ON COINCIDENCE INDEX FOR MULTIVALUED PERTURBATIONS OF NONLINEAR FREDHOLM MAPS AND SOME APPLICATIONS

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We define a nonoriented coincidence index for a compact, fundamentally restrictible, and condensing multivalued perturbations of a map which is nonlinear Fredholm of nonnegative index on the set of coincidence points. As an application, we consider an optimal controllability problem for a system governed by a second-order integro-differential equation.

1. Introduction

One of the most efficient methods for the study of boundary and periodic problems for nonlinear differential equations and inclusions, consists in the operator treatment of these problems in suitable functional spaces.

However, for a number of problems of this sort, the maps constructed in functional spaces do not possess "nice" properties on the whole domain, but only on some open neighborhood of the solutions set. As an example, we may note a Monge-Ampere problem arising in geometry of surfaces (see [10]). Moreover, the application of topological methods to the investigation of this kind of problems often requires the embedding of a given equation or inclusion into a corresponding parametric family. In such a situation, either solutions sets or their neighborhoods can vary in dependence of parameters.

In the present paper, we want to study an inclusion of the form

$$f(x) \in G(x), \tag{1.1}$$

where $f : Y \subseteq E \to E'$ is a map, $G : Y \subseteq E \multimap E'$ is a multivalued map (multimap), and E, E' are real Banach spaces.

In other words, we have to deal with the coincidence points of the triplet (f, G, Y). We assume that the set Q of the coincidence points of (f, G, Y) is compact, and that f is differentiable on some neighborhood of Q and is nonlinear Fredholm on Q.

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We suppose also that the multimap G is closed with convex, compact values and that the triplet (f, G, Y) satisfies some conditions of compact restrictibility on some neighborhood of Q; in particular, G may be compact or f-condensing with respect to a certain measure of noncompactness. For the case when G is a single-valued, completely continuous map, we refer the reader to [2]. For a pair of this type, a nonoriented index of solutions was defined and studied in [10]. For a single-valued and f-compactly restrictible map G, the nonoriented index of solutions was defined in [11].

In the case when G is a completely continuous multimap of acyclic type, f is a nonlinear Fredholm map of zero index, and the domain Y possesses the property of orientability of every Fredholm structure on it, the oriented coincidence index was constructed in [1]. Note that the situation considered in the present paper is essentially different since the orientability of Fredholm structures on arbitrary neighborhoods of the coincidence points set cannot be guaranteed in advance. We point out that for the case when f is a linear Fredholm operator, some topological characteristics of the couple (f, G) were studied in [3, 6, 8].

In the present paper we first define a relative coincidence index with respect to a convex, closed set K for triplets $(f, g, \bar{U})_K$ and $(f, G, \bar{U})_K$ where g and G are, respectively, single-valued and multivalued compact perturbations of a nonlinear Fredholm map f, and U is an open neighborhood of Q.

Using these results as a base, we define a nonoriented coincidence index $\operatorname{Ind}(f, G, U)$ for a triplet which is fundamentally restrictible on certain neighborhood of Q. We pay special attention to the important particular cases when (f, G, U) form a condensing or locally condensing triplet. We also describe the main properties of the constructed topological characteristic. As an application we consider an optimal controllability problem for a system governed by a second-order integro-differential equation.

2. Preliminaries

In the sequel E, E' denote real Banach spaces. Everywhere by Y we denote an open set: $U \subset E$ (case (i)) or $U_* \subset E \times [0,1]$ (case (ii)). We recall some notions.

Definition 2.1. A C^1-map $f : Y \to E'$ is *Fredholm of index* $n \geq 0$ on a set $S \subseteq Y$ ($f \in \Phi_n C^1(S)$) if for every $x \in S$ the Frechet derivative $f'(x)$ is a linear Fredholm map of index n, that is, $\dim \operatorname{Ker} f'(x) < +\infty$, $\dim \operatorname{Coker} f'(x) < +\infty$ and

$$\dim \ker f'(x) - \dim \operatorname{Coker} f'(x) = n. \tag{2.1}$$

For simplicity we will denote maps and their restrictions by the same symbols.

Definition 2.2. A map $f : Y \to E'$ is *proper* on a closed set $S \subset Y$, if $f^{-1}(K) \cap S$ is compact for each compact set $K \subset E'$.

Definition 2.3. A map $f : Y \to E'$ is *locally proper* on $S \subseteq Y$ if each point $x \in S$ has an open neighborhood $V = V(x) \subset Y$ such that the restriction $f_{|\bar{V}}$ is proper.

Definition 2.4. The maps $f, g : \bar{Y} \to E'$ and the set $\bar{Y}$ form an *s-admissible triplet* $(f, g, \bar{Y})$, if the following conditions are satisfied:

(h1) f is a proper, $\Phi_n C^1$ map, where $n \geq 0$ (case (i)) and $n > 0$ (case (ii));

(h2) g is a continuous, compact map, that is, $g(\bar{Y})$ is a relatively compact subset of E';

(h3) $\operatorname{Coin}(f,g) \cap \partial Y = \varnothing$, where $\operatorname{Coin}(f,g) = \{x \in \bar{Y} : f(x) = g(x)\}$ is the *coincidence points set*.

As a base of our construction, we use the coincidence index of s-admissible pair $\operatorname{Ind}(f, g, \bar{U})$ defined as an element of the Rohlin-Thom ring of bordisms which has the following main properties (see [9, 10]).

Proposition 2.5 (the coincidence point property). *If* $\operatorname{Ind}(f, g, \bar{U}) \neq 0$, *the zero element of the ring of bordisms, then* $\operatorname{Coin}(f,g) \neq \varnothing$.

Proposition 2.6 (the homotopy invariance property). *If* $(f_*, g_*, \bar{U}_*)$ *is an s-admissible triplet* $(U_* \subset E \times [0,1])$, *then*

$$\operatorname{Ind}\left(f_*(\cdot,0), g_*(\cdot,0), \overline{U_{*_0}}\right) = \operatorname{Ind}\left(f_*(\cdot,1), g_*(\cdot,1) \cdot \overline{U_{*_1}}\right), \tag{2.2}$$

where $U_{*_i} = U_* \cap (E \times \{i\})$, $i = 0, 1$.

Proposition 2.7 (additive dependence on the domain property). *Let* U_0 *and* U_1 *be disjoint open subsets of an open set* $U \subseteq E$, *and let* $(f, g, \bar{U})$ *be an s-admissible triplet such that*

$$\operatorname{Coin}(f,g) \cap \left(\bar{U} \backslash (U_0 \cup U_1)\right) = \varnothing. \tag{2.3}$$

Then

$$\operatorname{Ind}\left(f, g, \bar{U}\right) = \operatorname{Ind}\left(f, g, \overline{U_0}\right) + \operatorname{Ind}\left(f, g, \overline{U_1}\right). \tag{2.4}$$

We recall some notions for multivalued maps (cf. [5]). Denote by $Kv(E')$ the collection of all compact, convex subsets of E'.

Let $S \subseteq \bar{Y}$ be a closed subset.

Definition 2.8. A multivalued map (multimap) $G : S \to Kv(E')$ is

(a) *closed* if its graph Γ_G is a closed subset of $S \times E'$;

(b) *upper semicontinuous* (USC) if $G^{-1}(V) = \{x \in S : G(x) \subset V\}$ is an open subset of S for every open set $V \subset E'$.

Definition 2.9. A continuous map $g_\varepsilon : S \to E'$ $(\varepsilon > 0)$ is called an *ε-approximation* of the multimap $G : S \to Kv(E')$ if

(a) the graph Γ_{g_ε} is contained in the ε-neighborhood $W_\varepsilon(\Gamma_G)$ of the graph of G;

(b) $g_\varepsilon(S) \subset \operatorname{co} G(S)$.

The following statements are well known (cf. [5]).

Proposition 2.10. *If a multimap $G : S \to Kv(E')$ is closed and compact, that is, $G(S)$ is relatively compact in E', then G is USC.*

Proposition 2.11. *Every USC multimap $G : S \to Kv(E')$ admits an ε-approximation $g_\varepsilon : S \to E'$ for every $\varepsilon > 0$.*

3. Relative coincidence index

3.1. Single-valued perturbation. Let $K \subset E'$ be a closed, convex set.

Definition 3.1. The maps $f : \bar{Y} \to E'$, $g : f^{-1}(K) \to K$ form a Ks-admissible triplet $(f, g, \bar{Y})_K$ if conditions (h1), (h3) of Definition 2.4 hold together with the following condition:

($h2_{Ks}$) g is a continuous compact map.

Our aim is to define a relative coincidence index $\operatorname{Ind}(f, g, \bar{U})_K$. To this aim we consider first the trivial case

$$f^{-1}(K) = \varnothing. \tag{3.1}$$

We set, by definition

$$\operatorname{Ind}(f, g, \bar{U})_K = 0, \tag{3.2}$$

the zero element of the Rohlin-Thom ring of bordisms.

Now let $f^{-1}(K) \neq \varnothing$. Let $\hat{g} : \bar{U} \to K$ be an arbitrary extension of g such that $\hat{g}(\bar{U})$ is a relatively compact subset of K. Then

$$\operatorname{Ind}(f, g, \bar{U})_K := \operatorname{Ind}(f, \hat{g}, \bar{U}), \tag{3.3}$$

the coincidence index of the s-admissible triplet $(f, \hat{g}, \bar{U})$.

Lemma 3.2. *The definition of* $\operatorname{Ind}(f, g, \bar{U})_K$ *is consistent.*

Proof. (a) To verify that $(f, \hat{g}, \bar{U})$ is an s-admissible triplet it is sufficient to be sure that

$$\operatorname{Coin}(f, \hat{g}) \cap \partial U = \varnothing. \tag{3.4}$$

In fact, let $x \in \operatorname{Coin}(f, \hat{g})$, then

$$f(x) = \hat{g}(x) \in K, \tag{3.5}$$

hence $x \in f^{-1}(K)$ and $\hat{g}(x) = g(x)$, so $x \in \operatorname{Coin}(f, g)$ and $x \notin \partial U$.

(b) The index does not depend on the choice of the extension $\hat{g}$. In fact, let $\hat{g}_0, \hat{g}_1 : \bar{U} \to K$ be two extensions of g.

Consider the map $g_* : \bar{U} \times [0,1] \to E'$,

$$g_*(x, \lambda) = (1-\lambda)\hat{g}_0(x) + \lambda \hat{g}_1(x). \tag{3.6}$$

Suppose that

$$f(x) = g_*(x,\lambda) \tag{3.7}$$

for some $(x,\lambda) \in \bar{U} \times [0,1]$. Then $f(x) \in K$ and $x \in f^{-1}(K)$, hence

$$\hat{g}_0(x) = \hat{g}_1(x) = g(x), \quad x \in \operatorname{Coin}(f,g). \tag{3.8}$$

Therefore $x \notin \partial U$. By the homotopy property (Proposition 2.6)

$$\operatorname{Ind}(f,\hat{g}_0,\bar{U}) = \operatorname{Ind}(f,\hat{g}_1,\bar{U}). \tag{3.9}$$

□

We now describe the main properties of the defined characteristic.

Proposition 3.3 (the coincidence point property). *Let $(f,g,\bar{U})_K$ be a Ks-admissible triplet. If $\operatorname{Ind}(f,g,\bar{U})_K \neq 0$, then $\operatorname{Coin}(f,g) \neq \varnothing$.*

Proof. Let $\hat{g} : \bar{U} \to K$ be any extension of g. Then $\operatorname{Ind}(f,\hat{g},\bar{U}) \neq 0$ and by Proposition 2.5 $\operatorname{Coin}(f,\hat{g}) \neq \varnothing$. As we have seen earlier, $\operatorname{Coin}(f,\hat{g}) = \operatorname{Coin}(f,g)$. □

To formulate the topological invariance property of the relative coincidence index, it is convenient to give the following definition.

Definition 3.4. Two Ks-admissible triplets $(f_0,g_0,\bar{U}_0)_K$ and $(f_1,g_1,\bar{U}_1)_K$ are said to be homotopic

$$(f_0,g_0,\bar{U}_0)_K \sim (f_1,g_1,\bar{U}_1)_K \tag{3.10}$$

if there exists a Ks-admissible triplet $(f_*,g_*,\bar{U}_*)_K$, where $\bar{U}_* \subset E \times [0,1]$ is an open set, such that $U_i = U_* \cap (E \times \{i\})$, $f_i = f_*(\cdot,i)$, $g_i = g_*(\cdot,i)$, $i = 0,1$.

Proposition 3.5 (the homotopy invariance property). *If*

$$(f_0,g_0,\bar{U}_0)_K \sim (f_1,g_1,\bar{U}_1)_K, \tag{3.11}$$

then

$$\operatorname{Ind}(f_0,g_0,\bar{U}_0)_K = \operatorname{Ind}(f_1,g_1,\bar{U}_1)_K. \tag{3.12}$$

Proof. If we take any extension $\hat{g}_* : \bar{U}_* \to K$ of g_* such that $\hat{g}_*(\bar{U}_*)$ is a relatively compact subset of K, then by Proposition 2.6

$$\operatorname{Ind}(f_*(\cdot,0),\hat{g}_*(\cdot,0),\bar{U}_0) = \operatorname{Ind}(f_*(\cdot,1),\hat{g}_*(\cdot,1),\bar{U}_1), \tag{3.13}$$

which gives the desired equality. □

In the sequel we will need the following two properties of the relative coincidence index.

Let U_0, U_1 be disjoint open subsets of an open set $U \subseteq E$ and $(f,g,\bar{U})_K$ a Ks-admissible triplet such that

$$\operatorname{Coin}(f,g) \cap (\bar{U} \setminus (U_0 \cup U_1)) = \emptyset. \tag{3.14}$$

It is clear that $(f,g,\bar{U}_0)_K$, $(f,g,\bar{U}_1)_K$ are Ks-admissible triplets.

Proposition 3.6 (additive dependence on the domain property).

$$\operatorname{Ind}(f,g,\bar{U})_K = \operatorname{Ind}(f,g,\bar{U}_0)_K + \operatorname{Ind}(f,g,\bar{U}_1)_K. \tag{3.15}$$

This property follows immediately from Proposition 2.7.

Proposition 3.7 (the map restriction property). *Let K_1 be a closed convex subset of E', $K_1 \subset K$, and $(f,g,\bar{U})_K$ a Ks-admissible triplet such that $g(f^{-1}(K)) \subseteq K_1$. Then $(f,g,\bar{U})_{K_1}$ is a K_1s-admissible triplet and*

$$\operatorname{Ind}(f,g,\bar{U})_{K_1} = \operatorname{Ind}(f,g,\bar{U})_K. \tag{3.16}$$

Proof. The first sentence of the statement is evident. Let $\hat{g} : \bar{U} \to K$ be any extension of g from $f^{-1}(K)$ such that $\hat{g}(\bar{U}) \subseteq \overline{\operatorname{co}} g(f^{-1}(K)) \subseteq K_1$, and $\hat{g}_1 : \bar{U} \to K_1$ any extension of g from $f^{-1}(K_1)$ such that $\hat{g}_1(\bar{U}) \subseteq \overline{\operatorname{co}} g(f^{-1}(K_1))$. It is easy to see that the map $g_* : \bar{U} \times [0,1] \to E'$, $g_*(x,\lambda) = (1-\lambda)\hat{g}(x) + \lambda \hat{g}_1(x)$ gives the homotopy connection of s-admissible triplets $(f,\hat{g},\bar{U})$ and $(f,\hat{g}_1,\bar{U})$, hence by Proposition 2.6,

$$\operatorname{Ind}(f,\hat{g},\bar{U}) = \operatorname{Ind}(f,\hat{g}_1,\bar{U}). \tag{3.17}$$

□

3.2. Multivalued perturbation. Let K, Y, and $f : \bar{Y} \to E'$ be as in the previous section, $G : f^{-1}(K) \to Kv(E')$ a multimap.

Definition 3.8. The maps f, G, and the set Y form a *Km-admissible triplet* $(f,G,\bar{Y})_K$ if f satisfies condition (h1) of Definition 2.4 and the following assumptions hold:

- ($h2_{Km}$) G is a closed, compact multimap to K, that is, $G(f^{-1}(K))$ is a relatively compact subset of K;
- ($h3_{Km}$) $\operatorname{Coin}(f,G) \cap \partial Y = \emptyset$, where $\operatorname{Coin}(f,G) = \{x \in \bar{Y} : f(x) \in G(x)\}$ is the coincidence points set.

To define the relative coincidence index $\operatorname{Ind}(f,G,\bar{U})_K$ again we consider first the case $f^{-1}(K) = \emptyset$. In this situation, as before, we set by definition

$$\operatorname{Ind}(f,G,\bar{U}) = 0. \tag{3.18}$$

To consider the case $f^{-1}(K) \neq \emptyset$, we introduce the following notions.

Definition 3.9. Two Km-admissible triplets $(f_0, G_0, \bar{U}_0)_K$ and $(f_1, G_1, \bar{U}_1)_K$ are homotopic,

$$(f_0, G_0, \bar{U}_0)_K \sim (f_1, G_1, \bar{U}_1)_K \tag{3.19}$$

if there exists a Km-admissible triplet $(f_*, G_*, \bar{U}_*)_K$ where $U_* \subset E \times [0,1]$ is an open set, such that $U_i = U_* \cap (E \times \{i\})$, $f_i = f_*(\cdot, i)$, $G_i = G_*(\cdot, i)$, $i = 0, 1$.

Definition 3.10. A Ks-admissible triplet $(f_0, g, \bar{U}_0)_K$ is said to be a single-valued homotopic approximation of a Km-admissible triplet $(f, G, \bar{U})_K$ if

$$(f_0, g, \bar{U}_0)_K \sim (f, G, \bar{U})_K. \tag{3.20}$$

To prove the existence of a single-valued homotopic approximation and to study its properties, consider any Km-admissible triplet $(f, G, \bar{Y})_K$.

Let $g_\varepsilon : f^{-1}(K) \to K$ be any ε-approximation of G, $\varepsilon > 0$ (see Proposition 2.11). Consider the multimap $\Phi_\varepsilon : f^{-1}(K) \times [0,1] \to Kv(E')$ given by

$$\Phi_\varepsilon(x, \lambda, \mu) = (1-\mu)G(x, \lambda) + \mu g_\varepsilon(x, \lambda) \tag{3.21}$$

and denote by $\operatorname{Coin}(f, \Phi_\varepsilon)$ the set

$$\operatorname{Coin}(f, \Phi_\varepsilon) = \{(x, \lambda, \mu) \in f^{-1}(K) \times [0,1] : f(x, \lambda) \in \Phi_\varepsilon(x, \lambda, \mu)\}. \tag{3.22}$$

It is easy to see that, by construction, $\Phi_\varepsilon(f^{-1}(K) \times [0,1])$ is a relatively compact subset of K.

Proposition 3.11. *For $\varepsilon > 0$ small enough,* $\operatorname{Coin}(f, \Phi_\varepsilon) \cap (\partial Y \times [0,1]) = \emptyset$.

Proof. Suppose the contrary. Then, there exist sequences

$$\varepsilon_n \longrightarrow 0; \qquad (x_n, \lambda_n, \mu_n) \in \partial Y \times [0,1]; \tag{3.23}$$

with

$$(x_n, \lambda_n) \in f^{-1}(K), \tag{3.24}$$

such that

$$f(x_n, \lambda_n) \in \Phi_{\varepsilon_n}(x_n, \lambda_n, \mu_n). \tag{3.25}$$

It means that

$$f(x_n, \lambda_n) = (1-\mu_n)z_n + \mu_n g_{\varepsilon_n}(x_n, \lambda_n), \tag{3.26}$$

where $z_n \in G(x_n, \lambda_n)$.

From $(h2_{Km})$ it follows that we can assume, without loss of generality, that $z_n \to z_0 \in K$.

By construction, for every n we have

$$\Phi_{\varepsilon_n}(x_n, \lambda_n, \mu_n) \subset \overline{\text{co}}G(f^{-1}(K)), \tag{3.27}$$

where $\overline{\text{co}}G(f^{-1}(K))$ is a compact set and, since f is proper, we can assume, without loss of generality, that $(x_n, \lambda_n) \to (x_0, \lambda_0) \in f^{-1}(K) \cap \partial Y$. From the closedness of G we obtain that $z_0 \in G(x_0, \lambda_0)$.

We can also assume that μ_n converges to μ_0. As f is a continuous map, we get that $f(x_n, \lambda_n) \to f(x_0, \lambda_0)$.

Further, by definition of ε-approximation we have

$$\{g_{\varepsilon_n}(x_n, \lambda_n)\}_{n=1}^{\infty} \subset \overline{\text{co}}G(f^{-1}(K)), \tag{3.28}$$

hence we can assume, without loss of generality, that $g_{\varepsilon_n}(x_n, \lambda_n) \to y_0 \in K$.

Passing the limit in (3.26), we have

$$f(x_0, \lambda_0) = (1 - \mu_0)z_0 + \mu_0 y_0. \tag{3.29}$$

By definition of ε-approximation,

$$[(x_n, \lambda_n), g_{\varepsilon_n}(x_n, \lambda_n)] \in W_{\varepsilon_n}(\Gamma_G), \tag{3.30}$$

therefore

$$[(x_0, \lambda_0), y_0] \in \Gamma_G, \tag{3.31}$$

that is, $y_0 \in G(x_0, \lambda_0)$.

From (3.29) we get

$$f(x_0, \lambda_0) \in G(x_0, \lambda_0) \tag{3.32}$$

contrary to condition (h3_{Km}). □

Corollary 3.12. *Every Km-admissible triplet $(f, G, \bar{U})_K$ has a single-valued homotopic approximation.*

Proof. From Proposition 3.11, we see that we can take the triplet $(f, g_\varepsilon, \bar{U})_K$, where g_ε is an ε-approximation of G and $\varepsilon > 0$ is small enough, as a single-valued homotopic approximation. □

We can now justify the following definition.

Definition 3.13. Relative coincidence index $\text{Ind}(f, G, \bar{U})_K$ of a Km-admissible triplet $(f, G, \bar{U})_K$ is the relative coincidence index $\text{Ind}(f_0, g, \bar{U}_0)_K$ of an arbitrary single-valued homotopic approximation $(f_0, g, \bar{U}_0)_K$ of $(f, G, \bar{U})_K$.

This notion is well defined. In fact, we can prove the following statement.

Proposition 3.14. *Let* $(f_0, g_0, \bar{U}_0)_K$ *and* $(f_1, g_1, \bar{U}_1)_K$ *be two single-valued homotopic approximations of the* Km*-admissible triplet* $(f, G, \bar{U})_K$. *Then,*

$$(f_0, g_0, \bar{U}_0)_K \sim (f_1, g_1, \bar{U}_1)_K, \tag{3.33}$$

where the homotopy is in the class of Ks*-admissible triplets. Hence,*

$$\operatorname{Ind}(f_0, g_0, \bar{U}_0)_K = \operatorname{Ind}(f_1, g_1, \bar{U}_1)_K. \tag{3.34}$$

Proof. From the definition it follows that there exists a Km-admissible triplet $(f_*, G_*, U_*)_K$ where $U_* \subset E \times [0,1]$ is an open set such that $U_i = U_* \cap (E \times \{i\})$, $f_i = f_*(\cdot, i)$, $g_i = G_*(\cdot, i)$, $i = 0, 1$.

From Proposition 3.11, there exists a USC multimap $\Phi : f_*^{-1}(K) \times [0,1] \to Kv(K)$ with the following properties:

(a) $\Phi(\cdot, \cdot, 0) = G_*$;
(b) $\Phi(\cdot, \cdot, 1) = \varphi(\cdot, \cdot)$ is single-valued;
(c) $\Phi(x, \lambda, \mu) = (1-\mu)G_*(x, \lambda) + \mu\varphi(x, \lambda)$;
(d) Φ is compact;
(e) $\operatorname{Coin}(f_*, \Phi) \cap (\partial U_* \times [0,1]) = \varnothing$.

The single-valued map $h_0 : (\bar{U}_0 \cap f_*^{-1}(K)) \times [0,1] \to K$, given by $h_0(x, \mu) = \Phi(x, 0, \mu)$ defines the homotopy

$$(f_0, g_0, \bar{U}_0)_K \sim (f_0, \varphi(\cdot, 0), \bar{U}_0)_K. \tag{3.35}$$

Furthermore, we can consider the obvious homotopy

$$(f_0, \varphi(\cdot, 0), \bar{U}_0)_K \sim (f_1, \varphi(\cdot, 1), \bar{U}_1)_K, \tag{3.36}$$

and, at last, the single-valued map $h_1 : (\bar{U}_1 \cap f_*^{-1}(K)) \times [0,1] \to K$, defined by $h_1(x, \nu) = \Phi(x, 1, 1-\nu)$, implies the homotopy

$$(f_1, \varphi(\cdot, 1), \bar{U}_1)_K \sim (f_1, g_1, \bar{U}_1)_K, \tag{3.37}$$

so we have

$$(f_0, g_0, \bar{U}_0)_K \sim (f_1, g_1, \bar{U}_1)_K \tag{3.38}$$

and we can apply Proposition 3.5. □

As a direct consequence of the definition, we obtain the following property on the homotopy invariance of the relative coincidence index.

Proposition 3.15. *If*

$$(f_0, G_0, \bar{U}_0)_K \sim (f_1, G_1, \bar{U}_1)_K, \tag{3.39}$$

then

$$\operatorname{Ind}(f_0, G_0, \bar{U}_0)_K = \operatorname{Ind}(f_1, G_1, \bar{U}_1)_K. \tag{3.40}$$

We can now formulate the following coincidence point principle.

PROPOSITION 3.16. *Let* $(f,G,\bar{U})_K$ *be a* Km*-admissible triplet. If* $\operatorname{Ind}(f,G,\bar{U})_K \neq 0$ *then* $\operatorname{Coin}(f,G) \neq \emptyset$.

Proof. In fact, suppose the contrary, repeating the same arguments used in the proof of Proposition 3.11, we can find a single-valued homotopy approximation $(f,g,\bar{U})_K$ such that $\operatorname{Coin}(f,g) = \emptyset$ and hence, by Proposition 3.3, we have that $\operatorname{Ind}(f,g,\bar{U}) = 0$. □

The use of single-valued approximations in the definition of the index allows the following analogs of Propositions 3.6 and 3.7.

PROPOSITION 3.17. *Let* $(f,G,\bar{U})_K$ *be* Km*-admissible, and* $U_0, U_1 \subset E$ *be disjoint open sets such that* $\operatorname{Coin}(f,G) \cap (\bar{U} \backslash (U_0 \cup U_1)) = \emptyset$. *Then,*

$$\operatorname{Ind}(f,G,\bar{U})_K = \operatorname{Ind}(f,G,\bar{U}_0)_K + \operatorname{Ind}(f,G,\bar{U}_1)_K. \tag{3.41}$$

PROPOSITION 3.18. *Let* K_1 *be a closed, convex subset of* E', $K_1 \subset K$ *and* $(f,G,\bar{U})_K$ Km*-admissible such that* $G(f^{-1}(K)) \subseteq K_1$. *Then* $(f,G,\bar{U})_{K_1}$ *is* K_1m*-admissible and*

$$\operatorname{Ind}(f,G,\bar{U})_{K_1} = \operatorname{Ind}(f,G,\bar{U})_K. \tag{3.42}$$

4. Coincidence index for noncompact triplets

4.1. Coincidence index for fundamentally restrictible triplets. Let $f : S \subseteq \bar{Y} \to E'$ be a C^1-map, $G : S \subseteq \bar{Y} \to Kv(E')$ a closed multimap.

Definition 4.1. A convex, closed subset $T \subset E'$ is said to be *fundamental* for (f,G,S) if

(i) $G(f^{-1}(T)) \subseteq T$;
(ii) for any point $x \in S$, the inclusion $f(x) \in \overline{\operatorname{co}}(G(x) \cup T)$ implies that $f(x) \in T$.

It is easy to verify that this notion has the following properties (cf. [5]).

PROPOSITION 4.2. (a) *The set* $\operatorname{Coin}(f,G)$ *is included in* $f^{-1}(T)$ *for each fundamental set of* (f,G,S).

(b) *Let* T *be a fundamental set of* (f,G,S). *The set* $\tilde{T} = \overline{\operatorname{co}}G(f^{-1}(T))$ *is fundamental.*

(c) *Let* $\{T_\alpha\}$ *be a system of fundamental sets of* (f,G,S). *The set* $T = \cap_\alpha T_\alpha$ *is also fundamental.*

The entire space E' and $\overline{\operatorname{co}}G(S)$ are natural examples of fundamental sets of (f,G,S).

Definition 4.3. A triplet (f, G, S) is called *fundamentally restrictible* if there exists a fundamental set T such that the restriction $G_{|f^{-1}(T)}$ is compact. Such fundamental set is called *supporting*.

Definition 4.4. A triplet (f, G, Y) is said to be τ-*admissible* if the following conditions are satisfied:

(H_1) the set $Q = \text{Coin}(f, G)$ is compact;
(H_2) the map f is $\Phi_n C^1$ on the set Q ($n \geq 0$ in case (i) and $n > 0$ in case (ii));
(H_3) there exists an open neighborhood V of Q, $\bar{V} \subseteq Y$ such that $(f, G, \bar{V})$ is fundamentally restrictible.

Our goal is the definition of a coincidence index, $\text{Ind}(f, G, U)$ for a τ-admissible triplet (f, G, U).

First of all we consider the case when the set Q is empty. In this case we set by definition the index $\text{Ind}(f, G, U)$ as the zero element of the Rolin-Thom ring of nonoriented bordisms.

Suppose now that $Q \neq \emptyset$. We can assume, without loss of generality, that the restriction $f_{|\bar{V}}$ is a $\Phi_n C^1$-map. In fact, the set $\Phi_n(E, E')$ of linear Fredholm maps is open in $L(E, E')$ and the map $x \to f'(x)$ is continuous, hence every point $x \in Q$ has a neighborhood $V(x)$ such that $\bar{V}(x) \subset V$ and $f'(v) \in \Phi_n(E, E')$ for all $v \in \bar{V}(x)$. Selecting a finite subcover $\{V(x_1), \dots, V(x_m)\}$ from the cover $\{V(x)\}_{x \in Q}$ of Q, we can substitute V with the smaller neighborhood $V' = \cup_{i=1}^{m} V(x_i)$.

Furthermore, since every $\Phi_n C^1$-map is locally proper (see [7]) and Q is compact, we can also assume, without loss of generality, that the restriction $f_{|\bar{V}}$ is proper. Now, if T is any supporting fundamental set of the triplet $(f, g, \bar{V})$, we see that $(f, G, \bar{V})_T$ is a Tm-admissible triplet in the sense of Definition 3.8. We can now give the following definition of coincidence index.

Definition 4.5. Let (f, G, U) be τ-admissible with $Q \neq \emptyset$. Then,

$$\text{Ind}(f, G, U) := \text{Ind}\,(f, G, \bar{V})_T, \tag{4.1}$$

where T is any supporting fundamental set of $(f, G, \bar{V})$.

Lemma 4.6. *Definition 4.5 is consistent, that is, the coincidence index does not depend on the choice of the supporting fundamental set T and the neighborhood V with the above mentioned properties.*

Proof. (a) Let T_0 and T_1 be two supporting fundamental sets of $(f, G, \bar{V})$. Then, the intersection $T = T_0 \cap T_1$ is a supporting fundamental set of $(f, G, \bar{V})$ (see Proposition 4.2(c)). We prove that

$$\text{Ind}\,(f, G, \bar{V})_{T_i} = \text{Ind}\,(f, G, \bar{V})_T, \quad i = 0, 1. \tag{4.2}$$

Consider the retraction $\rho : E' \to T$ and the USC multimap $\hat{G} : \bar{V} \to K\nu(E')$, defined as $\hat{G}(x) = \overline{\text{co}}(\rho \circ G)(x)$. It is easy to verify that $(f, \hat{G}, \bar{V})_{T_0}$ forms a T_0m-admissible triplet. Moreover,

$$(f, G, \bar{V})_{T_0} \sim (f, \hat{G}, \bar{V})_{T_0}. \tag{4.3}$$

In fact, define $f_* : \bar{V} \times [0,1] \to E'$ by $f_*(x,\lambda) = f(x)$ for all $(x,\lambda) \in \bar{V} \times [0,1]$ and $G_* : \bar{V} \times [0,1] \to Kv(E')$ as $G_*(x,\lambda) = (1-\lambda)G(x) + \lambda\hat{G}(x)$. It is clear that the restriction $G_{*|f_*^{-1}(T_0)}$ is compact.

Now, let $f_*(x_0,\lambda_0) \in G_*(x_0,\lambda_0)$ for some $(x_0,\lambda_0) \in f_*^{-1}(T_0) \cap (\partial V \times [0,1])$. It means that

$$f_*(x_0,\lambda_0) = f(x_0) \in (1-\lambda_0)G(x_0) + \lambda_0\hat{G}(x_0) \subset \overline{\text{co}}(G(x_0) \cup T), \tag{4.4}$$

hence $f(x_0) \in T$, $\hat{G}(x_0) = G(x_0)$, and $f(x_0) \in G(x_0)$ giving a contradiction. Then, the property (h3_{Km}) of Definition 3.8 holds, and from the homotopy property of the relative index (see Proposition 3.15) we have

$$\text{Ind}\,(f, G, \bar{V})_{T_0} = \text{Ind}\,(f, \hat{G}, \bar{V})_{T_0}. \tag{4.5}$$

Applying the map restriction property (Proposition 3.18), we see that

$$\text{Ind}\,(f, \hat{G}, \bar{V})_{T_0} = \text{Ind}\,(f, \hat{G}, \bar{V})_{T}. \tag{4.6}$$

Since $\hat{G}_{|f^{-1}(T)} = G_{|f^{-1}(T)}$ we have, by definition,

$$\text{Ind}\,(f, \hat{G}, \bar{V})_{T} = \text{Ind}\,(f, G, \bar{V})_{T} \tag{4.7}$$

and therefore

$$\text{Ind}\,(f, G, \bar{V})_{T_0} = \text{Ind}\,(f, G, \bar{V})_{T}. \tag{4.8}$$

The equality

$$\text{Ind}\,(f, G, \bar{V})_{T_1} = \text{Ind}\,(f, G, \bar{V})_{T} \tag{4.9}$$

follows in the same way.

(b) Let V_0, V_1 be two open neighborhoods of Q with the necessary properties. We can assume without loss of generality that $V_0 \subset V_1$. Then the equality

$$\text{Ind}\,(f, G, \bar{V}_0)_{T} = \text{Ind}\,(f, G, \bar{V}_1)_{T} \tag{4.10}$$

is the consequence of the additive dependence on the domain and coincidence point properties (Propositions 3.16 and 3.17). □

The next two properties of the characteristics play a key role in the applications. They follow from Propositions 3.15 and 3.16.

Proposition 4.7 (topological invariance). *Let* (f_*, G_*, U_*) *be* τ*-admissible,* $U_* \subset E \times [0,1]$ *an open set. Then,*

$$\operatorname{Ind}(f_0, G_0, U_0) = \operatorname{Ind}(f_1, G_1, U_1), \tag{4.11}$$

where $U_i = U_* \cap (E \times \{i\})$; $f_i = f_*(\cdot, i)$; $G_i = G_*(\cdot, i)$; $i = 0, 1$.

Proposition 4.8 (coincidence point property). *Let* (f, G, U) *be* τ*-admissible. If* $\operatorname{Ind}(f, G, U) \neq 0$, *then* $\operatorname{Coin}(f, G) \neq \emptyset$.

4.2. Coincidence index for condensing triplets. We consider now some important examples of τ-admissible triplets.

The first one is rather simple. Suppose that (f, G, Y) satisfies assumptions (H_1), (H_2) of Definition 4.4 and the following one:

$(H_3{}')$ there exists an open neighborhood V of Q such that $V \subset Y$ and $G_{|V}$ is compact.

It is clear that in such a situation we can consider E' as a supporting fundamental set of (f, G, Y), hence (f, G, Y) is τ-admissible.

To deal with more consistent examples, we recall some notions (cf. [5]). Denote by $P(E')$ the collection of all nonempty subsets of E'.

Definition 4.9. A function $\beta : P(E') \to [0, +\infty]$ is called a (real) measure of noncompactness (MNC) in E' if

$$\beta(\overline{\operatorname{co}}\Omega) = \beta(\Omega) \tag{4.12}$$

for every $\Omega \in P(E')$, and $\beta(\Omega) < +\infty$ for each bounded set $\Omega \in P(E')$.

A MNC β is called:

(i) monotone if $\Omega_0, \Omega_1 \in P(E')$, $\Omega_0 \subseteq \Omega_1$ implies that $\beta(\Omega_0) \leq \beta(\Omega_1)$;
(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E', \Omega \in P(E')$;
(iii) semiadditive if $\beta(\Omega_0 \cup \Omega_1) = \max\{\beta(\Omega_0), \beta(\Omega_1)\}$ for every $\Omega_0, \Omega_1 \in P(E')$;
(iv) algebraically semiadditive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every Ω_0, $\Omega_1 \in P(E')$;
(v) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

Among the known examples of MNC satisfying all the above properties:

The Hausdorff MNC

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}. \tag{4.13}$$

The Kuratowskii MNC

$$\alpha(\Omega) = \inf\{d > 0 : \Omega \text{ has a finite partition with sets of diameter less than } d\}. \tag{4.14}$$

Let $M \subseteq Y$ be a bounded set; $f : M \to E'$ a map; $G : M \to Kv(E')$ a multimap; β a MNC in E'.

Definition 4.10. A triplet (f, G, M) is said to be β-condensing if, for every $\Omega \subseteq M$ such that $G(\Omega)$ is not relatively compact, we have

$$\beta(G(\Omega)) < \beta(f(\Omega)). \tag{4.15}$$

We now introduce the following important class of β-condensing triplets.

Definition 4.11. A triplet (f, G, M) is said to be (k, β)-condensing $(0 \leq k < 1)$ if

$$\beta(G(\Omega)) \leq k\beta(f(\Omega)) \tag{4.16}$$

for each $\Omega \subseteq M$.

We can now give new sufficient conditions under which (f, G, Y) is τ-admissible.

Theorem 4.12. *Let (f, G, Y) satisfy conditions (H_1), (H_2) of Definition 4.4 and the following:*

(H_3'') *there exists an open bounded neighborhood $V \subseteq Y$ of Q such that (f, G, V) is β-condensing with respect to a monotone MNC β in E'.*

Then (f, G, Y) is τ-admissible.

Proof. Let $\{T_\alpha\}$ be the collection of all fundamental sets of (f, G, V). Consider the set $T = \cap_\alpha T_\alpha$. From Proposition 4.2(b), (c) it follows that T is the fundamental set satisfying

$$T = \overline{\text{co}}G(f^{-1}(T)). \tag{4.17}$$

Then we have

$$\beta(f(f^{-1}(T))) \leq \beta(T) = \beta(G(f^{-1}(T))). \tag{4.18}$$

Hence $G(f^{-1}(T))$ is relatively compact. □

The condensivity condition may take only a local form.

Definition 4.13. A triplet (f, G, S), $S \subseteq Y$ is said to be *locally β-condensing* if every point $x \in S$ has a bounded open neighborhood $V(x)$ such that $(f, G, V(x))$ is β-condensing.

The notion of locally (k, β)-condensing triplet is defined analogously.

We can now formulate the following statement.

Theorem 4.14. *Let (f, G, Y) satisfy conditions (H_1), (H_2) of Definition 4.4 and the following one:*

(H_3''') *the triplet (f, G, Q) is locally β-condensing, where β is a monotone, semi-additive, and regular MNC in E'.*

Then (f, G, Y) is τ-admissible.

Proof. We prove that condition (H_3''') implies condition (H_3''). Choose a finite subcover $\{V(x_i)\}_{i=1}^m$ of Q from a cover $\{V(x)\}_{x\in Q}$. Then $V = \cup_{i=1}^m V(x_i)$ is the cover of condition (H_3''). In fact, let $\Omega \subset V$ be such that $G(\Omega)$ is not relatively compact. Let $\Omega_i = \Omega \cap V(x_i)$, $i = 1,\dots,m$, then

$$\beta(G(\Omega)) = \max_{1\le i\le m} \beta(G(\Omega_i)) = \beta(G(\Omega_{i_0})) \ne 0. \tag{4.19}$$

Further, the condition of local condensivity implies that

$$\beta(G(\Omega_{i_0})) < \beta(f(\Omega_{i_0})) \tag{4.20}$$

and, from the monotonicity of β we have

$$\beta(f(\Omega_{i_0})) \le \beta(f(\Omega)). \tag{4.21}$$

So, finally

$$\beta(G(\Omega)) < \beta(f(\Omega)). \tag{4.22}$$

□

So, if $Y = U \subset E$ (case (i)), and (f, G, U) satisfies conditions (H_1), (H_2) and either (H_3'), (H_3''), or (H_3'''), then (f, G, U) is τ-admissible and the coincidence index $\operatorname{Ind}(f, G, U)$ is well defined and satisfies all the properties described in Section 4.1.

We now select the property of topological invariance which we will use in applications.

THEOREM 4.15. *Let $U_* \subset E \times [0,1]$ be an open set; $f_* : U_* \to E'$ a C^1 map; $G_* : U_* \to Kv(E')$ a closed multimap satisfying conditions (H_1), (H_2), and (H_3'''). Then*

$$\operatorname{Ind}(f_0, G_0, U_0) = \operatorname{Ind}(f_1, G_1, U_1), \tag{4.23}$$

where $U_i = U_* \cap (E \times \{i\})$; $f_i = f_*(\cdot, i)$; $G_i = G_*(\cdot, i)$; $i = 0, 1$.

It may be convenient for applications to formulate the condensivity and local condensivity conditions in terms of Fréchet derivative f'.

We start from the following notion. Let $f : Y \to E'$ be any map, $G : Y \to Kv(E')$ a multimap; β a MNC in E'.

Definition 4.16. A triplet (f, G, Y) is said to be (k,β)-*bounded* at the point $x \in Y$, $k \ge 0$, if for every $\varepsilon > 0$ there exists a ball $B_\delta(x) \subset Y$ such that

$$\beta(G(\Omega)) \le (k+\varepsilon)\beta(f(\Omega)) \tag{4.24}$$

for each $\Omega \subset B_\delta(x)$.

We recall that a linear operator A is said to be a Φ_+-operator if $\operatorname{Im} A$ is closed and $\ker A$ is finite dimensional. It is clear that every linear Fredholm operator is a Φ_+-operator.

Denote by α_E, $\alpha_{E'}$ the Kuratowski MNC in spaces E and E', respectively.

Lemma 4.17 (see [4]). *Let $A : E \to E'$ be a Φ_+-operator. Then the number*

$$C_\alpha(A) = \sup\{c : \alpha_{E'}(A(\Omega)) \geq c\alpha_E(\Omega), \text{ for all bounded } \Omega \subset E\} \tag{4.25}$$

is finite and different from zero.

Theorem 4.18. *Let $V = V(x)$ be an open neighborhood of $x \in E$; $f : V \to E'$ a C^1-map such that $f'(x)$ is a Φ_+-operator and $G : V \to Kv(E)$ a multimap. If $(f'(x), G, V)$ is $(k, \alpha_{E'})$-bounded at x, then (f, G, V) is also $(k, \alpha_{E'})$-bounded at x.*

Proof. For each $\Omega \subset V - x$, we have

$$f'(x)(\Omega) \subseteq f(x+\Omega) - f(x) - \omega(x,\Omega), \tag{4.26}$$

where ω is the residual term in the representation

$$f(x+h) = f(x) + f'(x)h + \omega(x,h). \tag{4.27}$$

By [11, Lemma 2.3], for every ε, $0 < \varepsilon < 1$ there exists a ball $B_\delta(0)$ such that

$$\alpha_{E'}(\omega(x,\Omega)) \leq \varepsilon\alpha_{E'}(f'(x)(\Omega)), \quad \forall \Omega \subset B_\delta(0). \tag{4.28}$$

Without loss of generality, we can also suppose that

$$\alpha_{E'}(G(x+\Omega)) \leq (k+\varepsilon)\alpha_{E'}(f'(x)(\Omega)), \quad \forall \Omega \subset B_\delta(0). \tag{4.29}$$

Then, from (4.26) we have, for all $\Omega \subset B_\delta(0)$,

$$\begin{aligned} \alpha_{E'}(f'(x)(\Omega)) &\leq \alpha_{E'}(f(x+\Omega) - f(x) - \omega(x,\Omega)) \\ &\leq \alpha_{E'}(f(x+\Omega)) + \varepsilon\alpha_{E'}(f'(x)(\Omega)). \end{aligned} \tag{4.30}$$

So

$$\alpha_{E'}(f'(x)(\Omega)) \leq (1-\varepsilon)^{-1}\alpha_{E'}(f(x+\Omega)) \tag{4.31}$$

and therefore

$$\alpha_{E'}(G(x+\Omega)) \leq (1-\varepsilon)^{-1}(k+\varepsilon)\alpha_{E'}(f(x+\Omega)). \tag{4.32}$$

Since ε is arbitrary, we also have

$$\alpha_{E'}(G(x+\Omega)) \leq (k+\varepsilon)\alpha_{E'}(f(x+\Omega)), \tag{4.33}$$

proving the theorem. □

Corollary 4.19. *Let (f, G, Y) satisfy conditions (H_1), (H_2) of Definition 4.4 and suppose that $(f'(x), G, Y)$ are $(k(x), \alpha_{E'})$-bounded at every point $x \in Q$, where $0 \leq k(x) < 1$. Then (f, G, Y) satisfies (H_3''') of Theorem 4.14 with $\beta = \alpha_{E'}$, and hence is τ-admissible.*

Proof. From Theorem 4.18 it follows that (f, G, Y) is $(k(x), \alpha_{E'})$-bounded at every point $x \in Q$. Since $k(x) < 1$ it means that (f, G, Q) is locally $\alpha_{E'}$-condensing. □

Definition 4.20 (cf. [5, Definition 2.2.9]). A multimap $G : Y \to Kv(E')$ is said to be locally $(k(x), \alpha_E, \alpha_{E'})$-bounded on $S \subseteq Y$ if for each $x \in S$ there exists a bounded open neighborhood $V(x)$ such that

$$\alpha_{E'}(G(\Omega)) \leq k(x)\alpha_E(\Omega), \quad \forall \Omega \subset V(x). \tag{4.34}$$

Theorem 4.21. *Let (f, G, Y) satisfy conditions (H_1) and (H_2) of Definition 4.4. Let G be locally $(k(x), \alpha_E, \alpha_{E'})$-bounded on Q, with $k(x) < C_\alpha(f'(x))$ for all $x \in Q$. Then (f, G, Y) satisfies (H_3''') of Theorem 4.14 with $\beta = \alpha_{E'}$ and hence is τ-admissible.*

Proof. We prove that $(f'(x), G, Y)$ are $(k_1(x), \alpha_{E'})$-bounded at every point $x \in Q$, where $k_1(x) = k(x)C_\alpha^{-1}(f'(x)) < 1$. In fact, let $x \in Q$, then there exists a bounded open neighborhood $V(x)$ such that, for every $\Omega \subset V(x)$, we have

$$\begin{aligned} \alpha_{E'}(G(\Omega)) &\leq k(x)\alpha_E(\Omega) = k(x)C_\alpha^{-1}(f'(x))C_\alpha^{-1}(f'(x))\alpha_E(\Omega) \\ &\leq k_1(x)\alpha_{E'}(f'(x)(\Omega)). \end{aligned} \tag{4.35}$$

The statement now follows from Corollary 4.19. □

5. Application

5.1. An optimal controllability problem. We consider a control system governed by a second-order integro-differential equation. For simplicity we restrict ourselves to the one-dimensional model; the generalization to the n-dimensional case is straightforward.

Denote by $C[0,1]$ the space of continuous functions $x : [0,1] \to \mathbb{R}$ endowed with the usual norm $\|x\|_0 = \max_{t\in[0,1]} |x(t)|$, and by $C^k[0,1]$, $k = 1, 2$ the space of k times continuously differentiable functions with norms

$$\|x\|_1 = \|x\|_0 + \|\dot{x}\|_0, \qquad \|x\|_2 = \|x\|_0 + \|\dot{x}\|_0 + \|\ddot{x}\|_0, \tag{5.1}$$

respectively.

We suppose that the dynamic of the control system is the following:

$$\begin{aligned} &a_0\ddot{x}^m(t) + a_1\ddot{x}^{m-1}(t) + \cdots + a_m \\ &\qquad = \varphi(t, x(t), \dot{x}(t), \ddot{x}(t)) + \int_0^t \psi(s, x(s), \dot{x}(s), \ddot{x}(s), u(s))\, ds, \end{aligned} \tag{5.2}$$

where $x \in C^2[0,1]$ describes the state of the control system, and the control $u : [0,1] \to \mathbb{R}$ is a measurable function satisfying the feedback control relation

$$u(t) \in U\big(t, x(t), \dot{x}(t), \ddot{x}(t)\big), \quad t \in [0,1]. \tag{5.3}$$

We assume also that a lower semicontinuous cost functional

$$j_0 : C^2[0,1] \longrightarrow \mathbb{R}_+ \tag{5.4}$$

is given.

We want to find a control u_* such that the corresponding solution x_* of (5.2) and (5.3) satisfies the controllability relation

$$x(0) = c_0, \qquad x(1) = c_1, \tag{5.5}$$

for given $c_0, c_1 \in \mathbb{R}$, and minimizes j_0:

$$j_0(x_*) = \min_{x \in \Sigma} j_0(x), \tag{5.6}$$

where $\Sigma \subset C^2[0,1]$ denotes the set of all solutions of (5.2), (5.3), and (5.5).

We now describe the assumptions on the given control problem.

First of all, suppose the following.

(L) The polynomial

$$Ly = a_0 y^m + a_1 y^{m-1} + \cdots + a_m \tag{5.7}$$

has no multiple roots, its degree m is an odd number, and $a_0 > 0$.

Denote by $S = \{y_1, \dots, y_l\}$ the collection of all real roots of the derivative polynomial

$$L' y = m a_0 y^{m-1} + \cdots + a_{m-1}. \tag{5.8}$$

We consider constants $\varkappa > 0$ such that the $\varkappa$-neighborhoods $W_\varkappa(S)$ do not contain the roots of L and consist of disjoint intervals

$$(y_1 - \varkappa, y_1 + \varkappa), \dots, (y_l - \varkappa, y_l + \varkappa) \tag{5.9}$$

so the set $\mathbb{R} \setminus W_\varkappa(S)$ is partitioned in closed intervals

$$\bar{D}_i(\varkappa) = \big[\, y_i + \varkappa, y_{i+1} - \varkappa\big], \quad 0 \le i \le l. \tag{5.10}$$

(We set here $y_0 = -\infty$, $y_{l+1} = +\infty$.)

Denote

$$k_i(\varkappa) = \min_{y \in \bar{D}_i(\varkappa)} \big| L'(y) \big|, \quad 0 \le i \le l. \tag{5.11}$$

We assume that the function $\varphi : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the following hypotheses:

(φ_1) φ is continuous;

(φ_2) for some $\varkappa_0 > 0$ there exist numbers k_i, $0 \le k_i < k_i(\varkappa_0)$, $i = 0,\dots,l$ such that

$$|\varphi(t,v,w,y_1) - \varphi(t,v,w,y_0)| \le k_i |y_1 - y_0| \tag{5.12}$$

for $(t,v,w) \in [0,1]\times\mathbb{R}^2$, $y_0, y_1 \in \bar{D}_i(\varkappa_0)$;

(φ_3) for every $y \in \mathbb{R}$ there exist positive constants a, b, c such that

$$|\varphi(t,v,w,y)| \le a + b|v|^{m-1} + c|w|^{m-1} \tag{5.13}$$

for all $(t,v,w) \in [0,1]\times\mathbb{R}^2$.

We suppose that the integrand $\psi : [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ satisfies the following conditions:

(ψ_1) the function $\psi(\cdot,v,w,y,u) : [0,1] \to \mathbb{R}$ is measurable for all $(v,w,y,u) \in \mathbb{R}^4$;

(ψ_2) the function $\psi(t,\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^4 \to \mathbb{R}$ is continuous for a.a. $t \in [0,1]$;

(ψ_3) there exists a summable function $\mu : [0,1]\times\mathbb{R}_+$ such that

$$|\psi(t,v,w,y,u)| \le \mu(t), \qquad \text{for a.a. } t \in [0,1] \tag{5.14}$$

for all $(t,v,w,y) \in [0,1]\times\mathbb{R}^3$ and $u \in U(t,v,w,y)$;

(ψ_4) the set $\psi(t,v,w,y,U(t,v,w,y))$ is convex for all $(t,v,w,y) \in [0,1]\times\mathbb{R}^3$;

Denote by $K(\mathbb{R})$ the collection of all nonempty compact sets of $\mathbb{R}$. The feedback multimap $U : [0,1]\times\mathbb{R}^3 \to K(\mathbb{R})$ satisfies the following conditions:

(U_1) the multifunction $U(\cdot,v,w,y) : [0,1] \to K(\mathbb{R})$ is measurable for every $(v,w,y) \in \mathbb{R}^3$;

(U_2) the multifunction $U(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \to K(\mathbb{R})$ is continuous.

Moreover, we assume the following condition:

($\mathrm{L}\varphi\psi$) there exists $\varkappa_1 > \varkappa_0$ such that

$$|\varphi(t,v,w,y)| + \int_0^t \mu(s)\,ds < |Ly| \tag{5.15}$$

for all $(t,v,w) \in [0,1]\times\mathbb{R}^2$, $y \in W_{\varkappa_1}(S)$.

We are now in position to formulate the main result of this section.

THEOREM 5.1. *Under assumptions (L), (φ_1)–(φ_3), (ψ_1)–(ψ_4), (U_1), (U_2), and ($L\varphi\psi$) there exists a solution (x_*,u_*) of problems (5.2), (5.3), and (5.5).*

To arrive to the theorem we need to define some maps, multimaps and to prove preliminary lemmas.

From the general properties of multimaps (cf. [5]) it follows that, for every $x \in C^2[0,1]$, the multifunction

$$\begin{aligned}\Psi(x):[0,1] &\longrightarrow Kv(\mathbb{R}),\\ \Psi(x)(t) &= \psi\big(t,x(t),\dot{x}(t),\ddot{x}(t),U\big(t,x(t),\dot{x}(t),\ddot{x}(t)\big)\big)\end{aligned} \tag{5.16}$$

is measurable and hence by (ψ_3), integrable, and the multimap $\Pi : C^2[0,1] \to Kv(C[0,1])$, defined by

$$\Pi(x) = \left\{z : z(t) = \int_0^t v(s)\,ds : v(\cdot) \text{ is a summabe selection of } \Psi(x)(\cdot)\right\}, \tag{5.17}$$

is closed.

Further, consider the continuous superposition map $g : C^2[0,1] \to C[0,1]$,

$$g(x)(t) = \varphi\big(t,x(t),\dot{x}(t),\ddot{x}(t)\big). \tag{5.18}$$

Denote by $\mathscr{C}$ the Banach space $C[0,1]\times\mathbb{R}^2$ with norm $\|(x,a,b)\| = \|x\|_0 + |a| + |b|$. We can define the closed multimap $G : C^2[0,1] \to Kv(\mathscr{C})$ by

$$G(x) = \{g(x)+\Pi(x), c_0, c_1\}. \tag{5.19}$$

We define a map $f : C^2[0,1] \to \mathscr{C}$ as

$$f(x) = \{\tilde{f}(x), x(0), x(1)\}, \qquad \tilde{f}(x)(t) = L\ddot{x}(t). \tag{5.20}$$

To handle problems (5.2), (5.3), and (5.5), we study the solvability of the inclusion

$$f(x) \in G(x), \tag{5.21}$$

that is, we deal with the coincidence points of the pair (f,G).

Consider the closed sets

$$Z_i(\varkappa_1) = \{x \in C^2[0,1] : \ddot{x}(t) \in \bar{D}_i(\varkappa_1),\ \forall t \in [0,1]\}, \quad i = 0,\dots,l \tag{5.22}$$

and denote $Z(\varkappa_1) = \cup_{i=0}^{l} Z_i(\varkappa_1)$.

Lemma 5.2. *Under the assumptions of Theorem 5.1, the set Q of all solutions of the family of inclusions*

$$f(x) \in \lambda G(x), \quad \lambda \in [0,1] \tag{5.23}$$

is contained in $Z(\varkappa_1)$.

Proof. If $x_0 \in Q$ then $f(x_0) \in \lambda_0 G(x_0)$ for some $\lambda \in [0,1]$, and if we suppose that $x_0 \notin Z(\varkappa_1)$, then

$$\ddot{x}_0(t_0) \in W_{\varkappa_1}(S), \qquad \text{for some } t_0 \in [0,1]. \tag{5.24}$$

Then

$$L\ddot{x}_0(t_0) = \lambda_0\varphi(t_0, x_0(t_0), \dot{x}(t_0), \ddot{x}(t_0)) + \lambda_0 \int_0^t v_0(s)\,ds, \tag{5.25}$$

where v_0 is a summable selection of $\Psi(x_0)$.

Therefore,

$$|L\ddot{x}_0(t_0)| \le |\varphi(t_0, x_0(t_0), \dot{x}(t_0), \ddot{x}(t_0))| + \int_0^{t_0} \mu(s)\,ds, \tag{5.26}$$

contrary to condition (L$\varphi\psi$). □

Consider now, the sets

$$X_i(\varkappa_0) = \{x \in C^2[0,1] : \ddot{x}(t) \in D_i(\varkappa_0)\ \forall t \in [0,1]\}, \tag{5.27}$$

where $D_i(\varkappa_0) = (y_i + \varkappa_0, y_{i+1} - \varkappa_0)$, $i = 0,\dots,l$. Note that each $X_i(\varkappa_0)$ is an open neighborhood of $Z_i(\varkappa_1)$ ($i = 0,\dots,l$) and, hence, the set $X(\varkappa_0) = \cup_{i=1}^{l} X_i(\varkappa_0)$ is an open neighborhood of the set Q.

Lemma 5.3 (cf. [11]). *The map f is Φ_0C^1 on $X(\varkappa_0)$ and proper on $Z(\varkappa_1)$.*

Lemma 5.4. *The set Q is bounded.*

Proof. If $x \in Q$ then, using boundary conditions (5.5), we have

$$\begin{aligned} x(t) &= \lambda c_0 + \left[\lambda(c_1 - c_0)\int_0^1\int_0^\tau \ddot{x}(s)\,ds\,d\tau\right]t + \int_0^t\int_0^\tau \ddot{x}(s)\,ds\,d\tau, \\ \dot{x}(t) &= \lambda(c_1 - c_0) - \int_0^1\int_0^\tau \ddot{x}(s)\,ds\,d\tau + \int_0^t \ddot{x}(s)\,ds. \end{aligned} \tag{5.28}$$

Therefore

$$\|x\|_0 \le \lambda|c_0| + \lambda|c_1 - c_0| + 2\|\ddot{x}\|_0, \qquad \|\dot{x}\|_0 \le \lambda|c_1 - c_0| + 2\|\ddot{x}\|_0. \tag{5.29}$$

So, the boundedness of $\|\ddot{x}\|_0$ implies the boundedness of $\|x\|_2$.

Further

$$x \in Z_{i_0}(\varkappa_1) \quad \text{for some } i_0,\ 0 \le i_0 \le l. \tag{5.30}$$

It is clear that it is sufficient to consider the case when $Z_{i_0}(\varkappa_1)$ is unbounded, that is, when i_0 is either 0 or l.

All $z \in G(x)$ have the form $z = \{z(t), c_0, c_1\}$, where

$$z(t) = \varphi(t, x(t), \dot{x}(t), \ddot{x}(t)) + \int_0^t v(s)\,ds, \quad v(s) \in \Psi(x)(s), \tag{5.31}$$

hence

$$|z(t)| \le |\varphi(t, x(t), \dot{x}(t), \ddot{x}(t)) - \varphi(t, x(t), \dot{x}(t), y_0)| + |\varphi(t, x(t), \dot{x}(t), y_0)| + \mu_0 \tag{5.32}$$

for some $y_0 \in \bar{D}_{i_0}(\varkappa_1)$, where $\mu_0 = \int_0^1 \mu(s)ds$. Since $\bar{D}_{i_0}(\varkappa_1) \subset \bar{D}_{i_0}(\varkappa_0)$ and $k_{i_0}(\varkappa_0) \leq k_{i_0}(\varkappa_1)$ we can use condition (φ_2) to estimate $|z(t)|$:

$$\begin{aligned} |z(t)| &\leq k_{i_0} |\ddot{x}(t) - y_0| + |\varphi(t, x(t), \dot{x}(t), y_0)| + \mu_0 \\ &= k_{i_0}(k_{i_0}(\varkappa_1))^{-1} k_{i_0}(\varkappa_1) |\ddot{x}(t) - y_0| + |\varphi(t, x(t), \dot{x}(t), y_0)| + \mu_0. \end{aligned} \tag{5.33}$$

Using the mean value theorem and condition (φ_3), we obtain

$$|z(t)| \leq l_0 |L\ddot{x}(t) - Ly_0| + a' + b\|x\|_0^{m-1} + c\|\dot{x}\|_0^{m-1}, \tag{5.34}$$

where $l_0 = k_{i_0}(k_{i_0}(\varkappa_1))^{-1} < 1$ and $a' = a + \mu_0$.

Using estimate (5.29), we have

$$\begin{aligned} |z(t)| &\leq l_0 \|L\ddot{x}\|_0 + l_0 |Ly_0| + a' \\ &\quad + b(|c_0| + |c_1 - c_0| + 2\|\ddot{x}\|_0)^{m-1} + c(|c_1 - c_0| + 2\|\ddot{x}\|_0)^{m-1}. \end{aligned} \tag{5.35}$$

Supposing, without loss of generality, that $\|\ddot{x}\|_0 > 1$, we obtain

$$|z(t)| \leq l_0 \|L\ddot{x}\|_0 + a'' + b'' \|\ddot{x}\|_0^{m-1} \tag{5.36}$$

for some positive constants a'' and b''.

Therefore, since $f(x) \in \lambda G(x)$ for some $\lambda \in [0,1]$ we have the following estimate:

$$\|L\ddot{x}\|_0 \leq \lambda l_0 \|L\ddot{x}\|_0 + \lambda a'' + \lambda b'' \|\ddot{x}\|_0^{m-1} \leq l_0 \|L\ddot{x}\|_0 + a'' + b'' \|\ddot{x}\|_0^{m-1} \tag{5.37}$$

and hence

$$(1 - l_0) \|L\ddot{x}\|_0 \leq a'' + b'' \|\ddot{x}\|_0^{m-1}. \tag{5.38}$$

Taking into account condition (L), it becomes clear that the estimate (5.38) holds only if $\|\ddot{x}\|_0$ is a priori bounded and then $\|x\|_2$ is a priori bounded. □

Lemma 5.5. *The set Q is compact.*

Proof. The continuity of f and the closedness of G imply that Q is closed. Taking into account Lemma 5.3, it is sufficient to show that the set $f(Q)$ is compact in $\mathscr{C}$. To this aim we demonstrate that the set $\mathbb{L}(Q) = \{L\ddot{x}(\cdot) : x \in Q\} \subset C[0,1]$ is relatively compact.

It is clear that $\mathbb{L}(Q)$ is bounded since Q is bounded. Take $z \in \mathbb{L}(Q)$. We have

$$z(t) = \lambda\varphi(t, x(t), \dot{x}(t), \ddot{x}(t)) + \lambda \int_0^t v(s)\,ds, \tag{5.39}$$

for some $v(s) \in \Psi(x)(s)$, $\lambda \in [0,1]$, and $x \in Q$. Then taking any $t_0, t_1 \in [0,1]$, we obtain

$$
\begin{aligned}
|z(t_1) - z(t_0)| \leq {} & |\varphi(t_1, x(t_1), \dot{x}(t_1), \ddot{x}(t_1)) - \varphi(t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0))| \\
& + \int_{t_0}^{t_1} \mu(s)\, ds.
\end{aligned}
\tag{5.40}
$$

From Lemma 5.4 it follows, without loss of generality, that we can assume that the function φ is defined on the set $[0,1] \times [-N,N]^3$ for some $N > 0$, hence it is uniformly continuous. From the mean value theorem, it follows that

$$
|x(t_1) - x(t_0)| \leq N|t_1 - t_0|, \qquad |\dot{x}(t_1) - \dot{x}(t_0)| \leq N|t_1 - t_0|. \tag{5.41}
$$

Then,

$$
\begin{aligned}
&|z(t_1) - z(t_0)| \\
&\quad \leq |\varphi(t_1, x(t_1), \dot{x}(t_1), \ddot{x}(t_1)) - \varphi(t_1, x(t_1), \dot{x}(t_1), \ddot{x}(t_0))| \\
&\qquad + |\varphi(t_1, x(t_1), \dot{x}(t_1), \ddot{x}(t_0)) - \varphi(t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0))| + \int_{t_0}^{t_1} \mu(s)\, ds \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}
\tag{5.42}
$$

Using Lemma 5.2 and condition (φ_2) we can estimate

$$
I_1 \leq k_i |\ddot{x}(t_1) - \ddot{x}(t_0)| \tag{5.43}
$$

for some i, $0 \leq i \leq l$.

Taking into account the uniform continuity of φ and estimate (5.41), given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$
I_2 + I_3 \leq \varepsilon \quad \text{if } |t_1 - t_0| < \delta. \tag{5.44}
$$

On the other hand, since $x \in Q$, the function $z(\cdot)$ can be represented as

$$
z(t) = L\ddot{x}(t), \tag{5.45}
$$

and applying the mean value theorem, we obtain

$$
|z(t_1) - z(t_0)| \geq k_i(\varkappa_1)|\ddot{x}(t_1) - \ddot{x}(t_0)|, \tag{5.46}
$$

therefore

$$
(k_i(\varkappa_1) - k_i)|\ddot{x}(t_1) - \ddot{x}(t_0)| \leq \varepsilon, \qquad |\ddot{x}(t_1) - \ddot{x}(t_0)| \leq \varepsilon(k_i(\varkappa_1) - k_i)^{-1} \tag{5.47}
$$

if $|t_1 - t_0| < \delta$. Finally,

$$
|z(t_1) - z(t_0)| \leq \varepsilon k_i (k_i(\varkappa_1) - k_i)^{-1} + \varepsilon \tag{5.48}
$$

if $|t_1 - t_0| < \delta$. It follows that the set $\mathbb{L}(Q)$ is equicontinuous and hence relatively compact. □

We want to prove now that the multimap G is locally $(k_{V(x)}, \alpha_{C^2}, \alpha_C)$-bounded on Q with $k_{V(x)} \leq C_\alpha(f'(x))$ for every $x \in Q$. In order to get this result we need first to demonstrate the following statement.

Lemma 5.6. *The restrictions of G on each set $X_i(\varkappa_0)$, $i = 0,\ldots,l$ are $(k_i, \alpha_{C^2}, \alpha_C)$-Bounded.*

Proof. Let $\Omega \subset X_i(\varkappa_0)$ be a bounded set and $\alpha_{C^2}(\Omega) = d_0$. It means that for every $\varepsilon > 0$ there exists a partition

$$\Omega = \cup_{j=1}^{m} \Omega_j, \tag{5.49}$$

such that $\operatorname{diam}_{C^2}(\Omega_j) \leq d_0 + \varepsilon$ for all $j = 1,\ldots,m$.

We would like to estimate $\alpha_C(G(\Omega))$. Since the set $\Pi(\Omega)$ is relatively compact, the problem is reduced to the estimation of $\alpha_C(g(\Omega))$.

Let $\|x\|_2 \leq N'$ for all $x \in \Omega$. The function φ is uniformly continuous on the set $[0,1] \times [-N',N']^2 \times (\bar{D}_i(\varkappa_0) \cap [-N',N'])$, therefore we can find $\delta > 0$ such that

$$|\varphi(t_1, v_1, w_1, y_1) - \varphi(t_0, v_0, w_0, y_0)| < \varepsilon \tag{5.50}$$

if $(t_i, v_i, w_i, y_i) \in [0,1] \times [-N',N']^2 \times (\bar{D}_i(\varkappa_0) \cap [-N',N'])$, $i = 0,1$, $|t_1 - t_0| < \delta$, $|t_{v1} - v_0| < \delta$, $|w_1 - w_0| < \delta$, $|y_1 - y_0| < \delta$.

By the Ascoli-Arzelà theorem, the embedding $C^2[0,1] \hookrightarrow C^1[0,1]$ is completely continuous. It means that we can assume, without loss of generality, that the sets Ω_j are selected so that

$$\operatorname{diam}_{C^1}(\Omega_j) < \delta \quad \forall j = 1,\ldots,m. \tag{5.51}$$

Estimate $\operatorname{diam}_C(g(\Omega_j))$. If $x_0, x_1 \in \Omega_j$, then

$$\begin{aligned}
||g(x_1) - g(x_0)||_0 &= \max_{t\in[0,1]} |\varphi(t, x_1(t), \dot{x}_1(t), \ddot{x}_1(t)) - \varphi(t, x_0(t), \dot{x}_0(t), \ddot{x}_0(t))| \\
&\leq \max_{t\in[0,1]} |\varphi(t, x_1(t), \dot{x}_1(t), \ddot{x}_1(t)) - \varphi(t, x_1(t), \dot{x}_1(t), \ddot{x}_0(t))| \\
&\quad + \max_{t\in[0,1]} |\varphi(t, x_1(t), \dot{x}_1(t), \ddot{x}_0(t)) - \varphi(t, x_0(t), \dot{x}_0(t), \ddot{x}_0(t))| \\
&= J_1 + J_2.
\end{aligned} \tag{5.52}$$

Applying condition (φ_2) we obtain

$$\begin{aligned}
J_1 &\leq k_i \max_{t\in[0,1]} |\ddot{x}_1(t) - \ddot{x}_0(t)| = k_i||\ddot{x}_1 - \ddot{x}_0||_0 \\
&\leq k_i||x_1 - x_0||_2 \leq k_i(d_0 + \varepsilon).
\end{aligned} \tag{5.53}$$

By definition of δ we have $J_2 < \varepsilon$. Therefore

$$\operatorname{diam}_C\left(g(\Omega_j)\right) < k_i(d_0+\varepsilon)+\varepsilon, \tag{5.54}$$

and hence

$$\alpha_C\left(g(\Omega)\right) \le k_i d_0 = k_i \alpha_{C^2}(\Omega). \tag{5.55}$$

□

Lemma 5.7. *For every $x \in X_i(\varkappa_0)$,*

$$C_\alpha\left(f'(x)\right) \ge k_i(\varkappa_0). \tag{5.56}$$

Proof. For any bounded $\Omega \subset C^2[0,1]$ estimate $\alpha_C(\tilde{f}'(x)(\Omega))$, where $\tilde{f}(x)(t) = L\ddot{x}(t)$. Take any finite partition $\Omega = \cup_{j=1}^m \Omega_j$. Using again the compactness of the embedding $C^2[0,1] \hookrightarrow C^1[0,1]$, we can assume without loss of generality that $\operatorname{diam}_{C^1}(\Omega_j) < \delta$ for all $j = 1,\dots,m$, where δ is any prescribed number. Estimate $\operatorname{diam}_C(\tilde{f}'(x)(\Omega_j))$. If $h_0, h_1 \in \Omega_j$ then

$$\begin{aligned}
\left\|\tilde{f}'(x)h_1 - \tilde{f}'(x)h_0\right\|_0 &= \max_{t\in[0,1]} \left|\tilde{f}'(x)h_1(t) - \tilde{f}'(x)h_0(t)\right| \\
&= \max_{t\in[0,1]} \left|L'(\ddot{x}(t))\ddot{h}_1(t) - L(\ddot{x}(t))\ddot{h}_0(t)\right| \\
&= \max_{t\in[0,1]} \left|L'\ddot{x}(t)\right| \left\|\ddot{h}_1 - \ddot{h}_0\right\|_0 \ge k_i(\varkappa_0)\left\|\ddot{h}_1 - \ddot{h}_0\right\|_0 \qquad (5.57)\\
&= k_i(\varkappa_0)\left(\left\|h_1-h_0\right\|_2 - \left\|h_1-h_0\right\|_1\right) \\
&> k_i(\varkappa_0)\left(\left\|h_1-h_0\right\|_2 - \delta\right).
\end{aligned}$$

Since δ is arbitrary, we obtain that

$$\operatorname{diam}_C\left(\tilde{f}'(x)(\Omega_j)\right) \ge k_i(\varkappa_0)\operatorname{diam}_{C^2}\left(\Omega_j\right) \tag{5.58}$$

and hence

$$\alpha_C\left(\tilde{f}'(x)(\Omega)\right) \ge k_i(\varkappa_0)\alpha_{C^2}(\Omega). \tag{5.59}$$

Since Ω is bounded we have

$$\alpha_{\mathscr{C}}\left(f'(x)(\Omega)\right) \ge k_i(\varkappa_0)\alpha_{C^2}(\Omega), \tag{5.60}$$

and by definition of $C_\alpha(f'(x))$ we obtain

$$C_\alpha\left(f'(x)\right) \ge k_i(\varkappa_0). \tag{5.61}$$

□

Proof of Theorem 5.1. Define the following maps:

$$f_* : X(\varkappa_0) \times [0,1] \longrightarrow \mathscr{C} \qquad \text{by } f_*(x,\lambda) = f(x);$$
$$G_* : X(\varkappa_0) \times [0,1] \longrightarrow Kv(\mathscr{C}) \quad \text{by } G_*(x,\lambda) = \lambda G(x). \tag{5.62}$$

From Lemmas 5.2, 5.3, 5.4, 5.5, 5.6, and 5.7 it follows that $(f_*, G_*, X(\varkappa_0) \times [0,1])$ satisfies the conditions of Theorem 4.21, hence it is τ-admissible. Applying the homotopy invariance principle (Theorem 4.15) we obtain

$$\operatorname{Ind}(f, G, X(\varkappa_0)) = \operatorname{Ind}(f, 0, X(\varkappa_0)). \tag{5.63}$$

To evaluate $\operatorname{Ind}(f,0,X(\varkappa_0))$ consider $f^{-1}(0)$. This set consists of all functions $x \in Z(\varkappa_1)$ such that $x(0) = x(1) = 0$ and $L\ddot{x}(t) \equiv 0$. Hence each $\ddot{x}$ is constant equal to a root of L and the corresponding function x is a regular value of f. Taking into account condition (L), we conclude that the number of such functions is odd and hence $\operatorname{Ind}(f,0,X(\varkappa_0)) = 1$ (see [10]). Now applying Proposition 4.8, we conclude that the set $\Sigma = \operatorname{Coin}(f,G)$ is nonempty and compact. Let $x_* \in \Sigma$ be the minimizer of j_0:

$$j_0(x_*) = \min_{x \in \Sigma} j(x). \tag{5.64}$$

The existence of the corresponding control function u_* follows from Filippov implicit function lemma (cf. [5, Theorem 1.3.3]). □

5.2. Example. Consider the optimal control problem described by the relations

$$\begin{aligned} [\ddot{x}(t)]^3 - 3\ddot{x}(t) &= \sin\left(|\ddot{x}(t)| - \frac{\pi}{2}t\right) \cdot \cos\left(x(t)\dot{x}(t)\right) \\ &\quad + \frac{2}{9\pi}\int_0^t \sin\left[x(s)\cdot\ddot{x}(s)\right]\arctan\left[\dot{x}(s)\cdot u(s)\right] ds, \\ u(t) &\in \left[-x(t)e^{-at}, x(t)e^{-at}\right], \qquad x(0) = c_0; \qquad x(1) = c_1, \\ j_0(x) &= \int_0^1 \left(\|x(s)\|^2 + \|\dot{x}(s)\|^2 + \|\ddot{x}(s)\|^2\right) ds \longrightarrow \min. \end{aligned} \tag{5.65}$$

Here $Ly = y^3 - 3y$, $L'(y) = 3y^2 - 3$, $y_1 = -1$, $y_2 = 1$ and we can take $\varkappa_1 = 1/2$. Then

$$W_{\varkappa_1}(S) = \left(-\frac{3}{2}, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{3}{2}\right). \tag{5.66}$$

It is easy to see that

$$|Ly| > \frac{9}{8} \quad \text{for } y \in W_{\varkappa_1}(S). \tag{5.67}$$

Furthermore,

$$\left| \sin\left(|y| - \frac{\pi}{2}t\right)\cos(v \cdot w)\right| + \frac{2}{9\pi}\int_0^t \frac{\pi}{2}\,ds \le \frac{10}{9}, \tag{5.68}$$

hence, condition (L$\varphi\psi$) is fulfilled. A direct evaluation gives

$$\begin{gathered} \bar{D}_0(\varkappa_1) = \left(-\infty, -\frac{3}{2}\right], \qquad \bar{D}_1(\varkappa_1) = \left[-\frac{1}{2}, \frac{1}{2}\right], \qquad \bar{D}_2(\varkappa_1) = \left[\frac{3}{2}, +\infty\right), \\ k_0(\varkappa_1) = \frac{15}{4}, \qquad k_1(\varkappa_1) = \frac{9}{4}, \qquad k_2(\varkappa_1) = \frac{15}{4}. \end{gathered} \tag{5.69}$$

Choose now $\varkappa_0 < \varkappa_1$ so that $k_i(\varkappa_0) \ge 2$ for $i = 0, 1, 2$.

Then, for $y_0, y_1 \in \bar{D}_i(\varkappa_0)$ we have

$$\begin{aligned} &|\varphi(t,v,w,y_1) - \varphi(t,v,w,y_0)| \\ &\quad = \left|\sin\left(|y_1| - \frac{\pi}{2}t\right)\cos(v\cdot w) - \sin\left(|y_0| - \frac{\pi}{2}t\right)\cos(v\cdot w)\right| \\ &\quad \le \left|\sin\left(|y_1| - \frac{\pi}{2}t\right) - \sin\left(|y_0| - \frac{\pi}{2}t\right)\right| \\ &\quad \le \big||y_1| - |y_0|\big| \le |y_1 - y_0| < k_i(\varkappa_0)|y_1 - y_0|, \end{aligned} \tag{5.70}$$

hence condition (φ_2) is satisfied.

All other conditions of Theorem 5.1 can be easily checked up. Finally, we can conclude that problem (5.65) has a solution.

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