

Corollary 3.3. *If X is a Hilbert space, then the conditions that $0 \notin A\sigma(T_0)$ and T is surjective are necessary and sufficient for (1.1) to have an exponential dichotomy.*

This can be restated as follows.

If X is a Hilbert space, then the conditions

(1) *for all $f \in l_p$, there exists a solution $x \in l_p$ of (1.2);*

(2) *there exists a constant $c > 0$ such that all bounded solutions $x = \{x_n\}$ (with $x_0 = 0$ and $x \in l_p$) of (1.2) (with $f \in l_p$) satisfy $\sum_{n=0}^{\infty} \|x_n\|^p \leq c\sum_{n=0}^{\infty} \|f_n\|^p$*

are necessary and sufficient for (1.1) to have an exponential dichotomy.

Proof. The corollary is obvious in view of Corollary 2.2 and Theorem 3.2. □

If X is a finite-dimensional space, then every subspace of X is closed and complemented. Hence, by Theorem 3.2 we have the following corollary.

Corollary 3.4. *If X is a finite-dimensional space, then the condition that T is surjective is necessary and sufficient for existence of exponential dichotomy of (1.1).*

In our next result, we will characterize the exponential dichotomy of (1.1) using invertibility of a certain operator derived from the operator T. In order to obtain such a characterization, we have to know the subspace $\ker P_0$ in advance (see Theorem 3.6). Consequently, the exponential dichotomy of evolution family will be characterized by the invertibility of the restriction of T to a certain subspace of l_p. This restriction will be defined as follows.

Definition 3.5. For a closed linear subspace Z of X, define

$$l_p^Z := \{f = \{f_n\} \in l_p : f_0 \in Z\}. \tag{3.27}$$

Then, l_p^Z is a closed subspace of $(l_p, \|\cdot\|_p)$. Denote by T_Z the part of T in l_p^Z, that is, $D(T_Z) = l_p^Z$ and $T_Zu = Tu$ for $u \in l_p^Z$.

With these notations, we obtain the following characterization of exponential dichotomy of (1.1).

Theorem 3.6. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of bounded linear and uniformly bounded operators on the Banach space X and let Z be a closed linear subspace of X. Then the following assertions are equivalent.*

(i) *Equation (1.1) has an exponential dichotomy with the family of dichotomy projections $\{P_n\}_{n\in\mathbb{N}}$ satisfying* $\ker P_0 = Z$.

(ii) $T_Z : l_p^Z \to l_p$ *is invertible.*

Proof. We first note that the following proof is inspired by the proof of [14, Theorem 4.5].

(i)⇒(ii). Let P_n, $n \in \mathbb{N}$, be a family of projections given by the exponential dichotomy of (1.1) such that $\ker P_0 = Z$. Then $P_0X = X_0(0)$ and $X = X_0(0) \oplus Z$. Fix $f = \{f_n\} \in l_p$. By Theorem 3.2, there is $v = \{v_n\} \in D(T)$ such that $Tv = f$. On the other hand, by definition of $X_0(0)$, the sequence $u = \{u_n\}$ defined by $u_n = U_{n,0}P_0v_0$ belongs to l_p. By definition of T, we obtain that $Tu = 0$. Moreover, $v_0 - u_0 = v_0 - P_0v_0 \in Z$ since $X = P_0X \oplus Z$. Therefore, $v - u \in l_p^Z$ and $T_Z(v-u) = T(v-u) = Tv = f$. Hence, $T_Z : D(T_Z) \to l_p$ is surjective.

If now $w = \{w_n\} \in \ker T_Z$ then, by definition of T_Z, $w_n = U_{n,0}w_0$ with $w_0 \in Z \cap X_0(0) = \{0\}$. Thus, $w = 0$, that is, T_Z is injective.

(ii)⇒(i). Let $T_Z : l_p^Z \to l_p$ be invertible. Since T_Z is the restriction of T to l_p^Z, it follows that T is surjective. The boundedness of T_Z implies that T_Z^{-1} is bounded, and hence there is $\eta > 0$ such that $\eta\|Tv\|_p = \eta\|T_Zv\|_p \geq \|v\|_p$ for all $v \in l_p^Z$. Since T_0 is the part of T_Z in $l_p^0 = \{f \in l_p : f_0 = 0\}$, we obtain that $\eta\|T_0v\|_p \geq \|v\|_p$ for all $v \in D(T_0)$. Hence, $0 \notin A\sigma(T_0)$. By Corollary 2.2, $X_0(0)$ is closed. We now prove that $X = X_0(0) \oplus Z$. Let now $x \in X$. If $U_{n,0}x = 0$ for some $n = n_0 > 0$, then $U_{n,0}x = U_{n,n_0}U_{n_0,0}x = 0$ for all $n \geq n_0$ yielding $x \in X_0(0)$. Otherwise, $U_{n,0}x \neq 0$ for all $n \in \mathbb{N}$. Set $u = \{u_n\}$ with

$$u_n := \begin{cases} x & \text{for } n = 0 \\ 0 & \text{for } n > 0, \end{cases} \qquad f = \{f_n\} \quad \text{with } f_n := \begin{cases} -A_0x & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases} \tag{3.28}$$

Clearly, $u \in l_p$, $f \in l_p$, and $f_n = u_{n+1} - A_nu_n$. Therefore, $Tu = f$. On the other hand, since T_Z is invertible, there exists $v \in l_p^Z$ such that $T_Zv = f = Tv$. Thus, $u - v \in \ker T$, and hence

$$(u-v)_n = U_{n,0}(u_0 - v_0) = U_{n,0}(x - v_0), \quad n \in \mathbb{N}. \tag{3.29}$$

Since $u - v \in l_p$, this implies that $x - v_0 \in X_0(0)$. Thus $x = x - v_0 + v_0 \in X_0(0) + Z$.

If now $y \in X_0(0) \cap Z$, then the sequence $w = \{w_n\}$ defined by $w_n := U_{n,0}y$, $n \in \mathbb{N}$, belongs to $l_p^Z \cap \ker T$ (see definitions of $X_0(0)$ and T). Hence, $T_Zw = 0$ and by invertibility of T_Z, we have that $w = 0$. Thus $y = w_0 = 0$, that is, $X_0(0) \cap Z = \{0\}$. This yields that $X = X_0(0) \oplus Z$. The assertion now follows from Theorem 3.2. □

Using the above characterization of exponential dichotomy, we now study the robustness of the exponential dichotomy of (1.1) under small perturbations. Precisely, we have the following perturbation theorem.

Theorem 3.7. *Let (1.1) have an exponential dichotomy and let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of bounded linear operators from X to X which is uniformly bounded (i.e., there is constant*

$M > 0$ such that $\|B_n\| \le M$ for all $n \in \mathbb{N}$). Then, if $H := \sup_{n\in\mathbb{N}} \|B_n\|$ is sufficiently small, the equation

$$u_{n+1} = (A_n + B_n)u_n \tag{3.30}$$

has an exponential dichotomy as well.

Proof. Let (1.1) have an exponential dichotomy with the corresponding dichotomy projections $(P_n)_{n\in N}$. Put $\ker P_0 = Z$. Then, Z is a closed subspace of X. By Theorem 3.6, we have that the operator T_Z defined by Definition 3.5 is invertible. Let now $T_{B,Z}$ be the operator corresponding to the perturbed difference equation (3.30). That is, $T_{B,Z} : l_p^Z \to l_p$ is defined by $(T_{B,Z}u)_n = u_{n+1} - (A_n + B_n)u_n$. We now define the operator $\mathfrak{B}$ by $(\mathfrak{B}f)_n := B_n f_n$ for $f = \{f_n\}$ and all $n \in \mathbb{N}$. We then prove that $\mathfrak{B} : l_p \to l_p$ is a bounded linear operator and $\|\mathfrak{B}\| \le H$. Indeed, take $f \in l_p$. Then $\|(\mathfrak{B}f)_n\| = \|B_n f_n\| \le H\|f_n\|$ for all $n \in \mathbb{N}$. It follows that the sequence $\mathfrak{B}f$ belongs to l_p and $\|\mathfrak{B}f\|_p \le H\|f\|_p$. We thus obtain that $\mathfrak{B} : l_p \to l_p$ is a bounded linear operator and $\|\mathfrak{B}\| \le H$.

It is clear that $T_{B,Z} = T_Z - \mathfrak{B}$. Since T_Z is invertible, by a perturbation theorem of Kato [10, IV.1.16], we obtain that if $\|\mathfrak{B}\|$ is sufficiently small then $T_{B,Z} = T_Z - \mathfrak{B}$ is also invertible. By Theorem 3.6 we have that (3.30) has an exponential dichotomy. □

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References

[1] B. Aulbach and N. V. Minh, *The concept of spectral dichotomy for linear difference equations. II*, Journal of Difference Equations and Applications **2** (1996), no. 3, 251–262.

[2] D. D. Bainov, S. I. Kostadinov, and P. P. Zabreiko, *Stability of the notion of dichotomy of linear impulsive differential equations in a Banach space*, Italian Journal of Pure and Applied Mathematics **1** (1997), 43–50 (1998).

[3] A. G. Baskakov, *Semigroups of difference operators in the spectral analysis of linear differential operators*, Functional Analysis and Its Applications **30** (1996), no. 3, 149–157 (1997).

[4] S.-N. Chow and H. Leiva, *Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces*, Journal of Differential Equations **120** (1995), no. 2, 429–477.

[5] C. V. Coffman and J. J. Schäffer, *Dichotomies for linear difference equations*, Mathematische Annalen **172** (1967), no. 2, 139–166.

[6] Ju. L. Dalec'kiĭ and M. G. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Translations of Mathematical Monographs, vol. 43, American Mathematical Society, Rhode Island, 1974.

[7] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840, Springer, Berlin, 1981.

[8] N. T. Huy, *Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line*, Journal of Functional Analysis **235** (2006), no. 1, 330–354.

[9] N. T. Huy and N. V. Minh, *Exponential dichotomy of difference equations and applications to evolution equations on the half-line*, Computers & Mathematics with Applications **42** (2001), no. 3–5, 301–311.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York, 1980.

[11] Y. Latushkin and Y. Tomilov, *Fredholm differential operators with unbounded coefficients*, Journal of Differential Equations **208** (2005), no. 2, 388–429.
[12] T. Li, *Die Stabilitatsfrage bei Differenzengleichungen*, Acta Mathematica **63** (1934), 99–141.
[13] N. V. Minh and N. T. Huy, *Characterizations of dichotomies of evolution equations on the half-line*, Journal of Mathematical Analysis and Applications **261** (2001), no. 1, 28–44.
[14] N. V. Minh, F. Räbiger, and R. Schnaubelt, *Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line*, Integral Equations and Operator Theory **32** (1998), no. 3, 332–353.
[15] P. H. A. Ngoc and T. Naito, *New characterizations of exponential dichotomy and exponential stability of linear difference equations*, Journal of Difference Equations and Applications **11** (2005), no. 10, 909–918.
[16] A. C. Peterson and Y. N. Raffoul, *Exponential stability of dynamic equations on time scales*, Advances in Difference Equations **2005** (2005), no. 2, 133–144.
[17] A. Rodkina and H. Schurz, *Global asymptotic stability of solutions of cubic stochastic difference equations*, Advances in Difference Equations **2004** (2004), no. 3, 249–260.
[18] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences, vol. 143, Springer, New York, 2002.
[19] V. E. Slyusarchuk, *Exponential dichotomy of solutions of discrete systems*, Ukrainskiı Matematicheskiĭ Zhurnal **35** (1983), no. 1, 109–115, 137.
[20] G. Stefanidou and G. Papaschinopoulos, *Trichotomy, stability, and oscillation of a fuzzy difference equation*, Advances in Difference Equations **2004** (2004), no. 4, 337–357.

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