PERIODIC SOLUTIONS OF A DISCRETE-TIME DIFFUSIVE SYSTEM GOVERNED BY BACKWARD DIFFERENCE EQUATIONS

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A discrete-time delayed diffusion model governed by backward difference equations is investigated. By using the coincidence degree and the related continuation theorem as well as some priori estimates, easily verifiable sufficient criteria are established for the existence of positive periodic solutions.

1. Introduction

Recently, some biologists have argued that the ratio-dependent predator-prey model is more appropriate than the Gauss-type models for modelling predator-prey interactions where predation involves searching processes. This is strongly supported by numerous laboratory experiments and observations [1, 2, 3, 4, 10, 11, 12]. Many authors [1, 5, 7, 13, 14] have observed that the ratio-dependent predator-prey systems exhibit much richer, more complicated, and more reasonable or acceptable dynamics. In view of periodicity of the actual environment, Chen et al. [6] considered the following two-species ratio-dependent predator-prey nonautonomous diffusion system with time delay:

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left(a_1(t) - a_{11}(t)x_1(t) - \frac{a_{13}(t)x_3(t)}{m(t)x_3(t)+x_1(t)}\right) + D_1(t)\big(x_2(t)-x_1(t)\big),\\
\dot{x}_2(t) &= x_2(t)\big(a_2(t) - a_{22}(t)x_2(t)\big) + D_2(t)\big(x_1(t)-x_2(t)\big), \qquad (1.1)\\
\dot{x}_3(t) &= x_3(t)\left(-a_3(t) + \frac{a_{31}(t)x_1(t-\tau)}{m(t)x_3(t-\tau)+x_1(t-\tau)}\right),
\end{aligned}$$

where $x_i(t)$ represents the prey population in the ith patch $(i=1,2)$, and $x_3(t)$ represents the predator population, $\tau > 0$ is a constant delay due to gestation, and $D_i(t)$ denotes the dispersal rate of the prey in the ith patch $(i=1,2)$. $D_i(t)$ $(i=1,2)$, $a_i(t)$ $(i=1,2,3)$, $a_{11}(t)$, $a_{13}(t)$, $a_{22}(t)$, $a_{31}(t)$, and $m(t)$ are strictly positive continuous ω-periodic functions. They proved that system (1.1) has at least one positive ω-periodic solution if the conditions $a_{31}(t) > a_3(t)$ and $m(t)a_1(t) > a_{13}(t)$ are satisfied.

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One question arises naturally. Does the discrete analog of system (1.1) have a positive periodic solution? The purpose of this paper is to answer this question to some extent. More precisely, we consider the following discrete-time diffusion system governed by backward difference equations:

$$\begin{aligned}
x_1(k) &= x_1(k-1)\exp\left\{a_1(k) - a_{11}(k)x_1(k) - \frac{a_{13}(k)x_3(k)}{m(k)x_3(k)+x_1(k)} + D_1(k)\frac{x_2(k)-x_1(k)}{x_1(k)}\right\},\\
x_2(k) &= x_2(k-1)\exp\left\{a_2(k) - a_{22}(k)x_2(k) + D_2(k)\frac{x_1(k)-x_2(k)}{x_2(k)}\right\},\\
x_3(k) &= x_3(k-1)\exp\left\{-a_3(k) + \frac{a_{31}(k)x_1(k-l)}{m(k)x_3(k-l)+x_1(k-l)}\right\}
\end{aligned} \tag{1.2}$$

with initial condition

$$x_i(-m) \geq 0, \quad m = 1,2,\dots,l; \qquad x_i(0) > 0 \quad (i = 1,2,3), \tag{1.3}$$

where $D_i(k)$ $(i = 1,2)$, $a_i(k)$ $(i = 1,2,3)$, $a_{11}(k)$, $a_{13}(k)$, $a_{22}(k)$, $a_{31}(k)$, $m(k)$ are strictly positive ω-periodic sequence, that is,

$$\begin{gathered}
D_i(k+\omega) = D_i(k), \quad i = 1,2,\\
a_i(k+\omega) = a_i(k), \quad i = 1,2,3,\\
a_{11}(k+\omega) = a_{11}(k), \qquad a_{13}(k+\omega) = a_{13}(k),\\
a_{22}(k+\omega) = a_{22}(k), \qquad a_{31}(k+\omega) = a_{31}(k),\\
m(k+\omega) = m(k)
\end{gathered} \tag{1.4}$$

for arbitrary integer k, where ω, a fixed positive integer, denotes the prescribed common period of the parameters in (1.2).

It is well known that, compared to the continuous-time systems, the discrete-time ones are more difficult to deal with. To the best of our knowledge, no work has been done for the discrete-time system analogue of (1.1). Our purpose in this paper is, by using the continuation theorem of coincidence degree theory [9], to establish sufficient conditions for the existence of at least one positive ω-periodic solution of system (1.2).

Let $\mathbb{Z}$, Z^+, $\mathbb{R}$, $\mathbb{R}^+$, and R^3 denote the sets of all integers, nonnegative integers, real numbers, nonnegative real numbers, and the three-dimensional Euclidean vector space, respectively.

For convenience, we introduce the following notation:

$$I_\omega = \{1,2,\dots,\omega\}, \qquad \bar{u} = \frac{1}{\omega}\sum_{k=1}^{\omega} u(k),$$
$$u^L = \min_{k\in I_\omega} u(k), \qquad u^M = \max_{k\in I_\omega} u(k), \tag{1.5}$$

where $u(k)$ is an ω-periodic sequence of real numbers defined for $k \in \mathbb{Z}$.

Our main result in this paper is the following theorem.

Theorem 1.1. *Assume the following conditions are satisfied:*
(H_1) $\bar{a}_{31} > \bar{a}_3$;
(H_2) $m(k)a_1(k) > a_{13}(k)$.
Then system (1.2) has at least one ω-periodic solution, say $x^*(k) = (x_1^*(k), x_2^*(k), x_3^*(k))^T$ and there exist positive constants α_i and β_i, $i = 1,2,3$, such that

$$\alpha_i \le x_i^*(k) \le \beta_i, \quad i = 1,2,3,\ k \in \mathbb{Z}. \tag{1.6}$$

The proof of the theorem is based on the continuation theorem of coincidence degree theory [9]. For the sake of convenience, we introduce this theorem as follows.

Let X, Y be normed vector spaces, let $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and let $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. Suppose L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. Then $L \mid \operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 1.2 (continuation theorem). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*
(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \ne \lambda Nx$;*
(b) *$QNx \ne 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \ne 0$.
Then the operator equation $LX = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

Lemma 1.3 [8]. *Let $u : \mathbb{Z} \to \mathbb{R}$ be ω-periodic, that is, $u(k+\omega) = u(k)$. Then for any fixed k_1, $k_2 \in I_\omega$, and for any $k \in \mathbb{Z}$, it holds that*

$$u(k) \le u(k_1) + \sum_{s=1}^{\omega} |u(s) - u(s-1)|,$$
$$u(k) \ge u(k_2) - \sum_{s=1}^{\omega} |u(s) - u(s-1)|. \tag{1.7}$$

Lemma 1.4. *If the condition* (H_1) *holds, then the system of algebraic equations*

$$\begin{gathered} \bar{a}_1 - \bar{a}_{11} v_1 = 0, \\ \bar{a}_2 - \bar{a}_{22} v_2 = 0, \\ \bar{a}_3 - \frac{v_1}{\omega} \sum_{k=1}^{\omega} \frac{a_{31}(k)}{m(k) v_3 + v_1} = 0 \end{gathered} \tag{1.8}$$

has a unique solution $(v_1^*, v_2^*, v_3^*) \in R^3$ *with* $v_i^* > 0$.

Proof. From the first two equations of (1.8), we have

$$v_1^* = \frac{\bar{a}_1}{\bar{a}_{11}} > 0, \qquad v_2^* = \frac{\bar{a}_2}{\bar{a}_{22}} > 0. \tag{1.9}$$

Consider the function

$$f(u) = \bar{a}_3 - \frac{1}{\omega} \sum_{k=1}^{\omega} \frac{a_{31}(k)}{m(k) u + 1}, \qquad u \geq 0. \tag{1.10}$$

Obviously, $\lim_{u \to +\infty} f(u) = \bar{a}_3 > 0$. Since (H_1) implies $\bar{a}_{31} > \bar{a}_3$, it follows that

$$f(0) = \bar{a}_3 - \bar{a}_{31} < 0. \tag{1.11}$$

Then, by the zero-point theorem and the monotonicity of $f(u)$, there exists a unique $u^* > 0$ such that $f(u^*) = 0$. Let $v_3^* = u^* v_1^* > 0$. Then it is easy to see that $(v_1^*, v_2^*, v_3^*)^T$ is the unique positive solution of (1.8). The proof is complete. □

2. Priori estimates

In this section, we will give some priori estimates which are crucial in the proof of our theorem.

Lemma 2.1. *Suppose* $\lambda \in (0,1]$ *is a parameter, the conditions* (H_1)*-*(H_2) *hold,* $(y_1(k), y_2(k), y_3(k))^T$ *is an* ω*-periodic solution of the system*

$$\begin{aligned} & y_1(k) - y_1(k-1) \\ & \quad = \lambda \Bigg[a_1(k) - D_1(k) - a_{11}(k) \exp\{y_1(k)\} - \frac{a_{13}(k) \exp\{y_3(k)\}}{m(k) \exp\{y_3(k)\} + \exp\{y_1(k)\}} \\ & \qquad + D_1(k) \exp\{y_2(k) - y_1(k)\} \Bigg], \\ & y_2(k) - y_2(k-1) \\ & \quad = \lambda \big[a_2(k) - D_2(k) - a_{22}(k) \exp\{y_2(k)\} + D_2(k) \exp\{y_1(k) - y_2(k)\} \big], \\ & y_3(k) - y_3(k-1) = \lambda \Bigg[- a_3(k) + \frac{a_{31}(k) \exp\{y_1(k-l)\}}{m(k) \exp\{y_3(k-l)\} + \exp\{y_1(k-l)\}} \Bigg]. \end{aligned} \tag{2.1}$$

Then

$$|y_1(k)| + |y_2(k)| + |y_3(k)| \le R_1, \tag{2.2}$$

where $R_1 = 2M_1 + M_2$ *and*

$$\begin{aligned} M_1 &= \max\left\{ \left| \ln\left(\frac{a_1}{a_{11}}\right)^M \right|, \left| \ln\left(\frac{a_2}{a_{22}}\right)^M \right|, \left| \ln\left(\frac{a_2}{a_{22}}\right)^L \right|, \left| \ln\left(\frac{ma_1 - a_{13}}{ma_{11}}\right)^L \right| \right\}, \\ M_2 &= \max\left\{ \left| \ln\frac{1}{\bar{a}_3}\left(\frac{\overline{a_{31}}}{m}\right) + M_1 + 2\bar{a}_3\omega \right|, \left| \ln\frac{\bar{a}_{31} - \bar{a}_3}{\bar{a}_3 m^M} - M_1 - 2\bar{a}_3\omega \right| \right\}. \end{aligned} \tag{2.3}$$

Proof. Since $y_i(k)$ $(i = 1,2,3)$ are ω-periodic sequences, we only need to prove the result in I_ω. Choose $\xi_i \in I_\omega$ such that

$$y_i(\xi_i) = \max_{k \in I_\omega} y_i(k), \quad i = 1,2,3. \tag{2.4}$$

Then it is clear that

$$\nabla y_i(\xi_i) \ge 0, \quad i = 1,2,3, \tag{2.5}$$

where ∇ denotes the backward difference operator $\nabla y(k) = y(k) - y(k-1)$.

In view of this and the first two equations of (2.1), we obtain

$$\begin{aligned} &a_1(\xi_1) - D_1(\xi_1) - a_{11}(\xi_1)\exp\{y_1(\xi_1)\} \\ &\qquad - \frac{a_{13}(\xi_1)\exp\{y_3(\xi_1)\}}{m(\xi_1)\exp\{y_3(\xi_1)\} + \exp\{y_1(\xi_1)\}} + D_1(\xi_1)\exp\{y_2(\xi_1) - y_1(\xi_1)\} \ge 0, \\ &a_2(\xi_2) - D_2(\xi_2) - a_{22}(\xi_2)\exp\{y_2(\xi_2)\} + D_2(\xi_2)\exp\{y_1(\xi_2) - y_2(\xi_2)\} \ge 0. \end{aligned} \tag{2.6}$$

If $y_1(\xi_1) \ge y_2(\xi_2)$, then $y_1(\xi_1) \ge y_2(\xi_1)$. So from the first equation of (2.6), we have

$$a_{11}(\xi_1)\exp\{y_1(\xi_1)\} \le a_1(\xi_1) - D_1(\xi_1) + D_1(\xi_1)\exp\{y_2(\xi_1) - y_1(\xi_1)\} \le a_1(\xi_1), \tag{2.7}$$

which implies

$$y_2(\xi_2) \le y_1(\xi_1) \le \ln\frac{a_1(\xi_1)}{a_{11}(\xi_1)} \le \ln\left(\frac{a_1}{a_{11}}\right)^M. \tag{2.8}$$

Similarly, if $y_1(\xi_1) < y_2(\xi_2)$, then we will have

$$y_1(\xi_1) < y_2(\xi_2) \leq \ln \frac{a_2(\xi_2)}{a_{22}(\xi_2)} \leq \ln \left(\frac{a_2}{a_{22}} \right)^M. \tag{2.9}$$

Now choose $\eta_i \in I_\omega$ $(i = 1,2,3)$, such that

$$y_i(\eta_i) = \min_{k \in I_\omega} y_i(k), \quad i = 1,2,3. \tag{2.10}$$

Then

$$\nabla y_i(\eta_i) \leq 0, \quad i = 1,2,3. \tag{2.11}$$

A similar argument as that for $\nabla y_i(\xi_i) \geq 0$ will give us

$$\begin{aligned} y_1(\eta_1) &\geq y_2(\eta_2) \geq \ln \left(\frac{a_2}{a_{22}} \right)^L, \\ y_2(\eta_2) &\geq y_1(\eta_1) \geq \ln \left(\frac{ma_1 - a_{13}}{ma_{11}} \right)^L. \end{aligned} \tag{2.12}$$

In summary, we have shown

$$|y_i(k)| \leq M_1, \quad i = 1,2. \tag{2.13}$$

On the other hand, summing both sides of the third equation of (2.1) from 1 to ω with respect to k, we reach

$$\sum_{k=1}^{\omega} \frac{a_{31}(k) \exp\{y_1(k-l)\}}{m(k) \exp\{y_3(k-l)\} + \exp\{y_1(k-l)\}} = \bar{a}_3 \omega. \tag{2.14}$$

It follows from the third equation of (2.1) and (2.14) that

$$\begin{aligned} \sum_{k=1}^{\omega} |y_3(k) - y_3(k-1)| &\leq \bar{a}_3 \omega + \sum_{k=1}^{\omega} \frac{a_{31}(k) \exp\{y_1(k-l)\}}{m(k) \exp\{y_3(k-l)\} + \exp\{y_1(k-l)\}} \\ &= 2\bar{a}_3 \omega. \end{aligned} \tag{2.15}$$

From (2.13) and (2.14), we can derive that

$$\begin{aligned} \bar{a}_3 \omega &\leq \sum_{k=1}^{\omega} \frac{a_{31}(k) \exp\{y_1(k-l)\}}{m(k) \exp\{y_3(k-l)\}} \\ &\leq \sum_{k=1}^{\omega} \frac{a_{31}(k) \exp\{y_1(k-l)\}}{m(k) \exp\{y_3(\eta_3)\}} \\ &\leq \frac{\exp\{M_1\}}{\exp\{y_3(\eta_3)\}} \left(\overline{\frac{a_{31}}{m}} \right) \omega. \end{aligned} \tag{2.16}$$

Hence

$$y_3(\eta_3) \le \ln \frac{1}{\bar{a}_3}\left(\frac{\overline{a_{31}}}{m}\right) + M_1. \tag{2.17}$$

This, combined with (2.15) and Lemma 1.3, yields

$$\begin{aligned} y_3(k) &\le y_3(\eta_3) + \sum_{k=1}^{\omega} |y_3(k) - y_3(k-1)| \\ &\le \ln \frac{1}{\bar{a}_3}\left(\frac{\overline{a_{31}}}{m}\right) + M_1 + 2\bar{a}_3\omega. \end{aligned} \tag{2.18}$$

We can derive from (2.13) and (2.14) that

$$\begin{aligned} \bar{a}_3\omega &= \sum_{k=1}^{\omega} \frac{a_{31}(k)\exp\{y_1(k-l)\}}{m(k)\exp\{y_3(k-l)\} + \exp\{y_1(k-l)\}} \\ &\ge \sum_{k=1}^{\omega} \frac{a_{31}(k)\exp\{y_1(k-l)\}}{m^M\exp\{y_3(\xi_3)\} + \exp\{y_1(k-l)\}} \\ &\ge \frac{\exp\{-M_1\}}{m^M\exp\{y_3(\xi_3)\} + \exp\{-M_1\}}\bar{a}_{31}\omega. \end{aligned} \tag{2.19}$$

Then, it follows that

$$y_3(\xi_3) \ge \ln \frac{\bar{a}_{31} - \bar{a}_3}{m^M\bar{a}_3} - M_1. \tag{2.20}$$

Again, this, combined with (2.15) and Lemma 1.3, yields

$$\begin{aligned} y_3(k) &\ge y_3(\xi_3) - \sum_{k=1}^{\omega} |y_3(k) - y_3(k-1)| \\ &\ge \ln \frac{\bar{a}_{31} - \bar{a}_3}{m^M\bar{a}_3} - M_1 - 2\bar{a}_3\omega. \end{aligned} \tag{2.21}$$

Therefore, we have shown

$$|y_3(k)| \le \max\left\{\left|\ln \frac{1}{\bar{a}_3}\left(\frac{\overline{a_{31}}}{m}\right) + M_1 + 2\bar{a}_3\omega\right|, \left|\ln \frac{\bar{a}_{31} - \bar{a}_3}{m^M\bar{a}_3} - M_1 - 2\bar{a}_3\omega\right|\right\} = M_2. \tag{2.22}$$

Now, it follows from (2.13) and (2.22) that

$$|y_1(k)| + |y_2(k)| + |y_3(k)| \le R_1. \tag{2.23}$$

The proof is complete. $\square$

The following result can be proved in a similar way as for Lemma 2.1.

Lemma 2.2. *Suppose $\mu \in [0,1]$ is a parameter, the conditions (H_1)-(H_2) hold, and $(y_1, y_2, y_3)^T$ is a constant solution to the system of the equations*

$$\begin{aligned}
&\bar{a}_1 - \bar{a}_{11}\exp\{y_1\} \\
&\quad + \mu\left(- \bar{D}_1 - \frac{1}{\omega}\exp\{y_3\}\sum_{k=1}^{\omega}\frac{a_{13}(k)}{m(k)\exp\{y_3\}+\exp\{y_1\}} + \bar{D}_1\exp\{y_2 - y_1\}\right) = 0, \\
&\bar{a}_2 - \bar{a}_{22}\exp\{y_2\} + \mu\left(- \bar{D}_2 + \bar{D}_2\exp\{y_1 - y_2\}\right) = 0, \\
&- \bar{a}_3 + \frac{\exp\{y_1\}}{\omega}\sum_{k=1}^{\omega}\frac{a_{31}(k)}{m(k)\exp\{y_3\}+\exp\{y_1\}} = 0.
\end{aligned} \tag{2.24}$$

Then

$$|y_1| + |y_2| + |y_3| \le R_2, \tag{2.25}$$

where $R_2 = 2M_3 + M_4$ and

$$\begin{aligned}
M_3 &= \max\left\{\left|\ln\frac{\bar{a}_1}{\bar{a}_{11}}\right|, \left|\ln\frac{\bar{a}_2}{\bar{a}_{22}}\right|, \left|\ln\frac{\bar{a}_1 - (\overline{a_{13}/m})}{\bar{a}_{11}}\right|\right\}, \\
M_4 &= \max\left\{\left|\ln\frac{\bar{a}_{31} - \bar{a}_3}{m^M\bar{a}_3} - M_3\right|, \left|\ln\frac{\bar{a}_{31} - \bar{a}_3}{m^L\bar{a}_3} + M_3\right|\right\}.
\end{aligned} \tag{2.26}$$

3. Proof of the main result

Define

$$l_3 = \{y = \{y(k)\} : y(k) \in R^3,\ k \in \mathbb{Z}\}. \tag{3.1}$$

Let $l^\omega \subset l_3$ denote the subspace of all ω-periodic sequences equipped with the norm $\|\cdot\|$ defined by $\|y\| = \max_{k\in I_\omega}(|y_1(k)| + |y_2(k)| + |y_3(k)|)$ for $y = \{y(k)\} = \{(y_1(k), y_2(k), y_3(k))^T\} \in l^\omega$. It is not difficult to show that l^ω is a finite-dimensional Banach space.

Let

$$\begin{aligned}
l_0^\omega &= \left\{y = y(k) \in l^\omega : \sum_{k=1}^{\omega} y(k) = 0\right\}, \\
l_c^\omega &= \left\{y = y(k) \in l^\omega : y(k) = (y_1, y_2, y_3)^T \in R^3,\ k \in \mathbb{Z}\right\}.
\end{aligned} \tag{3.2}$$

Then, obviously, l_0^ω and l_c^ω are both closed linear subspaces of l^ω. Moreover,

$$l^\omega = l_0^\omega \bigoplus l_c^\omega, \quad \dim l_c^\omega = 3. \tag{3.3}$$

Now we reach the position to prove our main result.

Let $x_i(k) = \exp\{y_i(k)\}$, $i = 1,2,3$. Then system (1.2) can be rewritten as

$$\begin{aligned}
&y_1(k) - y_1(k-1) \\
&\quad = a_1(k) - D_1(k) - a_{11}(k)\exp\{y_1(k)\} \\
&\qquad - \frac{a_{13}(k)\exp\{y_3(k)\}}{m(k)\exp\{y_3(k)\} + \exp\{y_1(k)\}} + D_1(k)\exp\{y_2(k) - y_1(k)\}, \\
&y_2(k) - y_2(k-1) \\
&\quad = a_2(k) - D_2(k) - a_{22}(k)\exp\{y_2(k)\} + D_2(k)\exp\{y_1(k) - y_2(k)\}, \\
&y_3(k) - y_3(k-1) \\
&\quad = -a_3(k) + \frac{a_{31}(k)\exp\{y_1(k-l)\}}{m(k)\exp\{y_3(k-l)\} + \exp\{y_1(k-l)\}}.
\end{aligned} \tag{3.4}$$

So to complete the proof, it suffices to show that system (3.4) has at least one ω-periodic solution. To this end, we take $X = Y = l^\omega$, $(Ly)(k) = \nabla y(k) = y(k) - y(k-1)$, and

$$\begin{aligned}
&(Ny)(k) \\
&\quad = \begin{pmatrix}
a_1(k) - D_1(k) - a_{11}(k)\exp\{y_1(k)\} \\
\quad - \dfrac{a_{13}(k)\exp\{y_3(k)\}}{m(k)\exp\{y_3(k)\} + \exp\{y_1(k)\}} + D_1(k)\exp\{y_2(k) - y_1(k)\} \\
a_2(k) - D_2(k) - a_{22}(k)\exp\{y_2(k)\} + D_2(k)\exp\{y_1(k) - y_2(k)\} \\
\quad - a_3(k) + \dfrac{a_{31}(k)\exp\{y_1(k-l)\}}{m(k)\exp\{y_3(k-l)\} + \exp\{y_1(k-l)\}}
\end{pmatrix}
\end{aligned} \tag{3.5}$$

for any $y \in X$ and $k \in \mathbb{Z}$. It is trivial to see that L is a bounded linear operator and

$$\operatorname{Ker} L = l_c^\omega, \quad \operatorname{Im} L = l_0^\omega, \tag{3.6}$$

as well as

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 3. \tag{3.7}$$

So L is a Fredholm mapping of index zero.

Define

$$Py = \frac{1}{\omega}\sum_{k=1}^{\omega} y(k), \quad y \in X, \qquad Qz = \frac{1}{\omega}\sum_{k=1}^{\omega} z(k), \quad z \in Y. \tag{3.8}$$

It is not difficult to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q). \tag{3.9}$$

Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P(z) = \sum_{s=1}^{k} z(s) - \frac{1}{\omega}\sum_{s=1}^{k}(\omega - s + 1)z(s). \tag{3.10}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, and $K_P(I-Q)N$ is continuous, it follows that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \in X$. Particularly we take

$$\Omega := \{y = y(k) \in X : \|y\| < R_1 + R_2\}, \tag{3.11}$$

where R_1 and R_2 are as in Lemma 2.1 and Lemma 2.2. It is clear that Ω is an open bounded set in X, N is L-compact on $\bar{\Omega}$. Now we check the remaining three conditions of the continuation theorem of coincidence degree theory. Due to Lemma 2.1, we conclude that for each $\lambda \in (0,1)$, $y \in \partial\Omega \cap \operatorname{Dom} L$, $Ly \neq \lambda N y$. When $y = (y_1(k), y_2(k), y_3(k))^T \in \partial\Omega \cap \operatorname{Ker} L$, $(y_1(k), y_2(k), y_3(k))^T$ is a constant vector in R^3, we denote it by $(y_1, y_2, y_3)^T$ and $\|(y_1, y_2, y_3)^T\| = R_1 + R_2$. If $QNy = 0$, then $(y_1, y_2, y_3)^T$ is a constant solution to the following system of equations:

$$\begin{gathered}
\bar{a}_1 - \bar{a}_{11}\exp\{y_1\} \\
+ \left(-\bar{D}_1 - \frac{1}{\omega}\exp\{y_3\}\sum_{k=1}^{\omega}\frac{a_{13}(k)}{m(k)\exp\{y_3\} + \exp\{y_1\}} + \bar{D}_1\exp\{y_2 - y_1\}\right) = 0, \\
\bar{a}_2 - \bar{a}_{22}\exp\{y_2\} + \left(-\bar{D}_2 + \bar{D}_2\exp\{y_1 - y_2\}\right) = 0, \\
-\bar{a}_3 + \frac{\exp\{y_1\}}{\omega}\sum_{k=1}^{\omega}\frac{a_{31}(k)}{m(k)\exp\{y_3\} + \exp\{y_1\}} = 0.
\end{gathered} \tag{3.12}$$

From Lemma 2.2 with $\mu = 1$, we have $\|(y_1, y_2, y_3)^T\| \le R_2$. This contradiction implies for each $y \in \partial\Omega \cap \operatorname{Ker} L$, $QNy \neq 0$.

We select J, the isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$ as the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. In order to verify the condition (c) in the continuation theorem, we define ϕ : $(\operatorname{Dom} L \cap \operatorname{Ker} L) \times [0,1] \to X$ by

$$\phi(y_1, y_2, y_3, \mu) = \begin{pmatrix} \bar{a}_1 - \bar{a}_{11} \exp\{y_1\} \\ \bar{a}_2 - \bar{a}_{22} \exp\{y_2\} \\ -\bar{a}_3 + \dfrac{\exp\{y_1\}}{\omega} \displaystyle\sum_{k=1}^{\omega} \frac{a_{31}(k)}{m(k)\exp\{y_3\} + \exp\{y_1\}} \end{pmatrix} + \mu \begin{pmatrix} -\bar{D}_1 - \dfrac{1}{\omega} \exp\{y_3\} \displaystyle\sum_{k=1}^{\omega} \frac{a_{13}(k)}{m(k)\exp\{y_3\} + \exp\{y_1\}} + \bar{D}_1 \exp\{y_2 - y_1\} \\ -\bar{D}_2 + \bar{D}_2 \exp\{y_1 - y_2\} \\ 0 \end{pmatrix}, \tag{3.13}$$

where $\mu \in [0,1]$ is a parameter. When $y = (y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L$, $(y_1, y_2, y_3)^T$ is a constant vector with $\|(y_1, y_2, y_3)^T\| = R_1 + R_2$. From Lemma 2.2 we know $\phi(y_1, y_2, y_3, \mu) \neq 0$ on $\partial\Omega \cap \operatorname{Ker} L$. So, due to homotopy invariance theorem of topology degree we have

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{\phi(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{\phi(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\}. \end{aligned} \tag{3.14}$$

By Lemma 1.4, the algebraic equation (1.8) has a unique solution $(y_1^*, y_2^*, y_3^*)^T \in \Omega \cap \operatorname{Ker} L$. Thus, we have

$$\begin{aligned} &\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \\ &\quad = \operatorname{sign}\left(-\bar{a}_1 \bar{a}_2 \frac{\exp\{y_1^* + y_3^*\}}{\omega} \sum_{k=1}^{\omega} \frac{m(k) a_{13}(k)}{\left(m(k)\exp\{y_3^*\} + \exp\{y_1^*\}\right)^2} \right) \neq 0. \end{aligned} \tag{3.15}$$

By now, we have proved that Ω satisfies all the requirements of Lemma 1.2. So it follows that $Ly = Nx$ has at least one solution in $\operatorname{Dom} L \cap \bar{\Omega}$, that is to say, (3.4) has at least one ω-periodic solution in $\operatorname{Dom} L \cap \bar{\Omega}$, say $y^* = \{y^*(k)\} = \{(y_1^*(k), y_2^*(k), y_3^*(k))^T\}$. Let $x_i^*(k) = \exp\{y_i^*(k)\}$. Then $x^* = \{x^*(k)\} = \{(x_1^*(k), x_2^*(k), x_3^*(k))^T\}$ is an ω-periodic solution of system (1.2). The existence of positive constants α_i and β_i directly follows from the above discussion. The proof is complete.

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