

# BEHAVIOR OF THE POSITIVE SOLUTIONS OF FUZZY MAX-DIFFERENCE EQUATIONS

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We extend some results obtained in 1998 and 1999 by studying the periodicity of the solutions of the fuzzy difference equations  $x_{n+1} = \max\{A/x_n, A/x_{n-1}, \dots, A/x_{n-k}\}$ ,  $x_{n+1} = \max\{A_0/x_n, A_1/x_{n-1}\}$ , where  $k$  is a positive integer,  $A, A_i, i = 0, 1$ , are positive fuzzy numbers, and the initial values  $x_i, i = -k, -k+1, \dots, 0$  (resp.,  $i = -1, 0$ ) of the first (resp., second) equation are positive fuzzy numbers.

## 1. Introduction

Difference equations are often used in the study of linear and nonlinear physical, physiological, and economical problems (for partial review see [3, 6]). This fact leads to the fast promotion of the theory of difference equations which someone can find, for instance, in [1, 7, 9]. More precisely, max-difference equations have increasing interest since max operators have applications in automatic control (see [2, 11, 17, 18] and the references cited therein).

Nowadays, a modern and promising approach for engineering, social, and environmental problems with imprecise, uncertain input-output data arises, the fuzzy approach. This is an expectable effect, since fuzzy logic can handle various types of vagueness but particularly vagueness related to human linguistic and thinking (for partial review see [8, 12]).

The increasing interest in applications of these two scientific fields contributed to the appearance of fuzzy difference equations (see [4, 5, 10, 13, 14, 15, 16]).

In [17], Szalkai studied the periodicity of the solutions of the ordinary difference equation

$$x_{n+1} = \max \left\{ \frac{A}{x_n}, \frac{A}{x_{n-1}}, \dots, \frac{A}{x_{n-k}} \right\}, \quad (1.1)$$

where  $k$  is a positive integer,  $A$  is a real constant,  $x_i, i = -k, -k+1, \dots, 0$  are real numbers. More precisely, if  $A$  is a positive real constant and  $x_i, i = -k, -k+1, \dots, 0$  are positive real numbers, he proved that every positive solution of (1.1) is eventually periodic of period  $k+2$ .

In [2], Amleh et al. studied the periodicity of the solutions of the ordinary difference equation

$$x_{n+1} = \max \left\{ \frac{A_0}{x_n}, \frac{A_1}{x_{n-1}} \right\}, \quad (1.2)$$

where  $A_0, A_1$  are positive real constants and  $x_{-1}, x_0$  are real numbers. More precisely, if  $A_0, A_1$  are positive constants,  $x_{-1}, x_0$  are positive real numbers,  $A_0 > A_1$  (resp.,  $A_0 = A_1$ ) (resp.,  $A_0 < A_1$ ), then every positive solution of (1.2) is eventually periodic of period two (resp., three) (resp., four).

In this paper, our goal is to extend the above mentioned results for the corresponding fuzzy difference equations (1.1) and (1.2) where  $A, A_0, A_1$  are positive fuzzy numbers and  $x_i, i = -k, -k+1, \dots, 0, x_{-1}, x_0$  are positive fuzzy numbers. Moreover, we find conditions so that the corresponding fuzzy equations (1.1) and (1.2) have unbounded solutions, something that does not happen in case of the ordinary difference equations (1.1) and (1.2).

We note that, in order to study the behavior of a parametric fuzzy difference equation we use the following technique: we investigate the behavior of the solutions of a related family of systems of two parametric ordinary difference equations and then, using these results and the fuzzy analog of some concepts known by the theory of ordinary difference equations, we prove our main effects concerning the fuzzy difference equation.

## 2. Preliminaries

We need the following definitions.

For a set  $B$  we denote by  $\bar{B}$  the closure of  $B$ . We say that a function  $A$  from  $\mathbb{R}^+ = (0, \infty)$  into the interval  $[0, 1]$  is a fuzzy number if  $A$  is normal, convex fuzzy set (see [13]), upper semicontinuous and the support  $\text{supp} A = \overline{\bigcup_{a \in (0,1)} [A]_a} = \overline{\{x : A(x) > 0\}}$  is compact. Then from [12, Theorems 3.1.5 and 3.1.8] the  $a$ -cuts of the fuzzy number  $A$ ,  $[A]_a = \{x \in \mathbb{R}^+ : A(x) \geq a\}$  are closed intervals.

We say that a fuzzy number  $A$  is positive if  $\text{supp} A \subset (0, \infty)$ .

It is obvious that if  $A$  is a positive real number, then  $A$  is a positive fuzzy number and  $[A]_a = [A, A], a \in (0, 1]$ . In this case, we say that  $A$  is a trivial fuzzy number.

Let  $B_i, i = 0, 1, \dots, k, k$  is a positive integer, be fuzzy numbers such that

$$[B_i]_a = [B_{i,l,a}, B_{i,r,a}], \quad i = 0, 1, \dots, k, a \in (0, 1], \quad (2.1)$$

and for any  $a \in (0, 1]$

$$C_{l,a} = \max \{B_{i,l,a}, i = 0, 1, \dots, k\}, \quad C_{r,a} = \max \{B_{i,r,a}, i = 0, 1, \dots, k\}. \quad (2.2)$$

Then by [19, Theorem 2.1],  $(C_{l,a}, C_{r,a})$  determines a fuzzy number  $C$  such that

$$[C]_a = [C_{l,a}, C_{r,a}], \quad a \in (0, 1]. \quad (2.3)$$

According to [8] and [14, Lemma 2.3] we can define

$$C = \max \{B_i, i = 0, 1, \dots, k\}. \quad (2.4)$$

We say that  $x_n$  is a positive solution of (1.1) (resp., (1.2)) if  $x_n$  is a sequence of positive fuzzy numbers which satisfies (1.1) (resp., (1.2)).

We say that a sequence of positive fuzzy numbers  $x_n$  persists (resp., is bounded) if there exists a positive number  $M$  (resp.,  $N$ ) such that

$$\text{supp } x_n \subset [M, \infty), \quad (\text{resp.}, \text{supp } x_n \subset (0, N]), \quad n = 1, 2, \dots \tag{2.5}$$

In addition, we say that  $x_n$  is bounded and persists if there exist numbers  $M, N \in (0, \infty)$  such that

$$\text{supp } x_n \subset [M, N], \quad n = 1, 2, \dots \tag{2.6}$$

A solution  $x_n$  of (1.1) (resp., (1.2)) is said to be eventually periodic of period  $r$ ,  $r$  is a positive integer, if there exists a positive integer  $m$  such that

$$x_{n+r} = x_n, \quad n = m, m + 1, \dots \tag{2.7}$$

**3. Existence and uniqueness of the positive solutions of fuzzy difference equations (1.1) and (1.2)**

In this section, we study the existence and the uniqueness of the positive solutions of the fuzzy difference equations (1.1) and (1.2).

PROPOSITION 3.1. *Suppose that  $A, A_0, A_1$  are positive fuzzy numbers. Then for all positive fuzzy numbers  $x_{-k}, x_{-k+1}, \dots, x_0$  (resp.,  $x_{-1}, x_0$ ) there exists a unique positive solution  $x_n$  of (1.1) (resp., (1.2)) with initial values  $x_{-k}, x_{-k+1}, \dots, x_0$  (resp.,  $x_{-1}, x_0$ ).*

*Proof.* Suppose that

$$[A]_a = [A_{l,a}, A_{r,a}], \quad a \in (0, 1]. \tag{3.1}$$

Let  $x_i, i = -k, -k + 1, \dots, 0$  be positive fuzzy numbers such that

$$[x_i]_a = [L_{i,a}, R_{i,a}], \quad i = -k, -k + 1, \dots, 0, \quad a \in (0, 1] \tag{3.2}$$

and let  $(L_{n,a}, R_{n,a}), n = 0, 1, \dots, a \in (0, 1]$ , be the unique positive solution of the system of difference equations

$$\begin{aligned} L_{n+1,a} &= \max \left\{ \frac{A_{l,a}}{R_{n,a}}, \frac{A_{l,a}}{R_{n-1,a}}, \dots, \frac{A_{l,a}}{R_{n-k,a}} \right\}, \\ R_{n+1,a} &= \max \left\{ \frac{A_{r,a}}{L_{n,a}}, \frac{A_{r,a}}{L_{n-1,a}}, \dots, \frac{A_{r,a}}{L_{n-k,a}} \right\} \end{aligned} \tag{3.3}$$

with initial values  $(L_{i,a}, R_{i,a}), i = -k, -k + 1, \dots, 0$ . Using [19, Theorem 2.1] and relation (3.3) and working as in [13, Proposition 2.1] and [15, Proposition 1] we can easily prove that  $(L_{n,a}, R_{n,a}), n = 1, 2, \dots, a \in (0, 1]$  determines a sequence of positive fuzzy numbers  $x_n$  such that

$$[x_n]_a = [L_{n,a}, R_{n,a}], \quad n = 1, 2, \dots, a \in (0, 1]. \tag{3.4}$$

Now, we prove that  $x_n$  satisfies (1.1) with initial values  $x_i, i = -k, -k + 1, \dots, 0$ . From (3.1), (3.2), (3.3), (3.4), [15, Lemma 1], and by a slight generalization of [14, Lemma 2.3] we have

$$\begin{aligned} & \left[ \max \left\{ \frac{A}{x_n}, \frac{A}{x_{n-1}}, \dots, \frac{A}{x_{n-k}} \right\} \right]_a \\ &= \left[ \max \left\{ \frac{A_{l,a}}{R_{n,a}}, \frac{A_{l,a}}{R_{n-1,a}}, \dots, \frac{A_{l,a}}{R_{n-k,a}} \right\}, \max \left\{ \frac{A_{r,a}}{L_{n,a}}, \frac{A_{r,a}}{L_{n-1,a}}, \dots, \frac{A_{r,a}}{L_{n-k,a}} \right\} \right] \\ &= [L_{n+1,a}, R_{n+1,a}] = [x_{n+1}]_a, \quad a \in (0, 1). \end{aligned} \tag{3.5}$$

From (3.5) and arguing as in [13, Proposition 2.1] and [15, Proposition 1] we have that  $x_n$  is the unique positive solution of (1.1) with initial values  $x_i, i = -k, -k + 1, \dots, 0$ .

Now, suppose that

$$[A_i]_a = [A_{i,l,a}, A_{i,r,a}], \quad i = 0, 1, a \in (0, 1). \tag{3.6}$$

Arguing as above and using (3.6) we can easily prove that if  $x_i, i = -1, 0$  are positive fuzzy numbers which satisfy (3.2) for  $k = 1$ , then there exists a unique positive solution  $x_n$  of (1.2) with initial values  $x_i, i = -1, 0$  such that (3.4) holds and  $(L_{n,a}, R_{n,a})$  satisfies the system of difference equations

$$L_{n+1,a} = \max \left\{ \frac{A_{0,l,a}}{R_{n,a}}, \frac{A_{1,l,a}}{R_{n-1,a}} \right\}, \quad R_{n+1,a} = \max \left\{ \frac{A_{0,r,a}}{L_{n,a}}, \frac{A_{1,r,a}}{L_{n-1,a}} \right\}. \tag{3.7}$$

This completes the proof of the proposition. □

#### 4. Behavior of the positive solutions of fuzzy equation (1.1)

In this section, we study the behavior of the positive solutions of (1.1). Firstly, we study the periodicity of the positive solutions of (1.1). We need the following lemmas.

LEMMA 4.1. *Let  $A, a, b$  be positive numbers such that  $ab \neq A$ . If*

$$ab < A \quad (\text{resp.}, ab > A), \tag{4.1}$$

*then there exist positive numbers  $\bar{y}, \bar{z}$  such that*

$$\bar{y}\bar{z} = A, \tag{4.2}$$

$$a < \bar{y}, \quad b < \bar{z} \quad (\text{resp.}, a > \bar{y}, b > \bar{z}). \tag{4.3}$$

*Proof.* Suppose that (4.1) is satisfied. Then if  $\epsilon$  is a positive number such that

$$\begin{aligned} \epsilon &< \frac{A - ab}{b} && \left( \text{resp.}, \epsilon < \frac{ab - A}{b} \right), \\ \bar{y} &= a + \epsilon, \quad \bar{z} = \frac{A}{a + \epsilon} && \left( \text{resp.}, \bar{y} = a - \epsilon, \bar{z} = \frac{A}{a - \epsilon} \right), \end{aligned} \tag{4.4}$$

it is obvious that (4.2) and (4.3) hold. This completes the proof of the lemma. □

LEMMA 4.2. Consider the system of difference equations

$$y_{n+1} = \max \left\{ \frac{A}{z_n}, \frac{A}{z_{n-1}}, \dots, \frac{A}{z_{n-k}} \right\}, \quad z_{n+1} = \max \left\{ \frac{A}{y_n}, \frac{A}{y_{n-1}}, \dots, \frac{A}{y_{n-k}} \right\}, \quad (4.5)$$

where  $A$  is a positive real constant,  $k$  is a positive integer, and  $y_i, z_i, i = -k, -k + 1, \dots, 0$  are positive real numbers. Then every positive solution  $(y_n, z_n)$  of (4.5) is eventually periodic of period  $k + 2$ .

*Proof.* Let  $(y_n, z_n)$  be an arbitrary positive solution of (4.5). Firstly, suppose that there exists a  $\lambda \in \{1, 2, \dots, k + 2\}$  such that

$$y_\lambda z_\lambda < A. \quad (4.6)$$

Then from (4.6) and Lemma 4.1 there exist positive constants  $\bar{y}, \bar{z}$  such that (4.2) holds and

$$y_\lambda < \bar{y}, \quad z_\lambda < \bar{z}. \quad (4.7)$$

From (4.2), (4.5), and (4.7) we have, for  $i = \lambda + 1, \lambda + 2, \dots, k + \lambda + 1$ ,

$$y_i = \max \left\{ \frac{A}{z_{i-1}}, \frac{A}{z_{i-2}}, \dots, \frac{A}{z_{i-k-1}} \right\} \geq \frac{A}{z_\lambda} > \frac{A}{\bar{z}} = \bar{y}, \quad z_i > \bar{z}. \quad (4.8)$$

Then relations (4.2), (4.5), and (4.8) imply that

$$y_{k+\lambda+2} = \max \left\{ \frac{A}{z_{k+\lambda+1}}, \frac{A}{z_{k+\lambda}}, \dots, \frac{A}{z_{\lambda+1}} \right\} < \frac{A}{\bar{z}} = \bar{y}, \quad z_{k+\lambda+2} < \bar{z}. \quad (4.9)$$

Therefore, from (4.2), (4.5), (4.8), and (4.9) we take, for  $j = k + \lambda + 3, k + \lambda + 4, \dots, 2k + \lambda + 3$ ,

$$y_j = \max \left\{ \frac{A}{z_{j-1}}, \frac{A}{z_{j-2}}, \dots, \frac{A}{z_{j-k-1}} \right\} = \frac{A}{z_{k+\lambda+2}}, \quad z_j = \frac{A}{y_{k+\lambda+2}}. \quad (4.10)$$

So, from (4.5), (4.9), (4.10) and working inductively for  $i = 0, 1, \dots$  and  $j = 3, 4, \dots, k + 3$  we can easily prove that

$$\begin{aligned} y_{k+\lambda+2+i(k+2)} &= y_{k+\lambda+2}, & y_{k+\lambda+j+i(k+2)} &= \frac{A}{z_{k+\lambda+2}}, \\ z_{k+\lambda+2+i(k+2)} &= z_{k+\lambda+2}, & z_{k+\lambda+j+i(k+2)} &= \frac{A}{y_{k+\lambda+2}} \end{aligned} \quad (4.11)$$

and so it is obvious that  $(y_n, z_n)$  is eventually periodic of period  $k + 2$ .

Therefore, if relation

$$y_{k+2} z_{k+2} < A \quad (4.12)$$

holds, then  $(y_n, z_n)$  is eventually periodic of period  $k + 2$ .

Now, suppose that relation

$$y_{k+2}z_{k+2} > A \tag{4.13}$$

is satisfied. Then from (4.13) and Lemma 4.1 there exist positive constants  $\bar{y}, \bar{z}$  such that (4.2) holds and

$$y_{k+2} > \bar{y}, \quad z_{k+2} > \bar{z}. \tag{4.14}$$

Moreover, from (4.5) and (4.14) there exist  $\lambda, \mu \in \{1, 2, \dots, k+1\}$  such that

$$y_{k+2} = \max \left\{ \frac{A}{z_{k+1}}, \frac{A}{z_k}, \dots, \frac{A}{z_1} \right\} = \frac{A}{z_\lambda} > \bar{y}, \quad z_{k+2} = \frac{A}{y_\mu} > \bar{z}. \tag{4.15}$$

Hence, from (4.2) and (4.15) it follows that

$$z_\lambda < \bar{z}, \quad y_\mu < \bar{y}. \tag{4.16}$$

We prove that  $\lambda = \mu$ . Suppose on the contrary that  $\lambda \neq \mu$ . Without loss of generality we may suppose that  $1 \leq \mu \leq \lambda - 1$ . Then from (4.2), (4.5), and (4.16) we get

$$z_\lambda = \max \left\{ \frac{A}{y_{\lambda-1}}, \frac{A}{y_{\lambda-2}}, \dots, \frac{A}{y_{\lambda-k-1}} \right\} \geq \frac{A}{y_\mu} > \bar{z} \tag{4.17}$$

which contradicts to (4.16). Hence,  $\lambda = \mu$  and from (4.2) and (4.16) we have

$$y_\lambda z_\lambda < A \tag{4.18}$$

and so  $(y_n, z_n)$  is eventually periodic of period  $k+2$  if (4.13) holds.

Finally, suppose that

$$y_{k+2}z_{k+2} = A. \tag{4.19}$$

From (4.5) it is obvious that

$$y_{k+2} \geq \frac{A}{z_i}, \quad z_{k+2} \geq \frac{A}{y_i}, \quad i = 1, 2, \dots, k+1. \tag{4.20}$$

Therefore, relations (4.5), (4.19), and (4.20) imply that

$$y_{k+3} = \max \left\{ y_{k+2}, \frac{A}{z_{k+1}}, \dots, \frac{A}{z_2} \right\} = y_{k+2}, \quad z_{k+3} = z_{k+2}. \tag{4.21}$$

Hence, using (4.19), (4.20), (4.21) and working inductively we can easily prove that

$$y_{k+i} = y_{k+2}, \quad z_{k+i} = z_{k+2}, \quad i = 3, 4, \dots \tag{4.22}$$

and so it is obvious that  $(y_n, z_n)$  is eventually periodic of period  $k+2$  if (4.19) holds. This completes the proof of the lemma.  $\square$

PROPOSITION 4.3. Consider (1.1) where  $A$  is a positive real constant and  $x_{-k}, x_{-k+1}, \dots, x_0$  are positive fuzzy numbers. Then every positive solution of (1.1) is eventually periodic of period  $k + 2$ .

Proof. Let  $x_n$  be a positive solution of (1.1) with initial values  $x_{-k}, x_{-k+1}, \dots, x_0$  such that (3.2) and (3.4) hold. From Proposition 3.1,  $(L_{n,a}, R_{n,a}), n = 1, 2, \dots, a \in (0, 1]$  satisfies system (3.3). Using Lemma 4.2 we have that

$$L_{n+k+2,a} = L_{n,a}, \quad R_{n+k+2,a} = R_{n,a}, \quad n = 2k + 4, 2k + 5, \dots, a \in (0, 1]. \tag{4.23}$$

Therefore, from (3.4) and (4.23) we have that  $x_n$  is eventually periodic of period  $k + 2$ . This completes the proof of the proposition.  $\square$

Now, we find conditions so that every positive solution of (1.1) neither is bounded nor persists. We need the following lemma.

LEMMA 4.4. Consider the system of difference equations

$$y_{n+1} = \max \left\{ \frac{B}{z_n}, \frac{B}{z_{n-1}}, \dots, \frac{B}{z_{n-k}} \right\}, \quad z_{n+1} = \max \left\{ \frac{C}{y_n}, \frac{C}{y_{n-1}}, \dots, \frac{C}{y_{n-k}} \right\}, \tag{4.24}$$

where  $k$  is a positive integer,  $y_i, z_i, i = -k, -k + 1, \dots, 0$  are positive real numbers, and  $B, C$  are positive real constants such that

$$B < C. \tag{4.25}$$

Then for every positive solution  $(y_n, z_n)$  of (4.24) the following relations hold:

$$\lim_{n \rightarrow \infty} z_n = \infty, \quad \lim_{n \rightarrow \infty} y_n = 0. \tag{4.26}$$

Proof. Since for any  $n \geq 1$  we have

$$\frac{C}{y_n} = \frac{C}{\max \{B/z_{n-1}, B/z_{n-2}, \dots, B/z_{n-k-1}\}} = \lambda \min \{z_{n-1}, z_{n-2}, \dots, z_{n-k-1}\}, \tag{4.27}$$

where  $\lambda = C/B$ , from (4.24) we get

$$z_{n+1} = \max \left\{ \lambda \min \{z_{n-1}, z_{n-2}, \dots, z_{n-k-1}\}, \frac{C}{y_{n-1}}, \dots, \frac{C}{y_{n-k}} \right\} \tag{4.28}$$

and clearly

$$z_{n+1} \geq \lambda \min \{z_{n-1}, z_{n-2}, \dots, z_{n-k-1}\}, \quad n = 1, 2, \dots \tag{4.29}$$

Using (4.29) we can easily prove that

$$z_n \geq \lambda \min \{z_1, z_0, \dots, z_{-k}\}, \quad n = 2, 3, \dots, k + 3, \tag{4.30}$$

and so

$$z_n \geq \lambda^2 \min \{z_1, z_0, \dots, z_{-k}\}, \quad n = k + 4, k + 5, \dots, 2k + 5. \quad (4.31)$$

From (4.31) and working inductively we get, for  $r = 3, 4, \dots$ ,

$$z_n \geq \lambda^r \min \{z_1, z_0, \dots, z_{-k}\}, \quad n = (r - 1)k + 2r, (r - 1)k + 2r + 1, \dots, r(k + 2) + 1. \quad (4.32)$$

Obviously, from (4.25) and (4.32) we have that

$$\lim_{n \rightarrow \infty} z_n = \infty. \quad (4.33)$$

Hence, relations (4.24) and (4.33) imply that

$$\lim_{n \rightarrow \infty} y_n = 0 \quad (4.34)$$

and so from (4.33) and (4.34) we have that relations (4.26) are true. This completes the proof of the lemma.  $\square$

**PROPOSITION 4.5.** *Consider (1.1) where  $k$  is a positive integer,  $A$  is a nontrivial positive fuzzy number, and  $x_{-k}, x_{-k+1}, \dots, x_0$  are positive fuzzy numbers. Then every positive solution of (1.1) is unbounded and does not persist.*

*Proof.* Let  $x_n$  be a positive solution of (1.1) with initial values  $x_{-k}, x_{-k+1}, \dots, x_0$  such that (3.2) and (3.4) hold. Since  $A$  is a nontrivial positive fuzzy number there exists an  $\bar{a} \in (0, 1]$  such that

$$A_{l, \bar{a}} < A_{r, \bar{a}}. \quad (4.35)$$

Moreover, since (4.35) holds and  $(L_{n,a}, R_{n,a})$ ,  $a \in (0, 1]$  satisfies system (3.3), then from Lemma 4.4 we have that

$$\lim_{n \rightarrow \infty} R_{n, \bar{a}} = \infty, \quad \lim_{n \rightarrow \infty} L_{n, \bar{a}} = 0. \quad (4.36)$$

Therefore, from (4.36) there are no positive numbers  $M, N$  such that  $\bigcup_{a \in (0, 1]} [L_{n,a}, R_{n,a}] \subset [M, N]$ . This completes the proof of the proposition.  $\square$

From Propositions 4.3 and 4.5 the following corollary results.

**COROLLARY 4.6.** *Consider the fuzzy difference equation (1.1) where  $A$  is a positive fuzzy number. Then the following statements are true.*

(i) *Every positive solution of (1.1) is eventually periodic of period  $k + 2$  if and only if  $A$  is a trivial fuzzy number.*

(ii) *Every positive solution of (1.1) neither is bounded nor persists if and only if  $A$  is a nontrivial fuzzy number.*



**5. Behavior of the positive solutions of fuzzy equation (1.2)**

Firstly, we study the periodicity of the positive solutions of (1.2). We need the following lemma.

LEMMA 5.1. *Consider the system of difference equations*

$$y_{n+1} = \max \left\{ \frac{B}{z_n}, \frac{D}{z_{n-1}} \right\}, \quad z_{n+1} = \max \left\{ \frac{C}{y_n}, \frac{E}{y_{n-1}} \right\}, \tag{5.1}$$

where  $B, D, C, E$  are positive real constants and the initial values  $y_{-1}, y_0, z_{-1}, z_0$  are positive real numbers. Then the following statements are true.

(i) If

$$B = C, \quad B \geq E \geq D, \quad B, D, C, E \text{ are not all equal}, \tag{5.2}$$

then every positive solution of system (5.1) is eventually periodic of period two.

(ii) If

$$D = E, \quad D \geq C \geq B, \quad B, D, C, E \text{ are not all equal}, \tag{5.3}$$

then every positive solution of system (5.1) is eventually periodic of period four.

*Proof.* We give a sketch of the proof (for more details see the appendix). Let  $(y_n, z_n)$  be a positive solution of (5.1).

(i) Firstly, we prove that if there exists an  $m \in \{1, 2, \dots\}$  such that

$$E \leq y_m z_m \leq \frac{B^2}{E}, \tag{5.4}$$

then  $(y_n, z_n)$  is eventually periodic of period two.

Moreover, we prove that if for an  $m \in \{1, 2\}$  relation (5.4) does not hold, then there exists a  $w \in \{1, 2, 3\}$  such that

$$u_w = y_w z_w < E. \tag{5.5}$$

In addition, we prove that if

$$D \leq u_w < E, \tag{5.6}$$

then  $u_m$  for  $m = w + 2$  satisfies relation (5.4) which implies that  $(y_n, z_n)$  is eventually periodic of period two.

Finally, if

$$u_w < D, \tag{5.7}$$

then we prove that there exists an  $r \in \{0, 1, \dots\}$  such that

$$\left( \frac{DE}{B^2} \right)^{r+1} \leq \frac{u_w}{D} \leq \left( \frac{DE}{B^2} \right)^r \tag{5.8}$$

and  $u_m$  for  $m = w + 3r + 3$  satisfies relation (5.4) or (5.6) and so  $(y_n, z_n)$  is eventually periodic of period two.

(ii) Firstly, we prove that if there exists an  $m \in \{1, 2, \dots\}$  such that

$$\frac{C^2}{D} \leq y_m z_m \leq D, \tag{5.9}$$

then  $(y_n, z_n)$  is eventually periodic of period four.

In addition, we prove that if relation (5.9) does not hold for  $m \in \{1, 2, 3\}$  then there exists a  $p \in \{1, 2, 3, 4\}$  such that

$$u_p = y_p z_p < \frac{C^2}{D}. \tag{5.10}$$

Furthermore, if

$$\frac{B^2}{D} \leq u_p < \frac{C^2}{D}, \tag{5.11}$$

we prove that (5.9) holds for  $m = p + 4$  or  $m = p + 5$ . Therefore, the solution  $(y_n, z_n)$  is eventually periodic of period four.

Finally, if

$$u_p < \frac{B^2}{D}, \tag{5.12}$$

then we prove that there exists a  $q \in \{0, 1, \dots\}$  such that

$$\left(\frac{BC}{D^2}\right)^{q+1} \leq \frac{u_p D}{B^2} \leq \left(\frac{BC}{D^2}\right)^q \tag{5.13}$$

and either (5.9) or (5.11) holds for  $m = p + 3q + 3$  and so  $(y_n, z_n)$  is eventually periodic of period four.  $\square$

**PROPOSITION 5.2.** *Consider the fuzzy difference equation (1.2) where  $A_i$ ,  $i = 0, 1$  are nonequal positive fuzzy numbers such that (3.6) holds and the initial values  $x_i$ ,  $i = -1, 0$  are positive fuzzy numbers. Then the following statements are true.*

(i) *If  $A_0$  is a positive trivial fuzzy number such that*

$$A_{0,l,a} = A_{0,r,a} = A_0, \quad a \in (0, 1], \max\{A_0 - \epsilon, A_1\} = A_0 - \epsilon, \tag{5.14}$$

*where  $\epsilon$  is a real constant,  $0 < \epsilon < A_0$ , then every positive solution of (1.2) is eventually periodic of period two.*

(ii) *If  $A_1$  is a positive trivial fuzzy number such that*

$$A_{1,l,a} = A_{1,r,a} = A_1, \quad a \in (0, 1], \max\{A_0, A_1 - \epsilon\} = A_1 - \epsilon, \tag{5.15}$$

*where  $\epsilon$  is a real constant,  $0 < \epsilon < A_1$ , then every positive solution of (1.2) is eventually periodic of period four.*

*Proof.* Let  $x_n$  be a positive solution of (1.2) with initial values  $x_i, i = -1, 0$  such that relations (3.2) for  $k = 1$  and (3.4) hold, then  $(L_{n,a}, R_{n,a}), n = 1, 2, \dots, a \in (0, 1]$  satisfies system (3.7).

(i) Firstly, suppose that (5.14) is satisfied. We define the set  $E \subset (0, 1]$  as follows: for any  $a \in E$  there exists an  $m_a \in \{1, 2\}$  such that

$$A_{1,l,a} \leq u_{m_a,a} \leq \frac{A_0^2}{A_{1,r,a}}, \quad u_{n,a} = L_{n,a}R_{n,a}, \quad n = 1, 2, \dots, a \in E. \tag{5.16}$$

Then from statement (i) of Lemma 5.1 the sequences  $L_{n,a}, R_{n,a}, a \in E$  are periodic sequences of period two for  $n \geq 5$ . Moreover, since for any  $a \in (0, 1] - E$  the relation (5.16) does not hold, then from statement (i) of Lemma 5.1 for any  $a \in (0, 1] - E$  there exists a  $w_a \in \{1, 2, 3\}$  and an  $r_a \in \{0, 1, \dots\}$  such that

$$u_{w_a,a} < A_{1,l,a}, \quad \left( \frac{A_{1,l,a}A_{1,r,a}}{A_0^2} \right)^{r_a+1} \leq \frac{u_{w_a,a}}{A_{1,l,a}} \leq \left( \frac{A_{1,l,a}A_{1,r,a}}{A_0^2} \right)^{r_a}. \tag{5.17}$$

Hence, from statement (i) of Lemma 5.1,  $L_{n,a}, R_{n,a}, a \in (0, 1] - E$  are periodic sequences of period two for  $n \geq w_a + 3r_a + 3$  and so for  $n \geq 3r_a + 6$ .

We prove that there exists an  $r \in \{1, 2, \dots\}$  such that

$$r \geq r_a, \quad a \in (0, 1] - E. \tag{5.18}$$

Since  $x_i, i = 1, 2, 3$  are positive fuzzy numbers there exist positive real numbers  $K, L$  such that  $[L_{i,a}, R_{i,a}] \subset [K, L], i = 1, 2, 3, a \in (0, 1] - E$ . Then from (5.14) and (5.17) there exists an  $r \in \{1, 2, \dots\}$  such that, for  $a \in (0, 1] - E$ ,

$$\left( \frac{A_{1,l,a}A_{1,r,a}}{A_0^2} \right)^r \leq \left( \frac{A_0 - \epsilon}{A_0} \right)^{2r} \leq \frac{K^2}{A_0 - \epsilon} \leq \frac{u_{w_a,a}}{A_{1,l,a}} \leq \left( \frac{A_{1,l,a}A_{1,r,a}}{A_0^2} \right)^{r_a} \tag{5.19}$$

and so from (5.14) relation (5.18) is satisfied. Therefore, from (5.18) it follows that  $L_{n,a}, R_{n,a}, a \in (0, 1] - E$  are periodic sequences of period two for  $n \geq 3r + 6$  and so  $x_n$  is eventually periodic of period two.

Arguing as above and using statement (ii) of Lemma 5.1 we can easily prove that every positive solution of (1.2) is eventually periodic of period four if relation (5.15) holds. This completes the proof of the proposition. □

In the last proposition of this paper we find conditions so that every positive solution of (1.2) neither is bounded nor persists. We need the following lemma.

LEMMA 5.3. Consider system (5.1) where  $B, D, C, E$  are positive real constants,  $z_{-1}, z_0, y_{-1}, y_0$  are positive real numbers. If one of the following statements:

- (i)  $B < C, D < E,$
- (ii)  $B < C, D < C,$
- (iii)  $D < E, B < E,$

is satisfied, then for every positive solution  $(y_n, z_n)$  of (5.1) relations (4.26) hold.

*Proof.* Firstly, suppose that conditions (i) of Lemma 5.3 are satisfied then we have that either

$$C > D \tag{5.20}$$

or

$$E > B \tag{5.21}$$

holds. Suppose that (5.20) holds. From (5.1) it is obvious that for  $n = 1, 2, \dots$ ,

$$\frac{C}{y_n} = \frac{C}{\max\{B/z_{n-1}, D/z_{n-2}\}} \geq \lambda \min\{z_{n-1}, z_{n-2}\}, \quad \lambda = \min\left\{\frac{C}{B}, \frac{C}{D}\right\}. \tag{5.22}$$

Hence, from (5.1), (5.22) it follows that relation (4.29) holds for  $k = 1$ . Then arguing as in Lemma 4.4 we can prove relations (4.26).

Now, consider that relation (5.21) holds. From (5.1) it is obvious that for  $n = 2, 3, \dots$ ,

$$\frac{E}{y_{n-1}} = \frac{E}{\max\{B/z_{n-2}, D/z_{n-3}\}} \geq \mu \min\{z_{n-2}, z_{n-3}\}, \quad \mu = \min\left\{\frac{E}{B}, \frac{E}{D}\right\}, \tag{5.23}$$

then from (5.1), (5.23) it follows that

$$z_{n+1} \geq \mu \min\{z_{n-2}, z_{n-3}\}, \quad n = 2, 3, \dots \tag{5.24}$$

In view of (5.24) and using the same argument to prove (4.32) we get for  $r = 1, 2, \dots$ ,

$$z_n \geq \mu^r \min\{z_2, z_1, z_0, z_{-1}\}, \quad n = 4r - 1, 4r, 4r + 1, 4r + 2. \tag{5.25}$$

Thus, from (5.25) it is obvious that relations (4.26) are satisfied.

Now, suppose that relations (ii) (resp., (iii)) of Lemma 5.3 hold. Then relation (4.29) for  $k = 1$  (resp., (5.25)) holds which implies that (4.26) is true. This completes the proof of the lemma.  $\square$

**PROPOSITION 5.4.** *Consider the fuzzy difference equation (1.2) where  $A_i, i = 0, 1$  are positive fuzzy numbers such that (3.6) holds and the initial values  $x_i, i = -1, 0$  are positive fuzzy numbers. If there exists an  $a \in (0, 1]$  which satisfies one of the the following conditions:*

- (i)  $A_{0,l,a} < A_{0,r,a}, A_{1,l,a} < A_{1,r,a}$ ,
- (ii)  $A_{0,l,a} < A_{0,r,a}, A_{1,l,a} < A_{0,r,a}$ ,
- (iii)  $A_{0,l,a} < A_{1,r,a}, A_{1,l,a} < A_{1,r,a}$ ,

*then the solution  $x_n$  of (1.2) neither is bounded nor persists.*

*Proof.* Let  $x_n$  be a positive solution of (1.2) with initial values  $x_{-1}, x_0$  such that relations (3.2) for  $k = 1$  and (3.4) hold. Since there exists an  $a \in (0, 1]$  such that one of the relations (i), (ii), (iii) of Proposition 5.4 holds and  $(L_{n,a}, R_{n,a}), a \in (0, 1]$  satisfies (3.7) then from Lemma 5.3 and arguing as in Proposition 4.5 we can easily prove that the solution  $x_n$  of (1.2) neither is bounded nor persists. This completes the proof of the proposition.  $\square$

**Appendix**

*Proof of Lemma 5.1.* Let  $(y_n, z_n)$  be a positive solution of (5.1).

(i) Firstly, we prove that if there exists an  $m \in \{1, 2, \dots\}$  such that (5.4) holds, then  $(y_n, z_n)$  is eventually periodic of period two. Relations (5.1) and (5.2) imply that

$$z_n y_{n-1} \geq B, \quad y_n z_{n-1} \geq B, \quad n = 1, 2, \dots \tag{A.1}$$

From (5.2), (5.4), and (A.1) we get

$$\frac{D}{z_{m-1}} \leq \frac{D}{B} y_m \leq \frac{B}{z_m}, \quad \frac{E}{y_{m-1}} \leq \frac{E}{B} z_m \leq \frac{B}{y_m}. \tag{A.2}$$

Using (5.1), (5.2), and (A.2) it follows that

$$y_{m+1} = \max \left\{ \frac{B}{z_m}, \frac{D}{z_{m-1}} \right\} = \frac{B}{z_m}, \quad z_{m+1} = \frac{B}{y_m}. \tag{A.3}$$

From (5.1), (5.2), (5.4), and (A.3) we can easily prove that

$$\begin{aligned} y_{m+2} &= \max \left\{ y_m, \frac{D}{z_m} \right\} = y_m, & z_{m+2} &= z_m, \\ y_{m+3} &= \max \left\{ \frac{B}{z_m}, \frac{D}{B} y_m \right\} = \frac{B}{z_m} = y_{m+1}, & z_{m+3} &= z_{m+1}. \end{aligned} \tag{A.4}$$

Therefore, using (5.1), (A.4) and working inductively we can easily prove that

$$y_{n+2} = y_n, \quad z_{n+2} = z_n, \quad n = m + 2, m + 3, \dots \tag{A.5}$$

and so  $(y_n, z_n)$  is eventually periodic of period two.

Now, we prove that there exists an  $m \in \{1, 2, \dots\}$  such that (5.4) holds. If there exists an  $m \in \{1, 2\}$  such that (5.4) is satisfied, then the proof is completed. Now, suppose that for any  $m \in \{1, 2\}$  relation (5.4) is not true. We claim that there exists a  $w \in \{1, 2, 3\}$  such that (5.5) holds. If for  $w = 1, 2$  relation (5.5) does not hold, then from (5.2) and since (5.4) is not true for  $m = 1, 2$  we have

$$u_w > \frac{B^2}{E} > E, \quad w = 1, 2. \tag{A.6}$$

Hence, from (5.1), (5.2), (A.1), and (A.6) we get

$$y_3 z_3 = \max \left\{ \frac{B^2}{y_2 z_2}, \frac{BE}{y_1 z_2}, \frac{DB}{y_2 z_1}, \frac{DE}{y_1 z_1} \right\} < E \tag{A.7}$$

and so our claim is true.

Then since from (A.1) and (5.5), relations (A.2) for  $m = w$  hold, from (5.1) and (5.2) we have that relations (A.3) for  $m = w$  are true. Using (5.1), (5.2), (5.5), and (A.3) for  $m = w$  we can easily prove that

$$u_{w+2} = \max \left\{ u_w, \frac{BE}{y_w z_{w+1}}, \frac{DB}{y_{w+1} z_w}, \frac{DE}{u_w} \right\} = \max \left\{ E, \frac{DE}{u_w} \right\}. \tag{A.8}$$

Since (5.5) holds we have that either (5.6) or (5.7) is satisfied.

Firstly, suppose that (5.6) holds then from (A.8) we get  $u_{w+2} = E$  and so relation (5.4) is satisfied for  $m = w + 2$ , which means that  $(y_n, z_n)$  is eventually periodic of period two.

Now, suppose that (5.7) holds. From (5.2) and (5.7) there exists an  $r \in \{0, 1, \dots\}$  such that (5.8) holds. Now, we prove that, for all  $s = 0, 1, \dots, r + 1$ ,

$$y_{w+3s} = \frac{y_w B^s}{E^s}, \quad z_{w+3s} = \frac{z_w B^s}{D^s}, \quad y_{w+3s+1} = \frac{D^s}{z_w B^{s-1}}, \quad z_{w+3s+1} = \frac{E^s}{y_w B^{s-1}}. \quad (\text{A.9})$$

Relations (A.3) for  $m = w$  imply that (A.9) is true for  $s = 0$ . Suppose that (A.9) is true for an  $s = j \in \{0, 1, \dots, r\}$ . Then from (5.1), (5.2), (5.8), (A.9) we have

$$\begin{aligned} y_{w+3j+2} &= \max \left\{ \frac{B^j y_w}{E^j}, \frac{D^{j+1}}{z_w B^j} \right\} = \frac{D^{j+1}}{z_w B^j}, \\ z_{w+3j+2} &= \max \left\{ \frac{B^j z_w}{D^j}, \frac{E^{j+1}}{y_w B^j} \right\} = \frac{E^{j+1}}{y_w B^j}. \end{aligned} \quad (\text{A.10})$$

Moreover, using (5.1), (5.2), (5.8), (A.9), and (A.10) it follows that

$$y_{w+3j+3} = \frac{B^{j+1} y_w}{E^{j+1}}, \quad z_{w+3j+3} = \frac{B^{j+1} z_w}{D^{j+1}}, \quad y_{w+3j+4} = \frac{D^{j+1}}{B^j z_w}, \quad z_{w+3j+4} = \frac{E^{j+1}}{B^j y_w}. \quad (\text{A.11})$$

From relations (5.8) and (A.9) for  $j = r + 1$  we take that (A.9) is true for  $s = 0, 1, \dots, r + 1$ . Finally, from relations (A.11) it follows that

$$D \leq u_{w+3r+3} \leq \frac{B^2}{E} \quad (\text{A.12})$$

which means that either (5.4) or (5.6) holds for  $m = w + 3r + 3$ . Therefore,  $(y_n, z_n)$  is eventually periodic of period two. This completes the proof of statement (i).

(ii) Firstly, we prove that if there exists an  $m \in \{1, 2, \dots\}$  such that (5.9) holds, then  $(y_n, z_n)$  is eventually periodic of period four. Relations (5.1), (5.3) imply that

$$z_n y_{n-1} \geq C, \quad y_n z_{n-1} \geq B, \quad z_n y_{n-2} \geq D, \quad n = 1, 2, \dots \quad (\text{A.13})$$

Then from (5.1), (5.3), (5.9), and (A.13) we can easily prove that

$$y_{m+1} = \max \left\{ \frac{B}{z_m}, \frac{D}{z_{m-1}} \right\} \leq \frac{D}{B} y_m, \quad z_{m+1} \leq \frac{D}{C} z_m. \quad (\text{A.14})$$

In addition, from (5.1), (5.3), (5.9), and (A.13), we get

$$\frac{B}{z_{m+1}} \leq \frac{B}{C} y_m \leq \frac{BD}{C} \frac{1}{z_m} \leq \frac{D}{z_m} \quad (\text{A.15})$$

and so from (5.1) we have

$$y_{m+2} = \max \left\{ \frac{B}{z_{m+1}}, \frac{D}{z_m} \right\} = \frac{D}{z_m}. \quad (\text{A.16})$$

In what follows, we consider the following four cases:

- (A1)  $y_m z_m \leq BD/C$ ,
- (A2)  $y_m z_{m-1} \leq D^2/C, z_m/z_{m-1} \leq D/C$ ,
- (A3)  $y_m z_{m-1} \leq D^2/C, z_m/z_{m-1} > D/C$ ,
- (A4)  $y_m z_m > BD/C, y_m z_{m-1} > D^2/C$ .

Suppose that (A1) or (A2) is satisfied, then from (5.1) it is obvious that

$$\frac{C}{D}y_m \leq \max \left\{ \frac{B}{z_m}, \frac{D}{z_{m-1}} \right\} = y_{m+1} \tag{A.17}$$

which implies that

$$z_{m+2} = \max \left\{ \frac{C}{y_{m+1}}, \frac{D}{y_m} \right\} = \frac{D}{y_m}. \tag{A.18}$$

Also, since relations (5.3), (5.9), and (A.14) imply that

$$\frac{B}{D}y_m \leq \frac{B}{z_m} \leq \frac{BD}{C} \frac{1}{z_{m+1}} \leq \frac{D}{z_{m+1}}, \tag{A.19}$$

then from (5.1), (A.18), and (A.19) we have

$$y_{m+3} = \max \left\{ \frac{B}{D}y_m, \frac{D}{z_{m+1}} \right\} = \frac{D}{z_{m+1}}. \tag{A.20}$$

In addition, if  $z_m/z_{m-1} \leq D/C$ , then from (5.1), (5.3) we can easily prove that

$$z_m y_{m+1} = \max \left\{ B, D \frac{z_m}{z_{m-1}} \right\} \leq \frac{D^2}{C}. \tag{A.21}$$

Moreover, if (A1) is true then from (A.13), we get that  $z_m/z_{m-1} = z_m y_m / y_m z_{m-1} \leq D/C$  and so if (A1) or (A2) is satisfied, then from (5.1), (5.3), (A.16), and (A.21) we take

$$z_{m+3} = \max \left\{ \frac{C}{D}z_m, \frac{D}{y_{m+1}} \right\} = \frac{D}{y_{m+1}}. \tag{A.22}$$

According to relations (5.1), (5.3), (A.13), (A.14), (A.16), (A.18), (A.20), and (A.22) it is easy to prove that

$$y_{m+4} = y_m, \quad z_{m+4} = z_m, \quad y_{m+5} = y_{m+1}, \quad z_{m+5} = z_{m+1}. \tag{A.23}$$

Therefore, using (5.1), (5.3), (A.23) and working inductively we can easily prove that for  $n = m + 2, m + 3, \dots$  the following relations hold:

$$y_{n+4} = y_n, \quad z_{n+4} = z_n, \tag{A.24}$$

which means that  $(y_n, z_n)$  is eventually periodic of period four.

Now, suppose that condition (A3) holds then obviously, relations (A.18) and (A.20) are satisfied. From (5.1), (5.3) and arguing as in (A.21) we have that  $z_m y_{m+1} > D^2/C$  and

so from (5.1), (5.3), and (A.16) it follows that

$$z_{m+3} = \frac{C}{y_{m+2}} = \frac{C}{D}z_m. \tag{A.25}$$

Since from (A.13) and condition (A3) we get  $(C/D)z_m y_m > z_{m-1} y_m \geq B$  then from (5.1), (5.3), (5.9), (A.13), (A.14), (A.16), (A.18), (A.20), and (A.25) we can prove that

$$\begin{aligned} y_{m+4} = y_m, \quad z_{m+4} = z_m, \quad y_{m+5} = \frac{D^2}{Cz_m}, \quad z_{m+5} = z_{m+1}, \\ y_{m+6} = y_{m+2}, \quad z_{m+6} = \frac{D}{y_m} = z_{m+2}, \quad y_{m+7} = y_{m+3}, \quad z_{m+7} = \frac{C}{D}z_m = z_{m+3}. \end{aligned} \tag{A.26}$$

Using (5.1), (5.3), (A.26) and working inductively we can easily prove that (A.24) is true for  $n = m + 4, m + 5, \dots$  and so  $(y_n, z_n)$  is eventually periodic of period four.

Finally, consider that condition (A4) is satisfied. From (5.1), (5.3), (5.9), (A.14) and condition (A4) we get

$$y_{m+1} < \frac{C}{D}y_m, \quad y_{m+1}z_m < \frac{C}{D}y_mz_m \leq C \leq \frac{D^2}{C}, \quad z_{m+1}y_{m+1} < y_mz_m \leq D. \tag{A.27}$$

Then, in view of (5.1), (5.3), (A.13), (A.14), (A.16), and (A.27) we have

$$\begin{aligned} z_{m+2} = \frac{C}{y_{m+1}}, \quad y_{m+3} = \frac{D}{z_{m+1}}, \quad z_{m+3} = \frac{D}{y_{m+1}}, \quad y_{m+4} = \frac{Dy_{m+1}}{C}, \\ z_{m+4} = z_m, \quad y_{m+5} = y_{m+1}, \quad z_{m+5} = \max \left\{ \frac{C^2}{Dy_{m+1}}, z_{m+1} \right\}. \end{aligned} \tag{A.28}$$

If  $z_{m+1}y_{m+1} > C^2/D$ , then from (5.1), (5.3), and (A.28) we get

$$z_{m+5} = z_{m+1}, \quad y_{m+6} = y_{m+2}, \quad z_{m+6} = \frac{C}{y_{m+1}} = z_{m+2}. \tag{A.29}$$

Then from relations (5.1), (5.3), (A.28), (A.29) and working inductively we take relations (A.24) for  $n = m + 3, m + 4, \dots$  and so  $y_n, z_n$  is eventually periodic of period four.

Finally, if  $z_{m+1}y_{m+1} \leq C^2/D$  from the last relation of (A.28) we get

$$z_{m+5} = \frac{C^2}{D} \frac{1}{y_{m+1}}. \tag{A.30}$$

Then from (A.27), (A.28), and (A.30) we get

$$y_{m+5}z_{m+5} = \frac{C^2}{D}, \quad y_{m+5}z_{m+4} \leq \frac{D^2}{C}. \tag{A.31}$$

Therefore, from (A.31) it is obvious that relations (5.9) and (A2) or (A3) for  $m = m + 5$  are satisfied and so  $(y_n, z_n)$  is eventually periodic of period four.

Now, we prove that there exists an  $m \in \{1, 2, \dots\}$  such that (5.9) holds. If there exists an  $m \in \{1, 2, 3, 4\}$  such that (5.9) is satisfied, then the proof is completed. Now, suppose that



for any  $m \in \{1, 2, 3, 4\}$  relation (5.9) is not true. We claim that there exists a  $p \in \{1, 2, 3, 4\}$  such that relation (5.10) holds. If for  $p = 1, 2$  relation (5.10) does not hold and since (5.9) is not true for  $m = 1, 2$ , we have

$$u_1, u_2 > D. \tag{A.32}$$

Firstly, suppose that

$$z_1 y_2 > D. \tag{A.33}$$

Then since from (5.1) and (5.3) it follows that

$$u_{n+1} = \max \left\{ \frac{BC}{u_n}, \frac{BD}{z_n y_{n-1}}, \frac{CD}{y_n z_{n-1}}, \frac{D^2}{u_{n-1}} \right\}, \quad u_n = y_n z_n, \tag{A.34}$$

using relations (5.3), (A.13), (A.32), (A.33), (A.34) we have

$$u_3 < \max \left\{ \frac{BC}{D}, \frac{BD}{C}, C, D \right\} = D \tag{A.35}$$

and since (5.9) does not hold for  $m = 3$ , we get that (5.10) is true for  $p = 3$ .

Now, suppose that

$$z_1 y_2 \leq D, \quad u_3 \geq D. \tag{A.36}$$

Relations (5.1), (5.3), (A.32), and (A.36) imply that

$$y_3 z_2 = \max \left\{ B, \frac{Dz_2 y_2}{z_1 y_2} \right\} > D. \tag{A.37}$$

Then from (5.3), (A.13), (A.32), (A.34), (A.36), and (A.37) we can prove that

$$u_4 < \max \left\{ \frac{BC}{D}, \frac{BD}{C}, C, D \right\} = D \tag{A.38}$$

and so (5.10) is true for  $p = 4$ . Thus, our claim is true.

In view of (5.10) and (A.13) it follows that

$$\frac{D}{y_{p-1}} \leq \frac{D}{C} z_p < \frac{C}{y_p} \tag{A.39}$$

and so from (5.1) and (5.3) we get that

$$z_{p+1} = \frac{C}{y_p}. \tag{A.40}$$

Since (5.10) holds we have that either (5.11) or (5.12) is satisfied.

Firstly, suppose that (5.11) holds then using (5.1), (A.13) and arguing as in (A.14) we get

$$y_{p+1} \leq \frac{D}{B} y_p. \tag{A.41}$$

From (5.1), (5.3), (5.11), and (A.40) we get

$$y_{p+2} = \frac{D}{z_p}, \quad z_{p+2} = \max \left\{ \frac{C}{y_{p+1}}, \frac{D}{y_p} \right\}. \quad (\text{A.42})$$

Firstly, suppose

$$y_{p+1} < \frac{C}{D} y_p, \quad (\text{A.43})$$

then from (A.42) we have

$$z_{p+2} = \frac{C}{y_{p+1}}. \quad (\text{A.44})$$

Using (5.3), (5.11), (A.43) it follows that

$$\frac{C}{D} z_p < \frac{C^3}{D^2} \frac{1}{y_p} < \frac{C^4}{D^3} \frac{1}{y_{p+1}} \leq \frac{D}{y_{p+1}}, \quad \frac{BD}{C^2} y_p < \frac{B}{z_p} \leq y_{p+1} \quad (\text{A.45})$$

and so from relations (5.1), (5.3), (5.11), (A.40), (A.41), (A.42), (A.43), and (A.44) we get

$$y_{p+3} = \frac{Dy_p}{C}, \quad z_{p+3} = \frac{D}{y_{p+1}}, \quad y_{p+4} = \frac{Dy_{p+1}}{C}, \quad z_{p+4} = \frac{C^2}{Dy_p}, \quad (\text{A.46})$$

$$y_{p+5} = y_{p+1}, \quad z_{p+5} = \frac{C^2}{Dy_{p+1}}. \quad (\text{A.47})$$

From (A.47) clearly,

$$y_{p+5} z_{p+5} = \frac{C^2}{D} \quad (\text{A.48})$$

and so relation (5.9) holds for  $m = p + 5$  which means that  $(y_n, z_n)$  is eventually periodic of period four.

Now, suppose that

$$y_{p+1} \geq \frac{C}{D} y_p. \quad (\text{A.49})$$

Then (5.1), (5.3), and (A.49) imply that

$$z_{p+2} = \frac{D}{y_p}. \quad (\text{A.50})$$

Since from (A.13) and (A.41) it results that  $B/z_{p+3} \leq (B/D)y_{p+1} \leq y_p$  then using (5.1), (5.3), (5.11), (A.40), (A.42), and (A.50) we get

$$y_{p+3} = \frac{Dy_p}{C}, \quad z_{p+3} \geq \frac{D}{y_{p+1}}, \quad y_{p+4} = y_p, \quad z_{p+4} = \frac{C^2}{Dy_p}. \quad (\text{A.51})$$

From (A.51) we get

$$y_{p+4}z_{p+4} = \frac{C^2}{D} \quad (\text{A.52})$$

and so relation (5.9) holds for  $m = p + 4$ , which means that  $(y_n, z_n)$  is eventually periodic of period four.

Finally, suppose that relation (5.12) holds. From (5.3) and (5.12) there exists a  $q \in \{0, 1, \dots\}$  such that relation (5.13) holds. Using the same argument to prove (A.9), we can prove that, for all  $s = 0, 1, \dots, q + 1$ ,

$$y_{p+3s} = \frac{y_p D^s}{C^s}, \quad z_{p+3s} = \frac{z_p D^s}{B^s}, \quad y_{p+3s+1} = \frac{B^{s+1}}{z_p D^s}, \quad z_{p+3s+1} = \frac{C^{s+1}}{y_p D^s}. \quad (\text{A.53})$$

From (5.3), (5.13), and (A.53) for  $s = q + 1$  it easily results that

$$\frac{B^2}{D} \leq u_{p+3q+3} \leq \frac{BD}{C} \leq D \quad (\text{A.54})$$

and so we have that either (5.9) or (5.11) is satisfied for  $m = p + 3q + 3$ , which means that  $(y_n, z_n)$  is eventually periodic of period four. Thus, the proof of the lemma is completed.  $\square$

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