ON THE OSCILLATION OF CERTAIN THIRD-ORDER DIFFERENCE EQUATIONS

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We establish some new criteria for the oscillation of third-order difference equations of the form $\Delta((1/a_2(n))(\Delta(1/a_1(n))(\Delta x(n))^{\alpha_1})^{\alpha_2}) + \delta q(n) f(x[g(n)]) = 0$, where Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of the third-order difference equation

$$L_3 x(n) + \delta q(n) f(x[g(n)]) = 0, \tag{1.1;δ}$$

where $\delta = \pm 1$, $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$,

$$\begin{aligned} L_0 x(n) &= x(n), \qquad L_1 x(n) = \frac{1}{a_1(n)} (\Delta L_0 x(n))^{\alpha_1}, \\ L_2 x(n) &= \frac{1}{a_2(n)} (\Delta L_1 x(n))^{\alpha_2}, \qquad L_3 x(n) = \Delta L_2 x(n). \end{aligned} \tag{1.2}$$

In what follows, we will assume that

(i) $\{a_i(n)\}$, $i = 1, 2$, and $\{q(n)\}$ are positive sequences and

$$\sum^{\infty} (a_i(n))^{1/\alpha_i} = \infty, \quad i = 1, 2; \tag{1.3}$$

(ii) $\{g(n)\}$ is a nondecreasing sequence, and $\lim_{n\to\infty} g(n) = \infty$;

(iii) $f \in \mathscr{C}(\mathbb{R}, \mathbb{R})$, $x f(x) > 0$, and $f'(x) \ge 0$ for $x \ne 0$;

(iv) α_i, $i = 1, 2$, are quotients of positive odd integers.

The domain $\mathscr{D}(L_3)$ of L_3 is defined to be the set of all sequences $\{x(n)\}$, $n \ge n_0 \ge 0$ such that $\{L_j x(n)\}$, $0 \le j \le 3$ exist for $n \ge n_0$.

A nontrivial solution $\{x(n)\}$ of (1.1;δ) is called nonoscillatory if it is either eventually positive or eventually negative and it is oscillatory otherwise. An equation (1.1;δ) is called oscillatory if all its nontrivial solutions are oscillatory.

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The oscillatory behavior of second-order half-linear difference equations of the form

$$\Delta\left(\frac{1}{a_1(n)}(\Delta x(n))^{\alpha_1}\right)+\delta q(n)f(x[g(n)])=0, \tag{1.4;δ}$$

where δ, a_1, q, g, f, and α_1 are as in (1.1;δ) and/or related equations has been the subject of intensive study in the last decade. For typical results regarding (1.4;δ), we refer the reader to the monographs [1, 2, 4, 8, 12], the papers [3, 6, 11, 15], and the references cited therein. However, compared to second-order difference equations of type (1.4;δ), the study of higher-order equations, and in particular third-order equations of type (1.1;δ) has received considerably less attention (see [9, 10, 14]). In fact, not much has been established for equations with deviating arguments. The purpose of this paper is to present a systematic study for the behavioral properties of solutions of (1.1;δ), and therefore, establish criteria for the oscillation of (1.1;δ).

2. Properties of solutions of equation (1.1;1)

We will say that $\{x(n)\}$ is of type B_0 if

$$x(n)>0,\qquad L_1x(n)<0,\qquad L_2x(n)>0,\qquad L_3x(n)\le 0\quad\text{eventually,} \tag{2.1}$$

it is of type B_2 if

$$x(n)>0,\qquad L_1x(n)>0,\qquad L_2x(n)>0,\qquad L_3x(n)\le 0\quad\text{eventually.} \tag{2.2}$$

Clearly, any positive solution of (1.1;1) is either of type B_0 or B_2. In what follows, we will present some criteria for the nonexistence of solutions of type B_0 for (1.1;1).

Theorem 2.1. *Let conditions (i)–(iv) hold, $g(n)<n$ for $n\ge n_0\ge 0$, and*

$$-f(-xy)\ge f(xy)\ge f(x)f(y)\quad \text{for } xy>0. \tag{2.3}$$

Moreover, assume that there exists a nondecreasing sequence $\{\xi(n)\}$ such that $g(n)<\xi(n)$ $<n$ for $n\ge n_0$. If all bounded solutions of the second-order half-linear difference equation

$$\Delta\left(\frac{1}{a_2(n)}(\Delta y(n))^{\alpha_2}\right)-q(n)f\left(\sum_{k=g(n)}^{\xi(n)}a_1^{1/\alpha_1}(k)\right)f\left(y^{1/\alpha_1}[\xi(n)]\right)=0 \tag{2.4}$$

are oscillatory, then (1.1;1) has no solution of type B_0.

Proof. Let $\{x(n)\}$ be a solution of (1.1;1) of type B_0. There exists $n_0\in\mathbb{N}$ so large that (2.1) holds for all $n\ge n_0$. For $t\ge s\ge n_0$, we have

$$x(s)=x(t+1)-\sum_{j=s}^{t}a_1^{1/\alpha_1}(j)\frac{1}{a_1^{1/\alpha_1}(j)}\Delta x(j)\ge\left(\sum_{j=s}^{t}a_1^{1/\alpha_1}(j)\right)\left(-L_1^{1/\alpha_1}x(t)\right). \tag{2.5}$$

Replacing s and t by $g(n)$ and $\xi(n)$ respectively in (2.5), we have

$$x[g(n)] \geq \left(\sum_{j=g(n)}^{\xi(n)} a_1^{1/\alpha_1}(j)\right)\left(-L_1^{1/\alpha_1}x[\xi(n)]\right) \tag{2.6}$$

for $n \geq n_1 \in \mathbb{N}$ for some $n_1 \geq n_0$. Now using (2.3) and (2.6) in (1.1;1) and letting $y(n) = -L_1 x(n) > 0$ for $n \geq n_1$, we easily find

$$\Delta\left(\frac{1}{a_2(n)}(\Delta y(n))^{\alpha_2}\right) - q(n) f\left(\sum_{j=g(n)}^{\xi(n)} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}[\xi(n)]\right) \geq 0 \quad \text{for } n \geq n_1. \tag{2.7}$$

A special case of [16, Lemma 2.4] guarantees that (2.4) has a positive solution, a contradiction. This completes the proof. □

Theorem 2.2. *Let conditions (i)–(iv) and (2.3) hold, and assume that there exists a nondecreasing sequence $\{\xi(n)\}$ such that $g(n) < \xi(n) < n$ for $n \geq n_0$. Then, (1.1;1) has no solution of type B_0 if either one of the following conditions holds:*

(S_1)

$$\frac{f\left(u^{1/(\alpha_1\alpha_2)}\right)}{u} \geq 1 \quad \text{for } u \neq 0, \tag{2.8}$$

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1}\left\{q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=\xi(k)}^{\xi(n)} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right)\right\} > 1, \tag{2.9}$$

(S_2)

$$\frac{u}{f\left(u^{1/(\alpha_1\alpha_2)}\right)} \longrightarrow 0 \quad \text{as } u \longrightarrow 0, \tag{2.10}$$

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1}\left\{q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=\xi(k)}^{\xi(n)} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right)\right\} > 0. \tag{2.11}$$

Proof. Let $\{x(n)\}$ be a solution of (1.1;1) of type B_0. Proceeding as in the proof of Theorem 2.1 to obtain the inequality (2.7), it is easy to check that $y(n) > 0$ and $\Delta y(n) < 0$ for $n \geq n_1$. Let $n_2 > n_1$ be such that $\inf_{n\geq n_2} \xi(n) > n_1$. Now

$$\begin{aligned} y(\sigma) &= y(\tau+1) - \sum_{j=\sigma}^{\tau} a_2^{1/\alpha_2}(j)\left(\frac{1}{a_2(j)}(\Delta y(j))^{\alpha_2}\right)^{1/\alpha_2} \\ &\geq \left(\sum_{j=\sigma}^{\tau} a_2^{1/\alpha_2}(j)\right)\left(\frac{1}{a_2(\tau)}(-\Delta y(\tau))^{\alpha_2}\right)^{1/\alpha_2} \quad \text{for } \tau \geq \sigma \geq n_2. \end{aligned} \tag{2.12}$$

Replacing σ and τ by $\xi(k)$ and $\xi(n)$ respectively in (2.12), we have

$$y[\xi(k)] \geq \left(\sum_{j=\xi(k)}^{\xi(n)} a_2^{1/\alpha_2}(j)\right)\left(\frac{1}{a_2[\xi(n)]}\left(-\Delta y[\xi(n)]\right)^{\alpha_2}\right)^{1/\alpha_2} \quad \text{for } n \geq k \geq n_2. \tag{2.13}$$

Summing (2.7) from $\xi(n)$ to $(n-1)$ and letting $Y(n) = (-\Delta y(n))^{\alpha_2}/a_2(n)$ for $n \geq n_2$, we get

$$\begin{aligned} Y[\xi(n)] \geq Y(n) + \sum_{k=\xi(n)}^{n-1} q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) \\ \times f\left(\left[\left(\sum_{i=\xi(k)}^{\xi(n)} a_2^{1/\alpha_2}(i)\right) Y^{1/\alpha_2}[\xi(n)]\right]^{1/\alpha_1}\right) \quad \text{for } n \geq n_2. \end{aligned} \tag{2.14}$$

Using condition (2.3) in (2.14), we have

$$\begin{aligned} Y[\xi(n)] \geq{}& f\left(Y^{1/(\alpha_1\alpha_2)}[\xi(n)]\right) \\ &\times\left[\sum_{k=\xi(n)}^{n-1} q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=\xi(k)}^{\xi(n)} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right)\right], \quad n \geq n_2. \end{aligned} \tag{2.15}$$

Using (2.8) in (2.15) we have

$$1 \geq \sum_{k=\xi(n)}^{n-1} q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=\xi(k)}^{\xi(n)} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right). \tag{2.16}$$

Taking lim sup of both sides of the above inequality as $n \to \infty$, we obtain a contradiction to condition (2.9).

Next, using (2.10) in (2.15) and taking lim sup of the resulting inequality, we obtain a contradiction to condition (2.11). This completes the proof. □

Theorem 2.3. *Let the hypotheses of Theorem 2.2 hold. Then, (1.1;1) has no solutions of type B_0 if one of the following conditions holds:*

(O_1)

$$\frac{f^{1/\alpha_2}\left(u^{1/\alpha_1}\right)}{u} \geq 1 \quad \text{for } u \neq 0, \tag{2.17}$$

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1} a_2^{1/\alpha_2}(k)\left(\sum_{j=k}^{n-1} q(j) f\left(\sum_{i=g(j)}^{\xi(j)} a_1^{1/\alpha_1}(i)\right)\right)^{1/\alpha_2} > 1, \tag{2.18}$$

(O_2)

$$\frac{u}{f^{1/\alpha_2}(u^{1/\alpha_1})} \longrightarrow 0 \quad as\ u \longrightarrow 0, \tag{2.19}$$

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1} a_2^{1/\alpha_2}(k)\left(\sum_{j=k}^{n-1} q(j) f\left(\sum_{i=g(j)}^{\xi(j)} a_1^{1/\alpha_1}(i)\right)\right)^{1/\alpha_2} > 0. \tag{2.20}$$

Proof. Let $\{x(n)\}$ be a solution of (1.1;1) of type B_0. As in the proof of Theorem 2.1, we obtain the inequality (2.7) for $n \geq n_1$. Also, we see that $y(n) > 0$ and $\Delta y(n) < 0$ for $n \geq n_1$. Next, we let $n_2 \geq n_1$ be as in the proof of Theorem 2.2, and summing inequality (2.7) from $s \geq n_2$ to $(n-1)$, we have

$$\frac{1}{a_2(s)}(-\Delta y(s))^{\alpha_2} \geq \frac{1}{a_2(n)}(-\Delta y(n))^{\alpha_2} + \sum_{k=s}^{n-1} q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}[\xi(k)]\right), \tag{2.21}$$

which implies

$$-\Delta y(s) \geq a_2^{1/\alpha_2}(s)\left(\sum_{k=s}^{n-1} q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}[\xi(k)]\right)\right)^{1/\alpha_2}. \tag{2.22}$$

Now,

$$y(v) = y(n) + \sum_{s=v}^{n-1}(-\Delta y(s)) \geq \sum_{s=v}^{n-1}(-\Delta y(s)) \quad \text{for } n-1 \geq s \geq n_2. \tag{2.23}$$

Substituting (2.23) in (2.22) and setting $v = \xi(n)$, we have

$$\begin{aligned} y[\xi(n)] &\geq \sum_{s=\xi(n)}^{n-1} a_2^{1/\alpha_2}(s)\left(\sum_{k=s}^{n-1} q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}[\xi(k)]\right)\right)^{1/\alpha_2} \\ &\geq f^{1/\alpha_2}\left(y^{1/\alpha_1}[\xi(n)]\right) \sum_{s=\xi(n)}^{n-1} a_2^{1/\alpha_2}(s)\left(\sum_{k=s}^{n-1} q(k) f\left(\sum_{j=g(k)}^{\xi(k)} a_1^{1/\alpha_1}(j)\right)\right)^{1/\alpha_2}. \end{aligned} \tag{2.24}$$

The rest of the proof is similar to that of Theorem 2.2 and hence is omitted. □

Theorem 2.4. *Let conditions (i)–(iv), (2.3) hold, $g(n) = n - \tau$, where τ is a positive integer and assume that there exist two positive integers such that $\tau > \overline{\tau} > \tilde{\tau}$. If the first-order delay equation*

$$\Delta y(n) + q(n) f\left(\sum_{j=n-\tau}^{n-\overline{\tau}} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=n-\overline{\tau}}^{n-\tilde{\tau}} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right) f\left(y^{1/(\alpha_1\alpha_2)}[n-\tilde{\tau}]\right) = 0 \tag{2.25}$$

is oscillatory, then (1.1;1) has no solution of type B_0.

Proof. Let $\{x(n)\}$ be a solution of (1.1;1) of type B_0. As in the proof of Theorem 2.1, we obtain (2.6) for $n \geq n_1$, which takes the form

$$x[n-\tau] \geq \left(\sum_{j=n-\tau}^{n-\overline{\tau}} a_1^{1/\alpha_1}(j)\right)\left(-L_1^{1/\alpha_1} x[n-\overline{\tau}]\right) \quad \text{for } n \geq n_1. \tag{2.26}$$

Similarly, we find

$$-L_1 x[n-\overline{\tau}] \geq \left(\sum_{i=n-\overline{\tau}}^{n-\tilde{\tau}} a_2^{1/\alpha_2}(i)\right)\left(L_2^{1/\alpha_2} x[n-\tilde{\tau}]\right) \quad \text{for } n \geq n_2 \geq n_1. \tag{2.27}$$

Combining (2.26) with (2.27) we have

$$x[n-\tau] \geq \left(\sum_{j=n-\tau}^{n-\overline{\tau}} a_1^{1/\alpha_1}(j)\right)\left(\sum_{i=n-\overline{\tau}}^{n-\tilde{\tau}} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1} L_2^{1/(\alpha_1\alpha_2)} x[n-\tilde{\tau}] \quad \text{for } n \geq n_3 \geq n_2. \tag{2.28}$$

Using (2.3) and (2.28) in (1.1;1) and setting $Z(n) = L_2 x(n)$, we have

$$\begin{aligned} \Delta Z(n) + q(n) f\left(\sum_{j=n-\tau}^{n-\overline{\tau}} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=n-\overline{\tau}}^{n-\tilde{\tau}} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right) \\ \times f\left(Z^{1/(\alpha_1\alpha_2)}[n-\tilde{\tau}]\right) \leq 0 \quad \text{for } n \geq n_3. \end{aligned} \tag{2.29}$$

By a known result in [2, 12], we see that (2.25) has a positive solution which is a contradiction. This completes the proof. □

As an application of Theorem 2.4, we have the following result.

Corollary 2.5. *Let conditions (i)–(iv), (2.3) hold, $g(n) = n - \tau$, τ is a positive integer and let there exist two positive integers $\overline{\tau}$, $\tilde{\tau}$ such that $\tau > \overline{\tau} > \tilde{\tau}$. Then, (1.1;1) has no solution of type B_0 if either one of the following conditions holds:*

(I_1) *in addition to (2.8),*

$$\liminf_{n\to\infty} \sum_{k=n-\overline{\tau}}^{n-1} q(k) f\left(\sum_{j=k-\tau}^{k-\overline{\tau}} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=k-\overline{\tau}}^{k-\tilde{\tau}} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right) > \left(\frac{\tilde{\tau}}{1+\tilde{\tau}}\right)^{\tilde{\tau}+1}, \tag{2.30}$$

(I_2)

$$\int_{\pm 0} \frac{du}{f\left(u^{1/(\alpha_1\alpha_2)}\right)} < \infty, \tag{2.31}$$

$$\sum_{k=n_0}^{\infty} q(k) f\left(\sum_{j=k-\tau}^{k-\overline{\tau}} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=k-\overline{\tau}}^{k-\tilde{\tau}} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right) = \infty. \tag{2.32}$$

Next, we will present some criteria for the nonexistence of solutions of type B_2 of (1.1;1).

Theorem 2.6. *Let conditions (i)–(iv) and (2.3) hold. If*

$$\sum^{\infty} q(j) f\left(\sum_{i=n_0}^{g(j)-1} a_1^{1/\alpha_1}(i)\right) = \infty, \tag{2.33}$$

then (1.1;1) has no solution of type B_2.

Proof. Let $\{x(n)\}$ be a solution of (1.1;1). There exists an integer $n_0 \in \mathbb{N}$ so large that (2.2) holds for $n \geq n_0$. From (2.2), there exist a constant $c > 0$ and an integer $n_1 \geq n_0$ such that

$$\frac{1}{a_1(n)} \left(\Delta L_0 x(n)\right)^{\alpha_1} = L_1 x(n) \geq c, \tag{2.34}$$

or

$$\Delta x(n) \geq \left(c a_1(n)\right)^{1/\alpha_1} \quad \text{for } n \geq n_1. \tag{2.35}$$

Summing (2.35) from n_1 to $g(n) - 1 (\geq n_1)$ we obtain

$$x[g(n)] \geq c^{1/\alpha_1} \sum_{j=n_1}^{g(n)-1} a_1^{1/\alpha_1}(j). \tag{2.36}$$

Using (2.3) and (2.36) in (1.1;1) we have

$$\begin{aligned} -L_3 x(n) &= q(n) f(x[g(n)]) \\ &\geq q(n) f\left(c^{1/\alpha_1}\right) f\left(\sum_{j=n_1}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) \quad \text{for } n \geq n_2 \geq n_1. \end{aligned} \tag{2.37}$$

Summing (2.37) from n_2 to $n-1(>n_2)$ we obtain

$$\begin{aligned}\infty > L_2x(n_2) &\geq -L_2x(n)+L_2x(n_2)\\ &\geq f\left(c^{1/\alpha_1}\right)\sum_{k=n_2}^{n-1}q(k)f\left(\sum_{j=n_1}^{g(k)-1}a_1^{1/\alpha_1}(j)\right)\longrightarrow\infty \quad \text{as } n\longrightarrow\infty,\end{aligned}\tag{2.38}$$

a contradiction. This completes the proof. □

Theorem 2.7. *Let conditions (i)–(iv) and (2.3) hold, and $g(n)=n-\tau$, $n\geq n_0\geq 0$, where τ is a positive integer. If the first-order delay equation*

$$\Delta y(n)+q(n)f\left(\sum_{k=n_0}^{n-\tau-1}\left(a_1(k)\sum_{j=n_0}^{k-1}a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right)f\left(y^{1/(\alpha_1\alpha_2)}[n-\tau]\right)=0\tag{2.39}$$

is oscillatory, then (1.1;1) has no solution of type B_2.

Proof. Let $\{x(n)\}$ be a solution of (1.1;1) of type B_2. There exists an integer $n_0\geq 0$ so large that (2.2) holds for $n\geq n_0$. Now,

$$\begin{aligned}L_1x(n)&=L_1x(n_0)+\sum_{j=n_0}^{n-1}\Delta L_1x(j)\\ &=L_1x(n_0)+\sum_{j=n_0}^{n-1}a_2^{1/\alpha_2}(j)\left(a_2^{-1/\alpha_2}(j)\Delta L_1x(j)\right)\\ &=L_1x(n_0)+\sum_{j=n_0}^{n-1}a_2^{1/\alpha_2}(j)L_2^{1/\alpha_2}x(j)\\ &\geq L_2^{1/\alpha_2}x(n)\sum_{j=n_0}^{n-1}a_2^{1/\alpha_2}(j)\quad \text{for } n\geq n_1,\end{aligned}\tag{2.40}$$

or

$$\frac{1}{a_1(n)}(\Delta x(n))^{\alpha_1}\geq L_2^{1/\alpha_2}x(n)\sum_{j=n_0}^{n-1}a_2^{1/\alpha_2}(j).\tag{2.41}$$

Thus,

$$\Delta x(n)\geq\left(a_1(n)\sum_{j=n_0}^{n-1}a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}L_2^{1/(\alpha_1\alpha_2)}x(n)\quad \text{for } n\geq n_0.\tag{2.42}$$

Summing (2.42) from n_0 to $g(n)-1>n_0$, we have

$$x[g(n)]\geq\left(\sum_{k=n_0}^{g(n)-1}\left(a_1(k)\sum_{j=n_0}^{k-1}a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right)L_2^{1/(\alpha_1\alpha_2)}x[g(n)]\quad \text{for } n\geq n_1\geq n_0.\tag{2.43}$$

Using (2.3), (2.43), $g(n) = n - \tau$, and letting $y(n) = L_2x(n)$, $n \geq n_1$, we obtain

$$\Delta y(n) + q(n) f\left(\sum_{k=n_0}^{k-\tau-1}\left(a_1(k)\sum_{j=n_0}^{k-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) f\left(y^{1/(\alpha_1\alpha_2)}[n-\tau]\right) \leq 0. \tag{2.44}$$

The rest of the proof is similar to that of Theorem 2.4 and hence is omitted. □

Theorem 2.8. *Let conditions (i)–(iv) and (2.3) hold and $g(n) > n+1$ for $n \geq n_0 \in \mathbb{N}$. If the half-linear difference equation*

$$\Delta\left(\frac{1}{a_2(n)}(\Delta y(n))^{\alpha_2}\right) + q(n) f\left(\sum_{j=n}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}(n)\right) = 0 \tag{2.45}$$

is oscillatory, then (1.1;1) has no solution of type B_2.

Proof. Let $\{x(n)\}$ be a solution of (1.1;1) of type B_2. Then there exists an $n_0 \in \mathbb{N}$ sufficiently large so that (2.2) holds for $n \geq n_0$. Now, for $m \geq s \geq n_0$ we get

$$x(m) - x(s) = \sum_{j=s}^{m-1} a_1^{1/\alpha_1}(j) L_1^{1/\alpha_1} x(j), \tag{2.46}$$

or

$$x(m) \geq \left(\sum_{j=s}^{m-1} a_1^{1/\alpha_1}(j)\right) L_1^{1/\alpha_1} x(s). \tag{2.47}$$

Replacing m and s in (2.47) by $g(n)$ and n, respectively, we have

$$x[g(n)] \geq \left(\sum_{j=n}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) L_1^{1/\alpha_1} x(n) \quad \text{for } g(n) \geq n+1 \geq n_1 \geq n_0. \tag{2.48}$$

Using (2.3) and (2.48) in (1.1;1) and letting $Z(n) = L_1x(n)$ for $n \geq n_1$, we obtain

$$\Delta\left(\frac{1}{a_2(n)}(\Delta Z(n))^{\alpha_2}\right) + q(n) f\left(\sum_{j=n}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) f\left(Z^{1/\alpha_1}(n)\right) \leq 0 \quad \text{for } n \geq n_1. \tag{2.49}$$

By [16, Lemma 2.3], we see that (2.45) has a positive solution, a contradiction. This completes the proof. □

Remark 2.9. We note that a corollary similar to Corollary 2.5 can be deduced from Theorem 2.7. Here, we omit the details.

Remark 2.10. We note that the conclusion of Theorems 2.1–2.4 can be replaced by "all bounded solutions of (1.1;1) are oscillatory."

Next, we will combine our earlier results to obtain some sufficient conditions for the oscillation of (1.1;1).

THEOREM 2.11. *Let conditions (i)–(iv) and (2.3) hold, $g(n) < n$ for $n \ge n_0 \in \mathbb{N}$. Moreover, assume that there exists a nondecreasing sequence $\{\xi(n)\}$ such that $g(n) < \xi(n) < n$ for $n \ge n_0$. If either conditions (S_1) or (S_2) of Theorem 2.2 and condition (2.33) hold, the equation (1.1;1) is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (1.1;1), say, $x(n) > 0$ for $n \ge n_0 \in \mathbb{N}$. Then, $\{x(n)\}$ is either of type B_0 or B_2. By Theorem 2.2, $\{x(n)\}$ is not of type B_0 and by Theorem 2.6, $\{x(n)\}$ is not of type B_2. This completes the proof. □

THEOREM 2.12. *Let conditions (i)–(iv), (2.3) hold, $g(n) = n - \tau$, $n \ge n_0 \in \mathbb{N}$, where τ is a positive integer. Moreover, assume that there exist two positive integers $\overline{\tau}$ and $\tilde{\tau}$ such that $\tau > \overline{\tau} > \tilde{\tau}$. If both first-order delay equations (2.25) and (2.39) are oscillatory, then (1.1;1) is oscillatory.*

Proof. The proof follows from Theorems 2.4 and 2.7. □

Next, we will apply Theorems 2.11 and 2.12 to a special case of (1.1;1), namely, the equation

$$\Delta\left(\frac{1}{a_2(n)}\left(\Delta\frac{1}{a_1(n)}(\Delta x(n))^{\alpha_1}\right)^{\alpha_2}\right)+q(n)x^{\alpha}[g(n)]=0, \tag{2.50}$$

where α is the ratio of positive odd integers.

COROLLARY 2.13. *Let conditions (i)–(iv) hold, $g(n) < n$ for $n \ge n_0 \in \mathbb{N}$, and assume that there exists a nondecreasing sequence $\{\xi(n)\}$ such that $g(n) < \xi(n) < n$ for $n \ge n_0$. Equation (2.50) is oscillatory if either one of the following conditions holds:*

(A_1) $\alpha = \alpha_1\alpha_2$,

$$\sum_{j=n_0\ge 0}^{\infty} q(j)\left(\sum_{i=n_0}^{g(j)-1} a_1^{1/\alpha_1}(i)\right)^{\alpha} = \infty, \tag{2.51}$$

$$\limsup_{n\to\infty}\sum_{j=\xi(n)}^{n-1} q(j)\left(\sum_{i=g(j)}^{\xi(j)} a_1^{1/\alpha_1}(i)\right)^{\alpha}\left(\sum_{i=\xi(j)}^{\xi(n)} a_2^{1/\alpha_2}(i)\right)^{\alpha_2} > 1, \tag{2.52}$$

(A_2) $\alpha < \alpha_1\alpha_2$ *and condition (2.51) hold, and*

$$\limsup_{n\to\infty}\sum_{j=\xi(n)}^{n-1} q(j)\left(\sum_{i=g(j)}^{\xi(j)} a_1^{1/\alpha_1}(i)\right)^{\alpha}\left(\sum_{i=\xi(j)}^{\xi(n)} a_2^{1/\alpha_2}(i)\right)^{\alpha_2} > 0. \tag{2.53}$$

COROLLARY 2.14. *Let conditions (i)–(iv) hold, $g(n) = n - \tau$, $n \ge n_0 \in \mathbb{N}$, where τ is a positive integer, and assume that there exist two positive integers $\overline{\tau}$, $\tilde{\tau}$ such that $\tau > \overline{\tau} > \tilde{\tau}$. If* *the first-order delay equations*

$$\Delta y(n)+q(n)\left(\sum_{j=n-\tau}^{n-\overline{\tau}} a_1^{1/\alpha_1}(j)\right)^{\alpha}\left(\sum_{i=n-\overline{\tau}}^{n-\tilde{\tau}} a_2^{1/\alpha_2}(i)\right)^{\alpha_2} Z^{\alpha/(\alpha_1\alpha_2)}[n-\tilde{\tau}]=0, \tag{2.54}$$

$$\Delta Z(n)+q(n)\left(\sum_{j=n_0}^{n-\tau-1}\left(a_1(j)\sum_{i=n_0}^{j-1} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right)^{\alpha} Z^{\alpha/(\alpha_1\alpha_2)}[n-\tau]=0 \tag{2.55}$$

are oscillatory, then (2.50) is oscillatory.

For the mixed difference equations of the form

$$L_3x(t)+q_1(t)f_1(x[g_1(n)])+q_2(n)f_2(x[g_2(n)])=0, \tag{2.56}$$

where L_3 is defined as in (1.1;1), $\{a_i(n)\}$, $i=1,2$ are as in (i) satisfying (1.3), α_1 and α_2 are as in (iv), $\{q_i(n)\}$, $i=1,2$ are positive sequences, $\{g_i(n)\}$, $i=1,2$ are nondecreasing sequences with $\lim_{n\to\infty} g_i(n)=\infty$, $i=1,2$, $f_i\in\mathscr{C}(\mathbb{R},\mathbb{R})$, $xf_i(x)>0$ and $f_i(x)\geq 0$ for $x\neq 0$ and $i=1,2$. Also, f_1, f_2 satisfy condition (2.3) by replacing f by f_1 and/or f_2.

Now, we combine Theorems 2.1 and 2.8 and obtain the following interesting result.

Theorem 2.15. *Let the above hypotheses hold for (2.56), $g_1(n)<n$ and $g_2(n)>n+1$ for $n\geq n_0\in\mathbb{N}$ and assume that there exists a nondecreasing sequence $\{\xi(n)\}$ such that $g_1(n)<\xi(n)<n$ for $n\geq n_0$. If all bounded solutions of the equation*

$$\Delta\left(\frac{1}{a_2(n)}(\Delta y(n))^{\alpha_2}\right)-q_1(n)f_1\left(\sum_{k=g_1(n)}^{\xi(n)} a_1^{1/\alpha_1}(k)\right)f_1\left(y^{1/\alpha_1}[\xi(n)]\right)=0 \tag{2.57}$$

are oscillatory and all solutions of the equation

$$\Delta\left(\frac{1}{a_2(n)}(\Delta Z(n))^{\alpha_2}\right)+q_2(n)f_2\left(\sum_{j=n}^{g(n)-1} a_1^{1/\alpha_1}(j)\right)f_2\left(Z^{1/\alpha_1}(n)\right)=0 \tag{2.58}$$

are oscillatory, then (2.56) is oscillatory.

3. Properties of solutions of equation (1.1;-1)

We will say that $\{x(n)\}$ is of type B_1 if

$$x(n)>0,\qquad L_1x(n)>0,\qquad L_2x(n)<0,\qquad L_3x(n)\geq 0\quad\text{eventually,} \tag{3.1}$$

it is of type B_3 if

$$x(n)>0,\qquad L_ix(n)>0,\quad i=1,2,\qquad L_3x(n)\geq 0\quad\text{eventually.} \tag{3.2}$$

Clearly, any positive solution of (1.1;-1) is either of type B_1 or B_3. In what follows, we will give some criteria for the nonexistence of solutions of type B_1 for (1.1;-1).

Theorem 3.1. *Assume that conditions (i)–(iv) hold. If*

$$\sum^{\infty} q(j) = \infty, \tag{3.3}$$

then (1.1;-1) has no solution of type B_1.

Proof. Let $\{x(n)\}$ be a solution of (1.1;-1) of type B_1. Then there exists an $n_0 \in \mathbb{N}$ sufficiently large so that (3.1) holds for $n \geq n_0$. Next, there exist an integer $n_1 \geq n_0$ and a constant $c > 0$ such that

$$x[g(n)] \geq c \quad \text{for } n \geq n_1. \tag{3.4}$$

Summing (1.1;-1) from n_1 to $n-1 \geq n_1$ and using (3.4), we have

$$L_2x(n) - L_2x(n_1) = \sum_{j=n_1}^{n-1} q(j) f(x[g(j)]), \tag{3.5}$$

or

$$\infty > -L_2x(n_1) \geq f(c) \sum_{j=n_1}^{n-1} q(j) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \tag{3.6}$$

a contradiction. This completes the proof. □

Theorem 3.2. *Let conditions (i)–(iv) and (2.3) hold and $g(n) < n$ for $n \geq n_0 \in \mathbb{N}$. If all bounded solutions of the half-linear equation*

$$\Delta\left(\frac{1}{a_2(n)}(\Delta y(n))^{\alpha_2}\right) - q(n) f\left(\sum_{j=n_0}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}[g(n)]\right) = 0 \tag{3.7}$$

are oscillatory, then (1.1;-1) has no solutions of type B_1.

Proof. Let $\{x(n)\}$ be a solution of (1.1;-1) of type B_1. There exists an $n_0 \in \mathbb{N}$ such that (3.1) holds for $n \geq n_0$. Now

$$x(n) - x(n_0) = \sum_{j=n_0}^{n-1} \Delta x(j) = \sum_{j=n_0}^{n-1} a_1^{1/\alpha_1}(j) L_1^{1/\alpha_1} x(j). \tag{3.8}$$

Thus,

$$x(n) \geq \left(\sum_{j=n_0}^{n-1} a_1^{1/\alpha_1}(j)\right) L_1^{1/\alpha_1} x(n) \quad \text{for } n \geq n_0. \tag{3.9}$$

There exists an $n_1 \geq n_0$ such that

$$x[g(n)] \geq \left(\sum_{j=n_0}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) L_1^{1/\alpha_1} x[g(n)] \quad \text{for } n \geq n_1. \tag{3.10}$$

Using (2.3) and (3.10) in (1.1;-1) and letting $y(n) = L_1x(n)$ for $n \geq n_1$, we have

$$\Delta\left(\frac{1}{a_2(n)}(\Delta y(n))^{\alpha_2}\right) \geq q(n)f\left(\sum_{j=n_0}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}[g(n)]\right) \quad \text{for } n \geq n_1. \tag{3.11}$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. □

Next, we state the following criteria which are similar to Theorems 2.2, 2.3, and 2.4. Here, we omit the proofs.

Theorem 3.3. *Let conditions (i)–(iv) and (2.3) hold, and $g(n) < n$ for $n \geq n_0 \in \mathbb{N}$. Then, (1.1;-1) has no solution of type B_1 if either one of the following conditions holds:*

(C_1) *condition (2.8) holds, and*

$$\limsup_{n\to\infty} \sum_{k=g(n)}^{n-1} \left\{ q(k)f\left(\sum_{j=n_0\geq 0}^{g(k)-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=g(k)}^{g(n)} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right)\right\} > 1, \tag{3.12}$$

(C_2) *condition (2.10) holds, and*

$$\limsup_{n\to\infty} \sum_{k=g(n)}^{n-1} \left\{ q(k)f\left(\sum_{j=n_0\geq 0}^{g(k)-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{i=g(k)}^{g(n)} a_2^{1/\alpha_2}(i)\right)^{1/\alpha_1}\right)\right\} > 0. \tag{3.13}$$

Theorem 3.4. *Let the hypotheses of Theorem 3.3 be satisfied. Then, (1.1;-1) has no solutions of type B_1 if either one of the following conditions holds:*

(D_1) *condition (2.17) holds, and*

$$\limsup_{n\to\infty} \sum_{k=g(n)}^{n-1} a_2^{1/\alpha_2}(k)\left(\sum_{j=k}^{n-1} q(j)f\left(\sum_{i=n_0\geq 0}^{g(j)-1} a_1^{1/\alpha_1}(i)\right)\right)^{1/\alpha_2} > 1, \tag{3.14}$$

(D_2) *condition (2.19) holds, and*

$$\limsup_{n\to\infty} \sum_{k=g(n)}^{n-1} a_2^{1/\alpha_2}(k)\left(\sum_{j=k}^{n-1} q(j)f\left(\sum_{i=n_0\geq 0}^{g(j)-1} a_1^{1/\alpha_1}(i)\right)\right)^{1/\alpha_2} > 0. \tag{3.15}$$

THEOREM 3.5. *Let conditions (i)–(iv) and (2.3) hold, $g(n) = n - \tau$, $n \geq n_0 \in \mathbb{N}$ where τ is a positive integer, and assume that there exists an integer $\overline{\tau} > 0$ such that $\tau > \overline{\tau}$. If the first-order delay equation*

$$\Delta y(n) + q(n) f\left(\sum_{j=n_0}^{n-\tau-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=n-\tau}^{n-\overline{\tau}} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) f\left(y^{1/(\alpha_1\alpha_2)}[n-\overline{\tau}]\right) = 0 \tag{3.16}$$

is oscillatory, then (1.1;-1) has no solution of type B_1.

Next, we will present some results for the nonexistence of solutions of type B_3 for (1.1;-1).

THEOREM 3.6. *Let conditions (i)–(iv) and (2.3) hold, $g(n) > n+1$ for $n \geq n_0 \in \mathbb{N}$, and assume that there exists a nondecreasing sequence $\{\eta(n)\}$ such that $g(n) > \eta(n) > n+1$ for $n \geq n_0$. Then, (1.1;-1) has no solution of type B_3 if either one of the following conditions holds:*

(E_1) *condition (2.8) holds, and*

$$\limsup_{n\to\infty} \sum_{k=n}^{\eta(n)-1} q(k) f\left(\sum_{j=\eta(k)}^{g(k)-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=\eta(n)}^{\eta(k)-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) > 1, \tag{3.17}$$

(E_2)

$$\frac{u}{f\left(u^{1/(\alpha_1\alpha_2)}\right)} \longrightarrow 0 \quad \text{as } u \longrightarrow \infty, \tag{3.18}$$

$$\limsup_{n\to\infty} \sum_{k=n}^{\eta(n)-1} q(k) f\left(\sum_{j=\eta(k)}^{g(k)-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=\eta(n)}^{\eta(k)-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) > 0. \tag{3.19}$$

Proof. Let $\{x(n)\}$ be a solution of (1.1;-1) of type B_3. Then there exists a large integer $n_0 \in \mathbb{N}$ such that (3.2) holds for $n \geq n_0$. Now

$$\begin{aligned} x(\sigma) &= x(\tau) + \sum_{j=\tau}^{\sigma-1} \Delta x(j) = x(\tau) + \sum_{j=\tau}^{\sigma-1} a_1^{1/\alpha_1}(j) L_1^{1/\alpha_1} x(j) \\ &\geq \left(\sum_{j=\tau}^{\sigma-1} a_1^{1/\alpha_1}(j)\right) L_1^{1/\alpha_1} x(\tau) \quad \text{for } \sigma \geq \tau \geq n_0. \end{aligned} \tag{3.20}$$

Letting $\sigma = g(n)$, $\tau = \eta(n)$ in (3.20), we see that

$$x[g(n)] \geq \left(\sum_{j=\eta(n)}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) L_1^{1/\alpha_1} x[\eta(n)] \quad \text{for } n \geq n_1 \geq n_0. \tag{3.21}$$

Using (3.21) in (1.1;-1) and letting $y(n) = L_1 x(n)$, $n \geq n_1$ we have

$$\Delta\left(\frac{1}{a_2(n)}(\Delta y(n))^{\alpha_2}\right) \geq q(n) f\left(\sum_{j=\eta(n)}^{g(n)-1} a_1^{1/\alpha_1}(j)\right) f\left(y^{1/\alpha_1}[\eta(n)]\right) \quad \text{for } n \geq n_1. \tag{3.22}$$

Clearly, $y(n) > 0$ and $\Delta y(n) > 0$ for $n \geq n_1$. As in the above proof, we can easily find

$$y[\eta(k)] \geq \left(\sum_{j=\eta(n)}^{\eta(k)-1} a_2^{1/\alpha_2}(j)\right)\left(L^{1/\alpha_2} y[\eta(n)]\right) \quad \text{for } k \geq n-1 \geq n_1, \tag{3.23}$$

where $Ly(n) = (\Delta y(n))^{\alpha_2}/a_2(n)$. Using (2.3) and (3.23) in (3.22), we have

$$\Delta(Ly(k)) \geq q(k) f\left(\sum_{j=\eta(k)}^{g(k)-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=\eta(k)}^{\eta(k)-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) f\left(L^{1/(\alpha_1\alpha_2)} y[\eta(n)]\right) \tag{3.24}$$

for $k \geq n-1 \geq n_1$. Summing (3.24) from n to $\eta(n) - 1 \geq n$, we have

$$\begin{aligned} Ly[\eta(n)] &\geq Ly[\eta(n)] - Ly(n) \\ &\geq \sum_{k=n}^{\eta(k)-1} q(k) f\left(\sum_{j=\eta(k)}^{g(k)-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=\eta(n)}^{g(k)-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) f\left(L^{1/(\alpha_1\alpha_2)} y[\eta(k)]\right), \end{aligned} \tag{3.25}$$

or

$$\frac{Ly[\eta(k)]}{f\left(L^{1/(\alpha_1\alpha_2)} y[\eta(n)]\right)} \geq \sum_{k=n}^{\eta(k)-1} q(k) f\left(\sum_{j=\eta(n)}^{g(k)-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=\eta(n)}^{g(k)-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right). \tag{3.26}$$

Taking lim sup of both sides of (3.26) as $n \to \infty$ and applying the hypotheses, we arrive at the desired contradiction. □

THEOREM 3.7. *Let the hypotheses of Theorem 3.6 be satisfied. Then, (1.1;-1) has no solution of type B_3 if either one of the following conditions holds:*
(F_1) *condition (2.17) holds, and*

$$\limsup_{n\to\infty} \sum_{k=n}^{\eta(n)-1} a_2^{1/\alpha_2}(k)\left(\sum_{j=n}^{k-1} q(j) f\left(\sum_{i=\eta(j)}^{g(j)-1} a_1^{1/\alpha_1}(i)\right)\right)^{1/\alpha_2} > 1, \tag{3.27}$$

(F_2)

$$\frac{u}{f^{1/\alpha_2}(u^{1/\alpha_1})} \longrightarrow 0 \quad as\ u \longrightarrow \infty, \tag{3.28}$$

$$\limsup_{n\to\infty} \sum_{k=n}^{\eta(n)-1} a_2^{1/\alpha_2}(k)\left(\sum_{j=n}^{k-1} q(j) f\left(\sum_{i=\eta(j)}^{g(j)-1} a_1^{1/\alpha_1}(i)\right)\right)^{1/\alpha_2} > 0. \tag{3.29}$$

Proof. Let $\{x(n)\}$ be a solution of (1.1;-1) of type B_3. As in the proof of Theorem 3.6, we obtain the inequality (3.22) and we see that $y(n) > 0$ and $\Delta y(n) > 0$ for $n \geq n_1$. Summing inequality (3.22) from n to $k-1 \geq n \geq n_2 \geq n_1$, we have

$$\frac{1}{a_2(k)}(\Delta y(k))^{\alpha_2} \geq \sum_{j=n}^{k-1} q(j) f\left(\sum_{i=\eta(j)}^{g(j)-1} a_1^{1/\alpha_1}(i)\right) f\left(y^{1/\alpha_1}[\eta(j)]\right) \tag{3.30}$$

which implies that

$$\Delta y(k) \geq a_2^{1/\alpha_2}(k)\left(\sum_{j=n}^{k-1} q(j) f\left(\sum_{i=\eta(j)}^{g(j)-1} a_1^{1/\alpha_1}(i)\right) f\left(y^{1/\alpha_1}[\eta(j)]\right)\right)^{1/\alpha_2} \quad \text{for } n \geq n_2. \tag{3.31}$$

Combining (3.31) with the relation

$$y(s) = y(n) + \sum_{k=n}^{s-1} \Delta y(k) \quad \text{for } s-1 \geq n \geq n_2 \tag{3.32}$$

and setting $s = \eta(n)$, we have

$$\frac{y[\eta(n)]}{f^{1/\alpha_2}(u^{1/\alpha_1}[\eta(n)])} \geq \sum_{k=n}^{\eta(n)-1} a_2^{1/\alpha_2}(k)\left(\sum_{j=n}^{k-1} q(j) f\left(\sum_{i=\eta(j)}^{g(j)-1} a_1^{1/\alpha_1}(i)\right)\right)^{1/\alpha_2} \quad \text{for } n \geq n_2. \tag{3.33}$$

Taking lim sup of both sides of (3.33) as $n \to \infty$, we arrive at the desired contradiction. □

Theorem 3.8. *Let conditions (i)–(iv) and (3.2) hold, $g(n) = n + \sigma$ for $n \geq n_0 \in \mathbb{N}$, where σ is a positive integer, and assume that there exist two positive integers $\overline{\sigma}$ and $\tilde{\sigma} > 1$ such that $\sigma - 2 > \overline{\sigma} - 1 > \tilde{\sigma}$. If the first-order advanced equation*

$$\Delta y(n) - q(n) f\left(\sum_{j=n+\overline{\sigma}}^{n+\sigma-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=n+\tilde{\sigma}}^{n+\overline{\sigma}-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) f\left(y^{1/(\alpha_1\alpha_2)}[n+\tilde{\sigma}]\right) = 0 \tag{3.34}$$

is oscillatory, then (1.1;-1) has no solution of type B_3.

Proof. Let $\{x(n)\}$ be a solution of (1.1;-1) of type B_3. As in the proof of Theorem 3.6, we obtain the inequality (3.21) for $n \geq n_1$, that is,

$$x[n+\sigma] \geq \left(\sum_{j=n+\overline{\sigma}}^{n+\sigma-1} a_1^{1/\alpha_1}(j)\right) L_1^{1/\alpha_1} x[n+\overline{\sigma}] \quad \text{for } n \geq n_1. \tag{3.35}$$

Similarly, we see that

$$L_1 x[n+\overline{\sigma}] \geq \left(\sum_{j=n+\tilde{\sigma}}^{n+\overline{\sigma}-1} a_2^{1/\alpha_2}(j)\right) \left(L_2^{1/\alpha_2} x[n+\tilde{\sigma}]\right) \quad \text{for } n \geq n_2 \geq n_1. \tag{3.36}$$

Combining (3.35) and (3.36), we have

$$x[n+\sigma] \geq \left(\sum_{j=n+\overline{\sigma}}^{n+\sigma-1} a_1^{1/\alpha_1}(j)\right) \left(\sum_{j=n+\tilde{\sigma}}^{n+\overline{\sigma}-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1} L_1^{1/(\alpha_1\alpha_2)} x[n+\tilde{\sigma}] \quad \text{for } n \geq n_2. \tag{3.37}$$

Using (2.3) and (3.37) in (1.1;-1) and letting $Z(n) = L_1 x(n)$, $n \geq n_2$, we have

$$\Delta Z(n) \geq q(n) f\left(\sum_{j=n+\overline{\sigma}}^{n+\sigma-1} a_1^{1/\alpha_1}(j)\right) f\left(\left(\sum_{j=n+\tilde{\sigma}}^{n+\overline{\sigma}-1} a_2^{1/\alpha_2}(j)\right)^{1/\alpha_1}\right) f\left(Z^{1/(\alpha_1\alpha_2)}[n+\tilde{\sigma}]\right). \tag{3.38}$$

By a known result in [2, 12], we see that (3.34) has an eventually positive solution, a contradiction. This completes the proof. □

Next, we will combine our earlier results to obtain some sufficient conditions for the oscillation of (1.1;-1), as an example, we state the following result.

THEOREM 3.9. *Let conditions (i)–(iv) and (2.3) hold, $g(n) = n + \sigma$ for $n \geq n_0 \in \mathbb{N}$, and assume that there exist two positive integers $\overline{\sigma}$, $\tilde{\sigma}$ such that $\sigma - 2 > \overline{\sigma} - 1 > \tilde{\sigma}$. If condition (3.3) holds and equation (3.34) is oscillatory, then (1.1;-1) is oscillatory.*

Proof. The proof follows from Theorems 3.1 and 3.8. □

Now, we apply Theorem 3.9 to a special case of (1.1;-1), namely, the equation

$$\Delta\left(\frac{1}{a_2(n)}\left(\Delta\frac{1}{a_1(n)}(\Delta x(n))^{\alpha_1}\right)^{\alpha_2}\right) - q(n)x^{\alpha}[n+\sigma] = 0, \tag{3.39}$$

where α is the ratio of positive odd integers and σ is a positive integer, and obtain the following immediate result.

COROLLARY 3.10. *Let conditions (i)–(iv) hold and assume that there exist two positive integers $\overline{\sigma}$ and $\tilde{\sigma} > 1$ such that $\sigma - 2 > \overline{\sigma} - 1 > \tilde{\sigma}$. Then, (3.39) is oscillatory if either one of the following conditions is satisfied:*

(J_1) *condition (3.3) holds, and*

$$\liminf_{n\to\infty}\sum_{k=n+1}^{n+\tilde{\sigma}-1} q(k)\left(\sum_{j=k+\overline{\sigma}}^{k+\sigma-1} a_1^{1/\alpha_1}(j)\right)^{\alpha}\left(\sum_{j=k+\tilde{\sigma}}^{k+\overline{\sigma}-1} a_2^{1/\alpha_2}(j)\right)^{\alpha_2} > \left(\frac{\tilde{\sigma}-1}{\tilde{\sigma}}\right)^{\tilde{\sigma}} \quad \text{if } \alpha=\alpha_1\alpha_2, \tag{3.40}$$

(J_2) *condition (3.3) holds, and*

$$\limsup_{n\to\infty}\sum_{k=n+1}^{n+\tilde{\sigma}-1} q(k)\left(\sum_{j=k+\overline{\sigma}}^{k+\sigma-1} a_1^{1/\alpha_1}(j)\right)^{\alpha}\left(\sum_{j=k+\tilde{\sigma}}^{k+\overline{\sigma}-1} a_2^{1/\alpha_2}(j)\right)^{\alpha/\alpha_1} > 0 \quad \text{if } \alpha>\alpha_1\alpha_2. \tag{3.41}$$

Now we will combine Theorems 3.5 and 3.8 to obtain some interesting oscillation criteria for the mixed type of equations

$$L_3x(n)-q_1(n)f_1(x[g_1(n)])-q_2(n)f_2(x[g_2(n)])=0, \tag{3.42}$$

where L_3, q_i, g_i, and f_i, $i=1,2$ are as in (2.56).

Theorem 3.11. *Let the sequences $\{q_i(n)\}$, $\{g_i(n)\}$, and $f_i(x)$, $i=1,2$ be as in (2.56), let L_3 be defined as in (1.1;δ), and $\{a_i(n)\}$, α_i, $i=1,2$ are as in (i) and (iv), $g_1(n)=n-\tau$ and $g_2(n)=n+\sigma$, $n\ge n_0\in\mathbb{N}$, where τ and σ are positive integers. Moreover, assume that there exist positive integers $\overline{\tau}$, $\overline{\sigma}$, and $\tilde{\sigma}$ such that $\tau>\overline{\tau}$ and $\sigma-2>\overline{\sigma}-1>\tilde{\sigma}$. If (3.16) with q and f replaced by q_1 and f_1, respectively, and (3.34) with q and f replaced by q_2 and f_2, respectively, are oscillatory, then (3.42) is oscillatory.*

Remark 3.12. The results of this paper are presented in a form which is essentially new even if $\alpha_1=\alpha_2=1$.

4. Applications

We can apply our results to neutral equations of the form

$$L_3(x(n)+p(n)x[\tau(n)])+\delta f(x[g(n)])=0, \tag{4.1;δ}$$

where $\{p(n)\}$ and $\{\tau(n)\}$ are real sequences, $\tau(n)$ is increasing, $\tau^{-1}(n)$ exists, and $\lim_{n\to\infty}\tau(n)=\infty$. Here, we set

$$y(n)=x(n)+p(n)x[\tau(n)]. \tag{4.2}$$

If $x(n)>0$ and $p(n)\ge 0$ for $n\ge n_0\ge 0$, then $y(n)>0$ for $n\ge n_1\ge n_0$. We let $0\le p(n)\le 1$, $p(n)\not\equiv 1$ for $n\ge n_0$, and consider either (P_1) $\tau(n)<n$ when $\Delta y(n)>0$ for $n\ge n_1$, or (P_2) $\tau(n)>n$ when $\Delta y(n)<0$ for $n\ge n_1$. In both cases we see that

$$\begin{aligned} x(n)&=y(n)-p(n)x[\tau(n)]=y(n)-p(n)[y[\tau(n)]-p[\tau(n)]x[\tau\circ\tau(n)]]\\ &\ge y(n)-p(n)y[\tau(n)]\ge y(n)[1-p(n)] \quad \text{for } n\ge n_1. \end{aligned} \tag{4.3}$$

Next, we let $p(n) \geq 1$, $p(n) \not\equiv 1$ for $n \geq n_0$ and consider either (P_3) $\tau(n) > n$ if $\Delta y(n) > 0$ for $n \geq n_1$, or (P_4) $\tau(n) < n$ if $\Delta y(n) < 0$ for $n \geq n_1$. In both cases we see that

$$\begin{aligned} x(n) &= \frac{1}{p[\tau^{-1}(n)]}\left(y[\tau^{-1}(n)] - x[\tau^{-1}(n)]\right) \\ &= \frac{y[\tau^{-1}(n)]}{p[\tau^{-1}(n)]} - \frac{1}{p[\tau^{-1}(n)]}\left(\frac{y[\tau^{-1}\circ\tau^{-1}(n)]}{p[\tau^{-1}\circ\tau^{-1}(n)]} - \frac{x[\tau^{-1}\circ\tau^{-1}(n)]}{p[\tau^{-1}\circ\tau^{-1}(n)]}\right) \\ &\geq \frac{1}{p[\tau^{-1}(n)]}\left(1 - \frac{1}{p[\tau^{-1}\circ\tau^{-1}(n)]}\right)y[\tau^{-1}(n)] \quad \text{for } n \geq n_1. \end{aligned} \tag{4.4}$$

Using (4.3) or (4.4) in (4.1;δ), we see that the resulting inequalities are of type (1.1;δ). Therefore, we can apply our earlier results to obtain oscillation criteria for (4.1;δ). The formulation of such results are left to the reader.

In the case when $p(n) < 0$ for $n \geq n_0$, we let $p_1(n) = -p(n)$ and so

$$y(n) = x(n) - p_1(n)x[\tau(n)]. \tag{4.5}$$

Here, we may have $y(n) > 0$, or $y(n) < 0$ for $n \geq n_1 \geq n_0$. If $y(n) > 0$ for $n \geq n_0$, we see that

$$x(n) \geq y(n) \quad \text{for } n \geq n_1. \tag{4.6}$$

On the other hand, if $y(n) < 0$ for $n \geq n_1$, we have

$$x[\tau(n)] = \frac{1}{p_1(n)}[y(n) + x(n)] \geq \frac{y(n)}{p_1(n)}, \tag{4.7}$$

or

$$x(n) \geq \frac{y[\tau^{-1}(n)]}{p_1[\tau^{-1}(n)]} \quad \text{for } n \geq n_2 \geq n_1. \tag{4.8}$$

Next, using (4.6) or (4.8) in (4.1;δ), we see that the resulting inequalities are of the type (1.1;δ). Therefore, by applying our earlier results, we obtain oscillation results for (4.1;δ). The formulation of such results are left to the reader.

Next, we will present some oscillation results for all bounded solutions of (4.1;1) when $p(n) < 0$ and $\tau(n) = n - \sigma$, $n \geq n_0$ and σ is a positive integer.

Theorem 4.1. *Let $\tau(n) = n - \sigma$, σ is a positive integer, $p_1(n) = -p(n)$ and $0 < p_1(n) \leq p < 1$, $n \geq n_0$, p is a constant, and $g(n) < n$ for $n \geq n_0$. If*

$$\frac{u}{f^{1/(\alpha_1\alpha_2)}(u)} \leq 1 \quad \textit{for } u \neq 0, \tag{4.9}$$

$$\limsup_{n\to\infty} \sum_{k=g(n)}^{n-1}\left[a_1(k)\sum_{j=k}^{n-1}\left(a_2(j)\sum_{i=j}^{\infty} q(i)\right)^{1/\alpha_2}\right]^{1/\alpha_1} > 1, \tag{4.10}$$

then all bounded solutions of (4.1;1) are oscillatory.

Proof. Let $\{x(n)\}$ be a bounded nonoscillatory solution of (4.1;1), say, $x(n) > 0$ for $n \geq n_0 \geq 0$. Set

$$y(n) = x(n) - p_1(n)x[n-\sigma] \quad \text{for } n \geq n_1 \geq n_0. \tag{4.11}$$

Then,

$$L_3 y(n) = -q(n)f(x[g(n)]) \leq 0 \quad \text{for } n \geq n_1. \tag{4.12}$$

It is easy to see that $y(n)$, $L_1 y(n)$, and $L_2 y(n)$ are of one sign for $n \geq n_2 \geq n_1$. Now, we have two cases to consider: (M_1) $y(n) < 0$ for $n \geq n_2$, and (M_2) $y(n) > 0$ for $n \geq n_2$.

(M_1) Let $y(n) < 0$ for $n \geq n_2$. Then either $\Delta y(n) < 0$, or $\Delta y(n) > 0$ for $n \geq n_2$. If $\Delta y(n) < 0$ for $n \geq n_2$, then

$$x(n) < px[n-\sigma] < p^2 x[n-2\sigma] < \cdots < p^m x[n-m\sigma] \tag{4.13}$$

for $n \geq n_2 + m\sigma$, which implies that $\lim_{n\to\infty} x(n) = 0$. Consequently, $\lim_{n\to\infty} y(n) = 0$, a contradiction.

Now, we have $y(n) < 0$ and $\Delta y(n) > 0$ for $n \geq n_2$. Set $Z(n) = -y(n)$ for $n \geq n_2$. Then,

$$L_3 Z(n) = q(n)f(x[g(n)]) \geq 0 \quad \text{for } n \geq n_2 \tag{4.14}$$

and $\Delta Z(n) < 0$ for $n \geq n_2$. It is easy to derive at a contradiction if either $L_2 Z(n) > 0$ or $L_2 Z(n) < 0$ for $n \geq n_2$. The details are left to the reader.

(M_2) Let $y(n) > 0$ for $n \geq n_2$. Then, $x(n) \geq y(n)$ for $n \geq n_2$ and from (4.12), we have

$$L_3 y(n) \leq -q(n)f(y[g(n)]) \quad \text{for } n \geq n_2. \tag{4.15}$$

We claim that $\Delta y(n) < 0$ for $n \geq n_2$. Otherwise, $\Delta y(n) > 0$ for $n \geq n_2$ and hence we see that $y(n) \to \infty$ as $n \to \infty$, a contradiction. Thus, we have $y(n) > 0$ and $\Delta y(n) < 0$ for $n \geq n_2$. Summing (4.15) from $n \geq n_2$ to u and letting $u \to \infty$, we have

$$\Delta\left(\frac{1}{a_1(n)}(\Delta y(n))^{\alpha_1}\right) \geq f^{1/\alpha_2}(y[g(n)])\left(a_2(n)\sum_{i=n}^{\infty} q(i)\right)^{1/\alpha_2}. \tag{4.16}$$

Again summing (4.16) twice from $j = k$ to $n-1$, and from $k = g(n)$ to $n-1$, we obtain

$$1 \geq \frac{y[g(n)]}{f^{1/(\alpha_1\alpha_2)}(y[g(n)])} \geq \sum_{k=g(n)}^{n-1}\left[a_1(k)\sum_{j=k}^{n-1}\left(a_2(j)\sum_{i=j}^{\infty}q(i)\right)^{1/\alpha_2}\right]^{1/\alpha_1}. \tag{4.17}$$

Taking lim sup of both sides of the above inequality as $n \to \infty$, we arrive at the desired contradiction. This completes the proof. □

In the case when $p(n) \equiv -1$, we have the following result.

Theorem 4.2. *Let $\tau(n) = n - \sigma$, σ is a positive integer, $p(n) = -1$, and $g(n) < n$ for $n \geq n_2$. If*

$$\sum^{\infty}\left(a_1(k)\sum_{j=k}^{\infty}\left(a_2(j)\sum_{i=j}^{\infty}q(i)\right)^{1/\alpha_2}\right)^{1/\alpha_1} = \infty, \tag{4.18}$$

then all bounded solutions of (4.1;1) are oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (4.1;1), say, $x(n) > 0$ for $n \geq n_0 \geq 0$. Set

$$y(n) = x(n) - x[n-\sigma] \quad \text{for } n \geq n_1 \geq n_0. \tag{4.19}$$

Then,

$$L_3y(n) = -q(n)f(x[g(n)]) \leq 0 \quad \text{for } n \geq n_1. \tag{4.20}$$

It is easy to check that there are two possibilities to consider: (Z_1) $L_2y(n) \geq 0$, $\Delta y(n) \leq 0$, and $y(n) < 0$ for $n \geq n_2 \geq n_1$, or (Z_2) $L_2y(n) \geq 0$, $\Delta y(n) \leq 0$, and $y(n) > 0$ for $n \geq n_2$.

In case (Z_1), there exists a finite constant $b > 0$ such that $\lim_{n\to\infty} y(n) = -b$. Thus, there exists an $n_3 \geq n_2$ such that

$$-b < y(n) < -\frac{b}{2} \quad \text{for } n \geq n_3. \tag{4.21}$$

Hence,

$$x[n-\sigma] > \frac{b}{2} \quad \text{for } n \geq n_3, \tag{4.22}$$

then there exists an $n_4 \geq n_3$ such that

$$x[g(n)] > \frac{b}{2} \quad \text{for } n \geq n_4. \tag{4.23}$$

From (4.20), we have

$$L_3y(n) \leq -f\left(\frac{b}{2}\right)q(n) \quad \text{for } n \geq n_4. \tag{4.24}$$

In case (Z_2), we have

$$x(n) \geq x[n-\tau] \quad \text{for } n \geq n_2. \tag{4.25}$$

Then there exist a constant $b_1 > 0$ and an integer $n_3 \geq n_2$ such that

$$x[g(n)] \geq b_1 \quad \text{for } n \geq n_3. \tag{4.26}$$

Hence,

$$L_3 y(n) \leq -f(b_1)q(n) \quad \text{for } n \geq n_4 \geq n_3. \tag{4.27}$$

In both cases we are lead to the same inequality (4.27). Summing (4.27) from $n \geq n_4$ to $u \geq n$ and letting $u \to \infty$, we get

$$\Delta\left(\frac{1}{a_1(n)}(\Delta y(n))^{\alpha_1}\right) \geq f^{1/\alpha_2}(b_1)\left(a_2(n)\sum_{i=n}^{\infty} q(i)\right)^{1/\alpha_2}. \tag{4.28}$$

Once again, summing the above inequality from $n \geq n_4$ to $T \geq n$ and letting $T \to \infty$, we have

$$-\Delta y(n) \geq f^{1/(\alpha_1\alpha_2)}(b)\left(a_1(n)\sum_{n=k}^{\infty}\left(a_2(k)\sum_{i=k}^{\infty} q(i)\right)^{1/\alpha_2}\right)^{1/\alpha_1}. \tag{4.29}$$

Summing the above inequality from n_4 to $n-1 \geq n_4$, we get

$$\begin{aligned}\infty > y(n_4) > -y(n) + y(n_4) \geq f^{1/(\alpha_1\alpha_2)}(b_1)\sum_{k=n_4}^{n-1}\left(a_1(k)\sum_{j=k}^{\infty}\left(a_2(j)\sum_{j=i}^{\infty} q(i)\right)^{1/\alpha_2}\right)^{1/\alpha_1} \\ \longrightarrow \infty \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{4.30}$$

which is a contradiction. This completes the proof. □

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References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publishing, New York, in press.

[3] R. P. Agarwal and S. R. Grace, *Oscillation criteria for certain higher order difference equations*, Math. Sci. Res. J. **6** (2002), no. 1, 60–64.

[4] R. P. Agarwal, S. R. Grace, and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic, Dordrecht, 2000.

[5] ______, *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Academic, Dordrecht, 2002.

[6] ______, *On the oscillation of certain second order difference equations*, J. Differ. Equations Appl. **9** (2003), no. 1, 109–119.

[7] ______, *Oscillation Theory for Second Order Dynamic Equations*, Series in Mathematical Analysis and Applications, vol. 5, Taylor & Francis, London, 2003.

[8] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, Mathematics and Its Applications, vol. 404, Kluwer Academic, Dordrecht, 1997.

[9] Z. Došlá and A. Kobza, *Global asymptotic properties of third-order difference equations*, Comput. Math. Appl. **48** (2004), no. 1-2, 191–200.

[10] ______, *Oscillatory properties of third order linear adjoint difference equations*, to appear.

[11] S. R. Grace and B. S. Lalli, *Oscillation theorems for second order delay and neutral difference equations*, Utilitas Math. **45** (1994), 197–211.

[12] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, New York, 1991.

[13] Ch. G. Philos, *On Oscillations of Some Difference Equations*, Funkcial. Ekvac. **34** (1991), no. 1, 157–172.

[14] B. Smith, *Oscillation and nonoscillation theorems for third order quasi-adjoint difference equations*, Portugal. Math. **45** (1988), no. 3, 229–243.

[15] P. J. Y. Wong and R. P. Agarwal, *Oscillation theorems for certain second order nonlinear difference equations*, J. Math. Anal. Appl. **204** (1996), no. 3, 813–829.

[16] X. Zhou and J. Yan, *Oscillatory and asymptotic properties of higher order nonlinear difference equations*, Nonlinear Anal. **31** (1998), no. 3-4, 493–502.

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