SEMILINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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We establish the global existence of mild solutions to a class of nonlocal Cauchy problems associated with semilinear Volterra integrodifferential equations in a Banach space.

1. Introduction

We investigate the global existence of solutions to integrodifferential equations with nonlocal conditions of the general form

$$u'(t) + \int_0^t a(t-s)Au(s)\,ds = F(u)(t), \quad 0 < t < T,$$

$$u(0) = g(u) \tag{1.1}$$

in a Banach space X. Here, A is a closed, densely-defined linear unbounded operator on X, $g : C([0, T]; X) \to X$, $a \in L^1(0, T)$, and $F : C([0, T]; X) \to L^p(0, T; X)$ ($p \in [1, \infty]$) is a continuous hereditary mapping. The unknown u takes values in X. As a special case of (1.1), we consider the problem

$$u'(t) + \int_0^t a(t-s)\big[Au(s) + f\big(s, u(s)\big)\big]\,ds = h(t), \quad 0 < t < T, \tag{1.2a}$$

$$u(0) = g(u), \tag{1.2b}$$

where $h \in L^1(0, T; X)$ and $f : [0, T] \times X \to X$. This is obtained if one takes $F(u)(t) = h(t) - \int_0^t a(t-s)f(s, u(s))\,ds$ in (1.1).

Such problems are important from the viewpoint of applications since they cover nonlocal generalizations of integrodifferential equations arising in the mathematical modeling of heat conduction in materials with memory. Byszewski [6, 7] initiated the work concerning abstract nonlocal semilinear initial-value problems. He used fixed-point methods to prove the existence and uniqueness of mild solutions to the Cauchy

problem

$$u'(t) + Au(t) = f(t, u(t)), \quad 0 < t < T, \tag{1.3a}$$

$$u(0) + g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) = u_0, \tag{1.3b}$$

where $p \in \mathbb{N}$, $0 < t_1 < \cdots < t_p \leq T$ are given, $u_0 \in X$, $-A$ generates a linear C_0-semigroup on X, while $f : [0, T] \times X \to X$ and $g : [0, T]^p \times X^p \to X$ satisfy Lipschitz conditions. See also [4, 5, 8, 9, 13] for related results, including applications to integrodifferential equations. Next, Ntouyas and Tsamatos [17, 18] studied nonlocal semilinear problems in the absence of Lipschitz conditions by using compactness arguments. The extension of [6, 7] to the nonlinear case was carried out by Aizicovici and Gao [1]. Recently, Aizicovici and McKibben [3] have extended the results in [17, 18] to the fully nonlinear case under the crucial assumption that $-A$ generates a compact nonlinear semigroup on X.

The paper most closely related to the present one is by Lin and Liu [15]. They developed an existence theory for the nonlocal integrodifferential equation

$$u'(t) + A\left[u(t) + \int_0^t a(t-s)u(s)\,ds \right] = f(t, u(t)), \quad 0 < t < T,$$
$$u(0) + g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) = u_0 \tag{1.4}$$

in X. Here A, g, f, and u_0 are as in (1.3), and $a(t), 0 < t < T$, is a bounded linear operator on X.

Our results concerning (1.1) and (1.2) can be viewed as a counterpart to the results in [15], in the sense that the operator A appears now only behind the integral sign. We first establish the existence and uniqueness of mild solutions to (1.1) under Lipschitz conditions on F and g. We then replace the Lipschitz conditions by weaker sublinear growth conditions, at the expense of some compactness restrictions.

The outline of the paper is as follows. In Section 2, we review some basic facts concerning resolvent operators and mild solutions for abstract linear integrodifferential equations. Section 3 is concerned with the existence and uniqueness of solutions to (1.1) under Lipschitz conditions on F and g, while Section 4 is devoted to the existence theory for (1.1) and (1.2) under compactness assumptions. Finally, two examples involving integro-partial differential equations are discussed in Section 5.

2. Preliminaries

For further background and details of this section, we refer the reader to [10, 11, 12, 19, 20] and the references therein. Let X be a Banach space with norm $\| \cdot \|$, and $A : D(A) \subset X \to X$ be a linear, closed operator with dense domain $D(A)$. Consider the Volterra integrodifferential equation

$$u'(t) + \int_0^t a(t-s)Au(s)\,ds = h(t), \quad 0 \leq t \leq T,$$
$$u(0) = u_0 \in D(A), \tag{2.1}$$

where $a \in L^1(0, T)$ and $h \in L^1(0, T; X)$. It is easily seen that (2.1) can be rewritten in the equivalent form

$$u(t) + b * Au(t) = H(t), \quad 0 \le t \le T, \tag{2.2}$$

where $b(t) = \int_0^t a(s) \, ds$, $H(t) = u_0 + \int_0^t h(s) \, ds$, and "$*$" denotes the convolution over $(0, t)$.

Definition 2.1. A function $u \in C([0, T]; X)$ is called a *mild solution* of (2.1) if $b * u \in C([0, T]; Y)$ and $u(t) = H(t) - A(b * u)(t)$, for all $t \in [0, T]$, where Y stands for $D(A)$ equipped with the graph norm.

Definition 2.2. A family $\{S(t)\}_{0 \le t \le T}$ of bounded linear operators in X is said to be a *resolvent* for (2.1) (equivalently (2.2)) if the following conditions are satisfied:

   (i) $S(t)$ is strongly continuous on $[0, T]$ and $S(0) = I$;
   (ii) $S(t) D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$, for every $x \in D(A)$, $0 \le t \le T$;
   (iii) $S(t)x = x - \int_0^t b(t - s) AS(s)x \, ds$, for all $x \in D(A)$, $0 \le t \le T$.

Note that $\|S(t)\|_{\mathcal{B}(X)} \le M_S$ on $[0, T]$ due to (i) and the uniform boundedness principle. (Here $\mathcal{B}(X)$ designates the space of all bounded linear operators on X.)

   General conditions on a and A that guarantee the existence and uniqueness of a resolvent family for (2.1) can be found in [12].

   It is convenient to represent a mild solution of (2.1) using a variation of parameters type formula involving the resolvent $S(t)$. Specifically, we have the following proposition.

PROPOSITION 2.3. *Suppose that (2.1) admits a resolvent $S(t)$. Then, for any $x_0 \in X$ and $h \in L^1(0, T; X)$, equation (2.1) has a unique mild solution u, given by*

$$u(t) = S(t)u_0 + \int_0^t S(t - s)h(s) \, ds, \quad \forall t \in [0, T]. \tag{2.3}$$

Next we consider (2.1) under the following stronger restrictions on a and A:

(C1) $a \in L^1(0, T)$ is positive, nonincreasing, and convex;
(C2) A is a linear, densely-defined, closed, invertible operator such that A^{-1} is compact on X;
(C3) either
    (i) $-a'$ is convex, X is a separable Hilbert space, and A is self-adjoint and strictly positive definite, or
    (ii) a is log-convex (with a' locally absolutely continuous on $(0, T]$ in the case when $a(0^+) < \infty$), and $-A$ generates a strongly continuous cosine family on X.

These conditions ensure (cf. [10, 11, 14, 19]) that (2.1) has a resolvent $S(t)$, in the sense of Definition 2.2. Now let $\Phi : L^1(0, T; X) \to C([0, T]; X)$ be defined by

$$\Phi(v) = S * v, \quad \forall v \in L^1(0, T; X). \tag{2.4}$$

The following two compactness results have been established in [2].

PROPOSITION 2.4. *Let conditions (C1), (C2), and (C3)(either (i) or (ii)) be satisfied. In addition, assume that*

$$a'(0^+) = -\infty. \tag{2.5}$$

Then Φ maps each uniformly integrable subset of $L^1(0, T; X)$ into a precompact subset of $C([0, T]; X)$. In particular, Φ is compact as a map of $L^p(0, T; X)$ $(1 < p \leq \infty)$ into $C([0, T]; X)$.

PROPOSITION 2.5. *Let conditions (C1), (C2), and (C3)(either (i) or (ii)) be satisfied. In addition, assume*

$$a(0^+) < \infty. \tag{2.6}$$

Then the mapping $v \mapsto \Phi(a * v)$ from $L^1(0, T; X)$ into $C([0, T]; X)$ is compact.

Finally, we recall a useful alternative of the Leray-Schauder principle, which will play a key role in Section 4.

THEOREM 2.6 (Schaefer's fixed point theorem [21]). *Let $\mathscr{F} : X \to X$ be a continuous, compact map, and let $\xi(\mathscr{F}) = \{x \in X : \lambda x = \mathscr{F}x, \text{ for some } \lambda \geq 1\}$. If $\xi(\mathscr{F})$ is bounded, then $\mathscr{F}$ has a fixed point.*

3. The case of a Lipschitz continuous nonlinearity

Consider (1.1) in a Banach space X under the following assumptions:

(H1) $A : D(A) \subseteq X \to X$ is a closed, linear, densely-defined, unbounded operator;
(H2) $a \in L^1(0, T)$;
(H3) equation (2.1) admits a resolvent $S(t)$, with $\max_{t \in [0, T]} \|S(t)\|_{\mathscr{B}(X)} = M_S$;
(H4) $F : C([0, T]; X) \to L^p(0, T; X)$, $1 \leq p \leq \infty$, is a hereditary map; that is, for each fixed $t \in [0, T]$ and $u \in C([0, T]; X)$, the value $F(u)(t)$ is allowed to depend only on the restriction of u on $[0, t]$. In addition, there exists $M_F > 0$ such that

$$\|F(u) - F(v)\|_{L^p(0, T; X)} \leq M_F \|u - v\|_{C([0, T]; X)}, \quad \forall u, v \in C([0, T]; X); \tag{3.1}$$

(H5) $g : C([0, T]; X) \to X$ satisfies the following condition: there exists $M_g > 0$ such that $\|g(u) - g(v)\| \leq M_g \|u - v\|_{C([0, T]; X)}$, for all $u, v \in C([0, T]; X)$;
(H6) $M_S(M_g + M_F T^{1/q}) < 1$, where q is the conjugate of p (i.e., $p^{-1} + q^{-1} = 1$).

Remark 3.1. Naturally, by a mild solution $u \in C([0, T]; X)$ to (1.1), we mean a mild solution to (2.1) with $u_0 = g(u)$ and $h = F(u)$. According to Proposition 2.3, u is a mild solution of (1.1) if and only if

$$u(t) = S(t)g(u) + \int_0^t S(t - s)F(u)(s) \, ds, \quad \forall t \in [0, T]. \tag{3.2}$$

THEOREM 3.2. *Assume that (H1)–(H6) are satisfied. Then (1.1) has a unique mild solution.*

Proof. Let $v \in C([0, T]; X)$ and consider the problem

$$u'(t) + \int_0^t a(t-s) Au(s) \, ds = F(v)(t), \quad 0 < t < T,$$
$$u(0) = g(v).$$
(3.3)

Define the map $\mathscr{F} : C([0, T]; X) \to C([0, T]; X)$ by $\mathscr{F}v = u_v$, where u_v is the mild solution to (3.3). We show that $\mathscr{F}$ is a strict contraction. To this end, let $v, w \in C([0, T]; X)$. By using (2.3), (H2), (H3), (H4), and (H5), together with Hölder's inequality, we obtain

$$\|u_v - u_w\|_{C([0,T];X)} \leq M_S M_g \|v - w\|_{C([0,T];X)} + M_S \int_0^t \|(Fv)(s) - (Fw)(s)\| \, ds$$

$$\leq M_S M_g \|v - w\|_{C([0,T];X)} + M_S T^{1/q} \|Fv - Fw\|_{L^p(0,T;X)}$$

$$\leq M_S (M_g + M_F T^{1/q}) \|v - w\|_{C([0,T];X)}.$$
(3.4)

Since $M_S(M_g + M_F T^{1/q}) < 1$ (cf. (H6)), we conclude that $\mathscr{F}$ is indeed a strict contraction. So, by the contraction mapping principle, $\mathscr{F}$ has a unique fixed point u, which is clearly the mild solution to (1.1) that we seek. This completes the proof. $\qquad\square$

Very often, the functional F is generated by a function $f : (0, T) \times X \to X$ satisfying the Carathéodory conditions. Specifically, we consider (1.2), where $h \in L^1(0, T; X)$ and f satisfies the following conditions:

(H7) (i) $t \mapsto f(t, x)$ is measurable for every $x \in X$, and $f(\cdot, 0) \in L^1(0, T; X)$,
    (ii) there exists $M_f \in L^1(0, T)$ such that $\|f(t, x) - f(t, y)\| \leq M_f(t) \|x - y\|$, for all $x, y \in X$ and almost all $t \in (0, T)$.

We claim that the existence and uniqueness of mild solutions to (1.2) can be obtained as a corollary of Theorem 3.2. Indeed, define $F(u)(t) = h(t) - \int_0^t a(t-s) f(s, u(s)) \, ds$, for all $u \in C([0, T]; X)$ and note that F maps $C([0, T]; X)$ into $L^1(0, T; X)$. Moreover, $\|Fu - Fv\|_{L^1(0,T;X)} \leq \|a * M_f\|_{L^1(0,T)} \|u - v\|_{C([0,T];X)}$, so that (H4) is satisfied with $M_F = \|a * M_f\|_{L^1(0,T)}$. Consequently, (H6) reads

(H8) $M_S(M_g + \|a * M_f\|_{L^1(0,T)}) < 1$.

An application of Theorem 3.2 yields the following result.

THEOREM 3.3. *Assume that (H1)–(H3), (H5), (H7), and (H8) are satisfied. Then, for each $h \in L^1(0, T; X)$, (1.2) has a unique mild solution.*

We conclude this section with some comments on (1.2) where g is given by

$$g(u) = u(T), \quad \forall u \in C([0, T]; X). \tag{3.5}$$

Obviously g, as given by (3.5), satisfies (H5) with $M_g = 1$. Since always $M_S \geq 1$, it is

clear that condition (H8) does not hold. To incorporate (3.5) in our theory, we consider that the kernel a, and the functions f and h are defined on $[0, \infty)$ (rather than on a fixed interval $[0, T]$). Specifically, we assume

(H9) $a \in L^1(0, \infty)$;

(H10) $f : [0, \infty) \times X \to X$ is continuous and satisfies (for some $M > 0$)

$$\| f(t, x) - f(t, y) \| \le M \| x - y \|, \quad \forall x, y \in X \text{ and } t \ge 0; \tag{3.6}$$

(H11) $h \in L^1_{\text{loc}}([0, \infty); X)$.

We also suppose that the pair (a, A) generates a resolvent $S(t)$, in the sense of Definition 2.2, on $[0, \infty)$ such that

(H12) $\| S(t) \|_{\mathcal{B}(X)} \le L e^{-\omega t}$, for all $t \ge 0$,

for some constants $L \ge 1$ and $\omega > 0$. For conditions on a and A that guarantee (H12), see [12] and [19, pages 42–44]. (Note, in particular, that (H9) and (H12) are compatible.) We can now prove that the problem

$$u'(t) + \int_0^t a(t - s) \big[Au(s) + f(s, u(s)) \big] ds = h(t), \quad 0 < t < T, \tag{3.7}$$
$$u(0) = u(T)$$

has a unique mild solution, provided that T is large enough.

THEOREM 3.4. *Let (H1), (H9)–(H13) be satisfied. If also*

(H13) $L \exp(-\omega T + LM \|a\|_{L^1(0,\infty)}) < 1$,

then (3.7) has a unique mild solution on $[0, T]$.

Proof. By standard arguments, it follows that for each (fixed) $T > 0$ and $x \in X$, the initial-value problem

$$u'(t) + \int_0^t a(t - s) \big[Au(s) + f(s, u(s)) \big] ds = h(t), \quad 0 < t < T, \tag{3.8}$$
$$u(0) = x$$

has a unique mild solution u_x on $[0, T]$, with u_x given by

$$u_x(t) = S(t)x + S * h(t) - S * \big(a * f(\cdot, u_x(\cdot)) \big)(t). \tag{3.9}$$

On account of (H9), (H10), and (H12), (3.9) yields

$$\big\| u_x(t) - u_y(t) \big\| \le L e^{-\omega t} \| x - y \| + LM \int_0^t e^{-\omega(t-s)} \|a\|_{L^1(0,s)} \big\| u_x(s) - u_y(s) \big\| ds, \tag{3.10}$$

for all $x, y \in X$ and all $t \in [0, T]$. Employing (H9) and a Gronwall type inequality in (3.10) (cf., [15, Lemma 4.2]), we arrive at

$$\big\| u_x(T) - u_y(T) \big\| \le L \exp \big(-\omega T + LM \|a\|_{L^1(0,\infty)} \big) \| x - y \|, \quad \forall x, y \in X. \tag{3.11}$$

Define $Q_T : X \to X$ by $Q_T x = u_x(T)$, for all $x \in X$, and observe that (3.11) and (H13) imply that Q_T is a contraction on X for a sufficiently large T. Therefore, if T is chosen such that (H13) is satisfied, Q_T has a unique fixed point x_0. The corresponding $u_{x_0} = u$ is obviously the (unique) mild solution of (3.7) and the proof is complete.   □

Remark 3.5. It is easy to generalize Theorem 3.4 to the case when (1.2a) is coupled with a nonlocal condition of the form (1.3b). The details are left to the reader.

4. The case when A^{-1} is compact

We now investigate (1.1) in the case where the Lipschitz conditions on F and g are dropped. The following assumptions will be used instead.

(H14) $F : C([0, T]; X) \to L^p(0, T; X)$ ($p \in (1, \infty]$) is a continuous hereditary map satisfying

$$\|Fv\|_{L^p(0,T;X)} \le c_1 \|v\|_{C([0,T];X)} + c_2, \quad \forall v \in C([0, T]; X), \qquad (4.1)$$

where $c_1, c_2 > 0$;

(H15) $g : C([0, T]; X) \to X$ is a continuous, compact map such that

$$\|g(u)\| \le d_1 \|u\|_{C([0,T];X)} + d_2, \quad \forall u \in C([0, T]; X), \qquad (4.2)$$

where $d_1, d_2 > 0$.

THEOREM 4.1. *Assume that (C1), (C2), (C3)(either (i) or (ii)), (2.5), (H14), and (H15) are satisfied. If, in addition, $1 - M_S(d_1 + c_1 T^{1/q}) > 0$, then (1.1) has at least one mild solution on $[0, T]$.*

Proof. Consider (3.3) and define the solution map $\mathscr{F}$ as in the proof of Theorem 3.2. We appeal to Schaefer's theorem to prove that $\mathscr{F}$ has a fixed point. The continuity of $\mathscr{F}$ is easily verified. We show that $\mathscr{F}$ is a compact map. Let $K_r = \{v \in C([0, T]; X) : \|v\|_{C([0,T];X)} \le r\}$ and observe that (cf. (2.3) and (2.4)) $\mathscr{F}v = S(\cdot)g(v) + \Phi(F(v))$, for all $v \in K_r$. We claim that $\mathscr{F}(K_r)$ is precompact in $C([0, T]; X)$. First, $\{\Phi(F(v)) : v \in K_r\}$ is precompact in $C([0, T]; X)$ on account of Proposition 2.4 since $\{F(v) : v \in K_r\}$ is a bounded subset of $L^p(0, T; X)$ (cf. (H14)). It remains to show that $\{S(\cdot)g(v) : v \in K_r\}$ is precompact in $C([0, T]; X)$. To this end, note that by (H15) the set $K = \{g(v) : v \in K_r\}$ is precompact in X. Let $\tilde{K} = S(\cdot)K (\subset C([0, T]; X))$ and $\epsilon > 0$. Since K is precompact in X, there exist $\{x_1, \ldots, x_n\} \subset K$ such that $K \subset \cup_{i=1}^n B(x_i, \epsilon/M_S)$. (Here, $B(x_i, \epsilon/M_S)$ denotes the ball in X with radius ϵ/M_S centered at x_i.) Then $\tilde{K} \subset \cup_{i=1}^n S(\cdot)B(x_i, \epsilon/M_S)$. Set $\tilde{x}_i = S(\cdot)x_i (\in C([0, T]; X))$ and $\tilde{B}_i = \{y \in C([0, T]; X) : \|y - \tilde{x}_i\|_{C([0,T];X)} < \epsilon\}$. If $z \in \tilde{K}$, there exists $\xi \in K$ such that $z = S(\cdot)\xi$. Since $\xi \in K$, there is an $i \in \{1, \ldots, n\}$ such that $\|\xi - x_i\| < \epsilon/M_S$. Observe that $\|z - \tilde{x}_i\|_{C([0,T];X)} = \|S(\cdot)\xi - S(\cdot)x_i\|_{C([0,T];X)} \le M_S \|\xi - x_i\| < \epsilon$. It follows that $\tilde{K} \subset \cup_{i=1}^n \tilde{B}_i$ and therefore, it is totally bounded. Thus, $\tilde{K}$ is precompact in $C([0, T]; X)$, as desired.

To apply Schaefer's fixed point theorem, we also need to show that the set $\xi(\mathscr{F})$, as defined in Theorem 2.6 (with $C([0, T]; X)$ in place of X) is bounded. If $v \in \xi(\mathscr{F})$ and

$0 \leq t \leq T$, we have by (3.2), (H14), and (H15)

$$
\begin{aligned}
\lambda \|v(t)\| &\leq \|S(t)\|\|g(v)\| + \int_0^t \|S(t-s)\|\|(Fv)(s)\|\,ds \\
&\leq M_S\big[d_1\|v\|_{C([0,T];X)} + d_2 + T^{1/q}\|Fv\|_{L^p(0,T;X)}\big] \\
&\leq M_S\big[d_1 + c_1 T^{1/q}\big]\|v\|_{C([0,T];X)} + M_S\big[d_2 + c_2 T^{1/q}\big],
\end{aligned}
\tag{4.3}
$$

where M_S has the same meaning as in (H3). Taking into account that $\lambda \geq 1$ and $M_S[d_1 + c_1 T^{1/q}] < 1$ we conclude that $\|v\|_{C([0,T];X)} \leq C$. (We hereafter employ C to denote a generic positive constant.) Hence, $\xi(\mathscr{F})$ is bounded and consequently $\mathscr{F}$ has a fixed point $u \in C([0,T];X)$. Obviously, u satisfies (3.2) and the proof is complete. □

Next, we consider (1.2) under conditions similar to those in Theorem 4.1. Precisely, we replace assumption (H14) by

(H16) $f : [0,T] \times X \to X$ is a map satisfying the Carathéodory conditions (i.e., f is measurable in t and continuous in x) and $\|f(t,x)\| \leq c_1(t)\|x\| + c_2(t)$, for almost all $t \in (0,T)$ and all $x \in X$, where $c_1, c_2 \in L^1(0,T)$.

THEOREM 4.2. *Assume that (C1), (C2), (C3)(either (i) or (ii)), (2.6), and (H16) are satisfied. If, in addition, $1 - M_S(d_1 + \|a * c_1\|_{L^1(0,T)}) > 0$ and $h \in L^1(0,T;X)$, then (1.2) has at least one mild solution on $[0,T]$.*

Proof. Let $v \in C([0,T];X)$ and consider the problem

$$
\begin{aligned}
u'(t) + \int_0^t a(t-s)\big[Au(s) + f\big(s,v(s)\big)\big]\,ds &= h(t), \quad 0 < t < T, \\
u(0) &= g(v).
\end{aligned}
\tag{4.4}
$$

Define $\mathscr{F} : C([0,T];X) \to C([0,T];X)$ by $\mathscr{F}v = u_v$, where u_v is the mild solution of (4.4). Invoking (2.3) and (2.4), we can write, for $v \in C([0,T];X)$,

$$
(\mathscr{F}v)(t) = S(t)g(v) + S * h(t) - \Phi\big(a * f\big(\cdot, v(\cdot)\big)\big)(t).
\tag{4.5}
$$

Again, we use Schaefer's theorem to establish that $\mathscr{F}$ has at least one fixed point. The continuity of $\mathscr{F}$ follows easily. Let $r > 0$ and define the set K_r as in the proof of Theorem 4.1. We show that $\mathscr{F}(K_r)$ is precompact in $C([0,T];X)$. Since $\{S(\cdot)g(v) : v \in K_r\}$ was earlier shown to be precompact in $C([0,T];X)$ and $S * h$ is independent of v, we only focus on the last term on the right-hand side of (4.5). By (H16), we have $\|f(\cdot, v(\cdot))\|_{L^1(0,T;X)} \leq \|c_1\|_{L^1(0,T)} r + \|c_2\|_{L^1(0,T)}$, for all $v \in K_r$. Hence, Proposition 2.5 implies that $\{\Phi(a * f(\cdot, v(\cdot))) : v \in K_r\}$ is precompact in $C([0,T];X)$. Thus, $\mathscr{F}$ is a compact map.

Finally, let $\xi(\mathscr{F}) = \{v \in C([0,T];X) : \lambda v = \mathscr{F}v, \text{ for some } \lambda \geq 1\}$ and let $v \in \xi(\mathscr{F})$. We seek a constant C, independent of λ and v, such that $\|v\|_{C([0,T];X)} \leq C$, for all

$v \in \xi(\mathcal{F})$. Owing to (H15) and (H16), we have for $t \in [0, T]$,

$$
\begin{aligned}
\lambda \|v(t)\| &\leq \|S(t)g(v)\| + \int_0^t \|S(t-s)\| \left[\|h(s)\| + \int_0^s |a(s-\tau)| \|f(\tau, v(\tau))\| d\tau \right] ds \\
&\leq M_S \big(d_1 \|v\|_{C([0,T];X)} + d_2 \big) + M_S \|h\|_{L^1(0,T)} \\
&\quad + M_S \Big(\|a * c_1\|_{L^1(0,T)} \|v\|_{C([0,T];X)} + \|a\|_{L^1(0,T)} \|c_2\|_{L^1(0,T)} \Big) \\
&= M_S \bigg[\big(d_1 + \|a * c_1\|_{L^1(0,T)} \big) \|v\|_{C([0,T];X)} + d_2 + \|h\|_{L^1(0,T)} \\
&\quad + \|a\|_{L^1(0,T)} \|c_2\|_{L^1(0,T)} \bigg].
\end{aligned}
\tag{4.6}
$$

Recalling that $\lambda \geq 1$ and $M_S(d_1 + \|a * c_1\|_{L^1(0,T)}) < 1$, (4.6) yields the desired bound for $\|v\|_{C([0,T];X)}$. Therefore, we conclude by Theorem 2.6 that $\mathcal{F}$ has a fixed point, which is a mild solution to (1.2). This completes the proof. $\qquad \square$

We can further weaken the assumption on f by replacing (H16) with

(H17) $f : [0, T] \times X \to X$ satisfies the Carathéodory conditions, as well as
    (i) $\forall k \in \mathbb{N}$, $\exists g_k \in L^1(0, T; \mathbb{R}^+)$ such that for a.a. $t \in (0, T)$,
        $\sup_{\|x\| \leq k} \|f(t, x)\| \leq g_k(t)$;
    (ii) $\underline{\lim} k^{-1} \int_0^T (|a| * g_k)(s) \, ds = \beta < \infty$.

THEOREM 4.3. *Assume that (C1), (C2), (C3)(either (i) or (ii)), (2.6), (H15), and (H17) are satisfied. If, in addition, $M_S(d_1 + \beta) < 1$, then for every $h \in L^1(0, T; X)$, (1.2) has at least one mild solution on $[0, T]$.*

Proof. Consider (4.4) and define the solution map $\mathcal{F}$ by (4.5). We now use Schauder's fixed point theorem to show that $\mathcal{F}$ has a fixed point. First, we remark that $\mathcal{F}$ is continuous and compact in $C([0, T]; X)$; see the proof of Theorem 4.2. It remains to prove that $\mathcal{F}$ maps some ball in $C([0, T]; X)$ into itself. For each $n \in \mathbb{N}$, let $B_n = \{x \in C([0, T]; X) : \|x\|_{C([0,T];X)} \leq n\}$. We show that there is an $n \in \mathbb{N}$ such that $\mathcal{F} B_n \subseteq B_n$. Suppose, by contradiction, that for each $k \in \mathbb{N}$, there exists $u_k \in B_k$ such that $\mathcal{F} u_k \notin B_k$. Then

$$
1 \leq \varlimsup_{k \to \infty} k^{-1} \|\mathcal{F} u_k\|_{C([0,T];X)}.
\tag{4.7}
$$

Observe that

$$
\|\mathcal{F} u_k(t)\| \leq \|S(t)\| \|g(u_k)\| + \|S\| * \big(\|h\| + |a| * \|f(\cdot, u_k(\cdot))\| \big)(t), \quad \forall t \in [0, T].
\tag{4.8}
$$

Since $u_k \in B_k$, there exists $g_k \in L^1(0, T)$ such that $\|f(\tau, u_k(\tau)\| \leq g_k(\tau))$ for a.a. $\tau \in (0, T)$ (cf. (H17)(i)). Then, (4.8) leads to

$$
\|\mathcal{F} u_k(t)\| \leq M_S \Big(\|g(u_k)\| + \|h\|_{L^1(0,T;X)} + \|a * g_k\|_{L^1(0,T)} \Big), \quad \forall t \in [0, T].
\tag{4.9}
$$

Using (H15) and (H17)(ii) in (4.9), we arrive at $\underline{\lim}_{k\to\infty} k^{-1}\|\mathscr{F}u_k\|_{C([0,T];X)} \leq M_S(d_1+\beta) < 1$. This contradicts (4.7), so there must exist an $n_0 \in \mathbb{N}$ such that $\mathscr{F}B_{n_0} \subseteq B_{n_0}$. Thus, Schauder's fixed point theorem is applicable and the proof is complete. □

5. Examples

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary Γ. Consider the initial-boundary value problem

$$u_t(t,x) + \int_0^t a(t-s)\Big[-\Delta_x u(s,x) + \tilde{f}\big(s,u(s,x)\big)\Big]ds = h(t,x), \quad \text{a.e. on } (0,T)\times\Omega,$$
(5.1)

$$u(t,x) = 0, \quad \text{a.e. on } (0,T)\times\Gamma,$$
(5.2)

$$u(0,x) = \sum_{i=1}^p c_i u(t_i,x), \quad \text{a.e. on } \Omega.$$
(5.3)

Here, $0 < t_1 < t_2 < \cdots < t_p \leq T$ are given, $c_i > 0$ ($i = 1,\ldots,p$), $h : [0,T]\times\Omega \to \mathbb{R}$, $a : (0,T) \to \mathbb{R}$, and $\tilde{f} : [0,T]\times\Omega \to \mathbb{R}$.

THEOREM 5.1. *Let (C1) be satisfied. In addition, assume:*
  (H18) $h \in L^1(0,T;L^2(\Omega))$;
  (H19) $\tilde{f}$ *satisfies the Carathéodory conditions, $\tilde{f}(t,0) \in L^1(0,T)$, and there exists* $M_{\tilde{f}} \in L^1(0,T)$ *such that* $|\tilde{f}(t,x) - \tilde{f}(t,y)| \leq M_{\tilde{f}}(t)|x-y|$, $\forall x,y \in \Omega$, *a.e. on $(0,T)$*;
  (H20) $\sum_{i=1}^p c_i + \|M_{\tilde{f}}*a\|_{L^1(0,T)} < 1$.
Then the problem (5.1),(5.2), and (5.3) has a unique mild solution $u \in C([0,T];L^2(\Omega))$.

Proof. Let $X = L^2(\Omega)$ and define $A : D(A) \subseteq X \to X$ by

$$Au = -\Delta u, \quad D(A) = H^2(\Omega)\cap H_0^1(\Omega).$$
(5.4)

It is well known that A is a positive definite, self-adjoint operator in X. Moreover, by (C1) and [20, page 38], condition (H3) is satisfied with $M_S = 1$. Next, $\tilde{f}$ generates a function $f : [0,T]\times X \to X$ by the formula $f(t,v)(x) = \tilde{f}(t,v(x))$, a.e. on $(0,T)\times\Omega$, for all $v \in X$. Clearly, by (H19), f satisfies (H7). Finally, define $g : C([0,T];X) \to X$ by $g(u) = \sum_{i=1}^p c_i u(t_i,x)$ and observe that $\|g(u) - g(v)\| \leq M_g \|u-v\|_{C([0,T];X)}$, with $M_g = \sum_{i=1}^p c_i$ (that is, (H5) holds). Assumption (H8) is also satisfied because of (H20).

We can now rewrite (5.1), (5.2), and (5.3) in the form (1.2) in X, and apply Theorem 3.3 to conclude that the problem (5.1), (5.2), and (5.3) has a unique mild solution $u \in C([0,T];L^2(\Omega))$.

Next, consider (5.1) (with $\tilde{f}(s,u(s,x))$ replaced by $\tilde{f}(s,x,u(s,x))$) and (5.2), together with the new boundary condition

$$u(0,x) = \int_\Omega \int_0^T \tilde{g}\big(t,x,z,u(t,z)\big)dt\,dz \quad \text{a.e. on } \Omega,$$
(5.5)

where $\tilde{f} : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{g} : [0,T] \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ are given.

The following conditions will be imposed:

(H21) $\tilde{f} : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the Carathéodory conditions (that is, $\tilde{f}$ is measurable in (t,x) and continuous in the third variable), as well as the growth condition $|\tilde{f}(t,x,u)| \le m_1(t)|u| + m_2(t,x)$, for a.a. $(t,x) \in (0,T) \times \mathbb{R}$ and all $u \in \mathbb{R}$, where $m_1 \in L^1(0,T;\mathbb{R}^+)$ and $m_2 \in L^1(0,T;L^2(\Omega))$;

(H22) (i) for almost every $(t,x,z) \in (0,T) \times \Omega \times \Omega$, $\tilde{g}$ is a continuous function of r,

(ii) for each fixed $r \in \mathbb{R}$, $\tilde{g}$ is a measurable function of (t,x,z),

(iii) $|\tilde{g}(t,x,z,r) - \tilde{g}(t,\bar{x},z,r)| \le \gamma_k(t,x,\bar{x},r)$ for (t,x,z,r), $(t,\bar{x},z,r)$ in $[0,T] \times \Omega \times \Omega \times \mathbb{R}$ with $|r| \le k$, where $\gamma_k \in L^1((0,T) \times \Omega \times \Omega \times \mathbb{R};\mathbb{R}^+)$ satisfies $\lim_{x \to \bar{x}} \int_\Omega \int_0^T \gamma_k(t,x,\bar{x},r)\,dt\,dz = 0$ uniformly in $\bar{x} \in \Omega$,

(iv) $|\tilde{g}(t,x,z,r)| \le (\delta/Tm(\Omega))|r| + \sigma(t,x,z)$, $\forall r \in \mathbb{R}$, a.e. on $(0,T) \times \Omega \times \Omega$, where $\sigma \in L^2((0,T) \times \Omega \times \Omega;\mathbb{R}^+)$, $\delta > 0$, and $m(\Omega)$ denotes the Lebesgue measure of Ω;

(H23) $\beta + \delta < 1$, where $\beta = \|a * m_1\|_{L^1(0,T)}$. $\qquad\qquad\qquad\qquad$ $\square$

THEOREM 5.2. *Let (H18), (H21), (H22), and (H23) be satisfied. If also (C1) holds, and $-a'$ is convex, then the problem (5.1) (with $\tilde{f}(s,x,u)$ in place of $f(s,u)$), (5.2), (5.5) has at least one mild solution $u \in C([0,T];L^2(\Omega))$.*

Proof. Let $X = L^2(\Omega)$ and define A as in (5.4). Define $f : [0,T] \times X \to X$ and $g : C([0,T];X) \to X$ by

$$f(t,u)(x) = \tilde{f}(t,x,u(x)), \quad \forall u \in X, \text{ a.e. on } \Omega, \tag{5.6}$$

$$g(u)(x) = \int_\Omega \int_0^T \tilde{g}(t,x,z,u(t,z))\,dt\,dz, \quad \text{a.e. on } \Omega, \tag{5.7}$$

respectively, and remark that our problem can be rewritten in the abstract form (1.2) in X.

We show that Theorem 4.3 is applicable. First, note that from (5.6) and (H21), it follows that f is a well-defined mapping from $[0,T] \times X$ into X which satisfies the Carathéodory conditions. Let $k \in \mathbb{N}$ be fixed and $u \in X$ be such that $\|u\| \le k$. Then, (H21) implies that $\|f(t,u)\| \le m_1(t)k + \|m_2(t,\cdot)\| = g_k(t)$, with $g_k \in L^1(0,T;\mathbb{R}^+)$. Moreover,

$$\lim_{k \to \infty} k^{-1} \int_0^T (|a| * g_k)(s)\,ds = \lim_{k \to \infty} k^{-1} \int_0^T (|a| * (m_1(s)k + \|m_2(s,\cdot)\|))(s)\,ds$$

$$\le \lim_{k \to \infty} k^{-1} \left[k\|a * m_1\|_{L^1(0,T)} + \|a\|_{L^1(0,T)} \|m_2\|_{L^1(0,T;X)} \right]$$

$$= \|a * m_1\|_{L^1(0,T)} = \beta < \infty, \tag{5.8}$$

so that (H17) holds. Next, on account of (H22), it follows from Theorem 4.2 in [16, page 175] that g is a well-defined, continuous compact map. In addition, for $u \in C([0,T];X)$

we have (cf. (5.7) and (H22)(iv))

$$|g(u)(x)| \leq \frac{\delta}{Tm(\Omega)} \int_\Omega \int_0^T |u(t,z)| dt dz + \int_\Omega \int_0^T \sigma(t,x,z) dt dz$$

$$\leq \frac{\delta}{T\sqrt{m(\Omega)}} \int_0^T \left[\int_\Omega |u(t,z)|^2 dz \right]^{1/2} dt + \int_\Omega \int_0^T \sigma(t,x,z) dt dz \quad (5.9)$$

$$\leq \frac{\delta}{\sqrt{m(\Omega)}} \|u\|_{C([0,T];L^2(\Omega))} + \int_\Omega \int_0^T \sigma(t,x,z) dt dz.$$

Since $\sigma \in L^2((0,T) \times \Omega \times \Omega)$, we conclude that g satisfies all conditions of (H15). (In particular, $d_1 = \delta$.)

Finally, note that (C2) and (C3)(i) are fulfilled because of (5.4) and our assumptions on a, and that $M_S = 1$ in this case. Hence, all conditions of Theorem 4.3 are satisfied. As a result, the conclusion of Theorem 5.2 follows readily. $\square$

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