THE TURNPIKE PROPERTY FOR DYNAMIC DISCRETE TIME ZERO-SUM GAMES

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We consider a class of dynamic discrete-time two-player zero-sum games. We show that for a generic cost function and each initial state, there exists a pair of overtaking equilibria strategies over an infinite horizon. We also establish that for a generic cost function f, there exists a pair of stationary equilibria strategies (x_f, y_f) such that each pair of "approximate" equilibria strategies spends almost all of its time in a small neighborhood of (x_f, y_f).

1. Introduction

The study of variational and optimal control problems defined on infinite intervals has recently been a rapidly growing area of research [4, 6, 9, 10, 15, 16, 17]. These problems arise in engineering [1, 19], in models of economic dynamics [11, 13, 18], in continuum mechanics [5, 10, 12], and in game theory [3, 4, 7].

In this paper, we study the existence and the structure of "approximate" equilibria for dynamic two-player zero-sum games.

Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^m$. Let $X \subset \mathbb{R}^{m_1}$ and $Y \subset \mathbb{R}^{m_2}$ be nonempty convex compact sets. Denote by $\mathfrak{M}$ the set of all continuous functions $f : X \times X \times Y \times Y \to \mathbb{R}^1$ such that:

- for each $(y_1, y_2) \in Y \times Y$ the function $(x_1, x_2) \to f(x_1, x_2, y_1, y_2)$, $(x_1, x_2) \in X \times X$ is convex;
- for each $(x_1, x_2) \in X \times X$ the function $(y_1, y_2) \to f(x_1, x_2, y_1, y_2)$, $(y_1, y_2) \in Y \times Y$ is concave.

For the set $\mathfrak{M}$ we define a metric $\rho : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}^1$ by

$$\rho(f, g) = \sup\left\{\left|f\left(x_1, x_2, y_1, y_2\right) - g\left(x_1, x_2, y_1, y_2\right)\right| : x_1, x_2 \in X,\ y_1, y_2 \in Y\right\}, \quad f, g \in \mathfrak{M}. \tag{1.1}$$

Clearly $\mathfrak{M}$ is a complete metric space.

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Given $f \in \mathfrak{M}$ and an integer $n \geq 1$, we consider a discrete-time two-player zero-sum game over the interval $[0,n]$. For this game $\{\{x_i\}_{i=0}^n : x_i \in X, i = 0,\dots,n\}$ is the set of strategies for the first player, $\{\{y_i\}_{i=0}^n : y_i \in Y,\ i = 0,\dots,n\}$ is the set of strategies for the second player, and the cost for the first player associated with the strategies $\{x_i\}_{i=0}^n$, $\{y_i\}_{i=0}^n$ is given by $\sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_i, y_{i+1})$.

Definition 1.1. Let $f \in \mathfrak{M}$, $n \geq 1$ be an integer and let $M \in [0,\infty)$. A pair of sequences $\{\bar{x}_i\}_{i=0}^n \subset X$, $\{\bar{y}_i\}_{i=0}^n \subset Y$ is called (f,M)-good if the following properties hold:

(i) for each sequence $\{x_i\}_{i=0}^n \subset X$ satisfying $x_0 = \bar{x}_0$, $x_n = \bar{x}_n$

$$M + \sum_{i=0}^{n-1} f\big(x_i, x_{i+1}, \bar{y}_i, \bar{y}_{i+1}\big) \geq \sum_{i=0}^{n-1} f\big(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}\big); \tag{1.2}$$

(ii) for each sequence $\{y_i\}_{i=0}^n \subset Y$ satisfying $y_0 = \bar{y}_0$, $y_n = \bar{y}_n$

$$M + \sum_{i=0}^{n-1} f\big(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}\big) \geq \sum_{i=0}^{n-1} f\big(\bar{x}_i, \bar{x}_{i+1}, y_i, y_{i+1}\big). \tag{1.3}$$

If a pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ is $(f,0)$-good, then it is called (f)-optimal.

Our first main result in this paper deals with the so-called "turnpike property" of "good" pairs of sequences. To have this property means, roughly speaking, that the "good" pairs of sequences are determined mainly by the cost function, and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics and optimal control (see [11, 13, 15, 16, 17, 18, 19] and the references therein).

Consider any $f \in \mathfrak{M}$. We say that the function f has the *turnpike property* if there exists a unique pair $(x_f, y_f) \in X \times Y$ for which the following assertion holds.

For each $\epsilon > 0$ there exist an integer $n_0 \geq 2$ and a number $\delta > 0$ such that, for each integer $n \geq 2n_0$ and each (f,δ)-good pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ the relations $\|x_i - x_f\|$, $\|y_i - y_f\| \leq \epsilon$ holds for all integers $i \in [n_0, n-n_0]$.

In this paper, our goal is to show that the turnpike property holds for a generic $f \in \mathfrak{M}$. We prove the existence of a set $\mathfrak{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}$ such that each $f \in \mathfrak{F}$ has the turnpike property (see Theorem 2.1). Results of this kind for classes of single-player control systems have been established in [15, 16, 17]. Thus, instead of considering the turnpike property for a single function, we investigate it for a space of all such functions equipped with some natural metric, and show that this property holds for most of these functions. This allows us to establish the turnpike property without restrictive assumptions on the functions.

We also study the existence of equilibria over an infinite horizon for the class of zero-sum games considered in the paper. We employ the following version of the overtaking optimality criterion which was introduced in the economic literature by Gale [8] and von Weizsacker [14] and used in control and game theory [1, 3, 4, 19].

Definition 1.2. Let $f \in \mathfrak{M}$. A pair of sequences $\{\bar{x}_i\}_{i=0}^{\infty} \subset X$, $\{\bar{y}_i\}_{i=0}^{\infty} \subset Y$ is called (f)-*overtaking optimal* if the following properties hold:

(i) for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying $x_0 = \bar{x}_0$

$$\limsup_{T\to\infty}\left[\sum_{i=0}^{T-1} f\big(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}\big) - \sum_{i=0}^{T-1} f\big(x_i, x_{i+1}, \bar{y}_i, \bar{y}_{i+1}\big)\right] \le 0; \tag{1.4}$$

(ii) for each sequence $\{y_i\}_{i=0}^{\infty} \subset Y$ satisfying $y_0 = \bar{y}_0$

$$\limsup_{T\to\infty}\left[\sum_{i=0}^{T-1} f\big(\bar{x}_i, \bar{x}_{i+1}, y_i, y_{i+1}\big) - \sum_{i=0}^{T-1} f\big(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}\big)\right] \le 0. \tag{1.5}$$

Our second main result (see Theorem 2.2) shows that for a generic $f \in \mathfrak{M}$ and each $(x, y) \in X \times Y$ there exists an (f)-overtaking optimal pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ such that $x_0 = x$, $y_0 = y$.

2. Main results

In this section we present our main results.

THEOREM 2.1. *There exists a set $\mathfrak{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}$ such that for each $f \in \mathfrak{F}$ the following assertions hold.*

(1) *There exists a unique pair $(x_f, y_f) \in X \times Y$ for which*

$$\sup_{y\in Y} f\big(x_f, x_f, y, y\big) = f\big(x_f, x_f, y_f, y_f\big) = \inf_{x\in X} f\big(x, x, y_f, y_f\big). \tag{2.1}$$

(2) *For each $\epsilon > 0$ there exist a neighborhood U of f in $\mathfrak{M}$, an integer $n_0 \ge 2$, and a number $\delta > 0$ such that for each $g \in U$, each integer $n \ge 2n_0$, and each (g, δ)-good pair of sequences $\{x_i\}_{i=0}^{n} \subset X$, $\{y_i\}_{i=0}^{n} \subset Y$ the relation*

$$\|x_i - x_f\|,\ \|y_i - y_f\| \le \epsilon \tag{2.2}$$

holds for all integers $i \in [n_0, n-n_0]$. Moreover, if $\|x_0 - x_f\|, \|y_0 - y_f\| \le \delta$, then (2.2) holds for all integers $i \in [0, n-n_0]$, and if $\|x_n - x_f\|, \|y_n - y_f\| \le \delta$, then (2.2) is valid for all integers $i \in [n_0, n]$.

THEOREM 2.2. *There exists a set $\mathfrak{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}$ such that for each $f \in \mathfrak{F}$ the following assertion holds.*

For each $x \in X$ and each $y \in Y$ there exists an (f)-overtaking optimal pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ such that $x_0 = x$, $y_0 = y$.

3. Definitions and notations

Let $f \in \mathfrak{M}$. Define a function $\bar{f} : X \times Y \to \mathbb{R}^1$ by

$$\bar{f}(x, y) = f(x, x, y, y), \quad x \in X,\ y \in Y. \tag{3.1}$$

Then there exists a saddle point $(x_f, y_f) \in X \times Y$ for $\bar{f}$. We have

$$\sup_{y \in Y} \bar{f}\big(x_f, y\big) = \bar{f}\big(x_f, y_f\big) = \inf_{x \in X} \bar{f}\big(x, y_f\big). \tag{3.2}$$

Set

$$\mu(f) = \bar{f}\big(x_f, y_f\big). \tag{3.3}$$

Definition 3.1. Let $f \in \mathfrak{M}$. A pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ is called (f)-minimal if for each integer $n \geq 2$ the pair of sequences $\{x_i\}_{i=0}^{n}$, $\{y_i\}_{i=0}^{n}$ is (f)-optimal.

We show in Section 5 (see Proposition 5.3) that for each $f \in \mathfrak{M}$, each $x \in X$, and each $y \in Y$ there exists an (f)-minimal pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ such that $x_0 = x$, $y_0 = y$.

Let $f \in \mathfrak{M}$, $n \geq 1$ be an integer, and let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in X \times X \times Y \times Y$. Define

$$\Lambda_X(\xi, n) = \Big\{\{x_i\}_{i=0}^{n} \subset X : x_0 = \xi_1,\ x_n = \xi_2\Big\}, \tag{3.4}$$

$$\Lambda_Y(\xi, n) = \Big\{\{y_i\}_{i=0}^{n} \subset Y : y_0 = \xi_3,\ y_n = \xi_4\Big\}, \tag{3.5}$$

$$\begin{gathered} f^{(\xi,n)}\Big(\big(x_0, \ldots, x_i, \ldots, x_n\big), \big(y_0, \ldots, y_i, \ldots, y_n\big)\Big) = \sum_{i=0}^{n-1} f\big(x_i, x_{i+1}, y_i, y_{i+1}\big), \\ \{x_i\}_{i=0}^{n} \in \Lambda_X(\xi, n), \quad \{y_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n). \end{gathered} \tag{3.6}$$

4. Preliminary results

Let M, N be nonempty sets and let $f : M \times N \to \mathbb{R}^1$. Set

$$f^a(x) = \sup_{y \in N} f(x, y), \quad x \in M, \qquad f^b(y) = \inf_{x \in M} f(x, y), \quad y \in N, \tag{4.1}$$

$$v_f^a = \inf_{x \in M} \sup_{y \in N} f(x, y), \qquad v_f^b = \sup_{y \in N} \inf_{x \in M} f(x, y). \tag{4.2}$$

Clearly

$$v_f^b \leq v_f^a. \tag{4.3}$$

We have the following result (see [2, Chapter 6, Section 2, Proposition 1]).

Proposition 4.1. *Let $f : M \times N \to \mathbb{R}^1$, $\bar{x} \in M$, $\bar{y} \in N$. Then*

$$\sup_{y \in N} f(\bar{x}, y) = f(\bar{x}, \bar{y}) = \inf_{x \in M} f(x, \bar{y}) \tag{4.4}$$

if and only if

$$v_f^a = v_f^b, \quad \sup_{y \in N} f(\bar{x}, y) = v_f^a, \quad \inf_{x \in M} f(x, \bar{y}) = v_f^b. \tag{4.5}$$

Let $f: M \times N \to \mathbb{R}^1$. If $(\bar{x}, \bar{y}) \in M \times N$ satisfies (4.4) that it is called a saddle point (for f). We have the following result (see [2, Chapter 6, Section 2, Theorem 8]).

PROPOSITION 4.2. *Let $M \subset \mathbb{R}^m$, $N \subset \mathbb{R}^n$ be convex compact sets and let $f: M \times N \to \mathbb{R}^1$ be a continuous function. Assume that for each $y \in N$, the function $x \to f(x, y)$, $x \in M$ is convex and for each $x \in M$, the function $y \to f(x, y)$, $y \in N$ is concave. Then there exists a saddle point for f.*

PROPOSITION 4.3. *Let M, N be nonempty sets, $f: M \times N \to \mathbb{R}^1$ and*

$$-\infty < v_f^a = v_f^b < +\infty, \quad x_0 \in M,\ y_0 \in N,\ \Delta_1, \Delta_2 \in [0, \infty), \tag{4.6}$$

$$\sup_{y \in N} f\big(x_0, y\big) \le v_f^a + \Delta_1, \quad \inf_{x \in M} f\big(x, y_0\big) \ge v_f^b - \Delta_2. \tag{4.7}$$

Then

$$\sup_{y \in N} f\big(x_0, y\big) - \Delta_1 - \Delta_2 \le f\big(x_0, y_0\big) \le \inf_{x \in M} f\big(x, y_0\big) + \Delta_1 + \Delta_2. \tag{4.8}$$

Proof. By (4.7) and (4.6)

$$\begin{aligned} &\sup_{y \in N} f\big(x_0, y\big) - \Delta_1 - \Delta_2 \\ &\quad \le v_f^a - \Delta_2 = v_f^b - \Delta_2 \le \inf_{x \in M} f\big(x, y_0\big) \le f\big(x_0, y_0\big) \\ &\quad \le \sup_{y \in N} f\big(x_0, y\big) \le v_f^a + \Delta_1 = v_f^b + \Delta_1 \le \inf_{x \in M} f\big(x, y_0\big) + \Delta_1 + \Delta_2. \end{aligned} \tag{4.9}$$

This completes the proof. □

PROPOSITION 4.4. *Let M, N be nonempty sets and let $f: M \times N \to \mathbb{R}^1$. Assume that (4.6) is valid, $x_0 \in M$, $y_0 \in N$, Δ_1, $\Delta_2 \in [0, \infty)$, and*

$$\sup_{y \in N} f\big(x_0, y\big) - \Delta_2 \le f\big(x_0, y_0\big) \le \inf_{x \in M} f\big(x, y_0\big) + \Delta_1. \tag{4.10}$$

Then

$$\sup_{y \in N} f\big(x_0, y\big) \le v_f^a + \Delta_1 + \Delta_2, \qquad \inf_{x \in M} f\big(x, y_0\big) \ge v_f^b - \Delta_1 - \Delta_2. \tag{4.11}$$

Proof. It follows from (4.10), (4.2), (4.6), and (4.3) that

$$v_f^b - \Delta_2 = v_f^a - \Delta_2 \le \sup_{y \in N} f\big(x_0, y\big) - \Delta_2 \le \inf_{x \in M} f\big(x, y_0\big) + \Delta_1 \le v_f^b + \Delta_1. \tag{4.12}$$

This implies (4.11). The proposition is thus proved. □

5. The existence of a minimal pair of sequences

Let $f \in \mathfrak{M}$, $x_f \in X$, $y_f \in Y$, and

$$\sup_{y \in Y} \bar{f}(x_f, y) = \bar{f}(x_f, y_f) = \inf_{x \in X} \bar{f}(x, y_f). \tag{5.1}$$

PROPOSITION 5.1. *Let $n \geq 2$ be an integer and*

$$\bar{x}_i = x_f, \quad \bar{y}_i = y_f, \quad i = 0, \dots, n. \tag{5.2}$$

Then the pair of sequences $\{\bar{x}_i\}_{i=0}^n$, $\{\bar{y}_i\}_{i=0}^n$ is (f)-optimal.

Proof. Assume that $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$, and

$$x_0, x_n = x_f, \qquad y_0, y_n = y_f. \tag{5.3}$$

By (5.1), (5.2), and (5.3)

$$\begin{aligned}
\sum_{i=0}^{n-1} f(x_i, x_{i+1}, \bar{y}_i, \bar{y}_{i+1}) &= \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) \\
&\geq nf\left(n^{-1}\sum_{i=0}^{n-1} x_i, n^{-1}\sum_{i=0}^{n-1} x_{i+1}, y_f, y_f\right) \\
&= nf\left(n^{-1}\sum_{i=0}^{n-1} x_i, n^{-1}\sum_{i=0}^{n-1} x_i, y_f, y_f\right) \\
&\geq nf(x_f, x_f, y_f, y_f), \\
\sum_{i=0}^{n-1} f(\bar{x}_i, \bar{x}_{i+1}, y_i, y_{i+1}) &= \sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) \\
&\leq nf\left(x_f, x_f, n^{-1}\sum_{i=0}^{n-1} y_i, n^{-1}\sum_{i=0}^{n-1} y_{i+1}\right) \\
&= nf\left(x_f, x_f, n^{-1}\sum_{i=0}^{n-1} y_i, n^{-1}\sum_{i=0}^{n-1} y_i\right) \\
&\leq nf(x_f, x_f, y_f, y_f).
\end{aligned} \tag{5.4}$$

This completes the proof of the proposition. □

PROPOSITION 5.2. *Let $n \geq 2$ be an integer and let*

$$\left(\left\{x_i^{(k)}\right\}_{i=0}^n, \left\{y_i^{(k)}\right\}_{i=0}^n\right) \subset X \times Y, \quad k = 1, 2, \dots \tag{5.5}$$

be a sequence of (f)-optimal pairs. Assume that

$$\lim_{k \to \infty} x_i^{(k)} = x_i, \quad \lim_{k \to \infty} y_i^{(k)} = y_i, \quad i = 0, 1, 2, \dots, n. \tag{5.6}$$

Then the pair of sequences $(\{x_i\}_{i=0}^n, \{y_i\}_{i=0}^n)$ *is* (f)*-optimal.*

Proof. Let

$$\{u_i\}_{i=0}^n \subset X, \quad u_0 = x_0,\ u_n = x_n. \tag{5.7}$$

We show that

$$\sum_{i=0}^{n-1} f\big(x_i, x_{i+1}, y_i, y_{i+1}\big) \le \sum_{i=0}^{n-1} f\big(u_i, u_{i+1}, y_i, y_{i+1}\big). \tag{5.8}$$

Assume the contrary. Then there exists $\epsilon > 0$ such that

$$\sum_{i=0}^{n-1} f\big(x_i, x_{i+1}, y_i, y_{i+1}\big) > \sum_{i=0}^{n-1} f\big(u_i, u_{i+1}, y_i, y_{i+1}\big) + 8\epsilon. \tag{5.9}$$

There exists a number $\delta \in (0, \epsilon)$ such that

$$\big| f\big(z_1, z_2, \xi_1, \xi_2\big) - f\big(\bar{z}_1, \bar{z}_2, \bar{\xi}_1, \bar{\xi}_2\big)\big| \le \epsilon (8n)^{-1} \tag{5.10}$$

for each $z_1, z_2, \bar{z}_1, \bar{z}_2 \in X,\ \xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2 \in Y$ satisfying $\|z_i - \bar{z}_i\|, \|\xi_i - \bar{\xi}_i\| \le \delta,\ i = 1, 2.$ There exists an integer $q \ge 1$ such that

$$\big\|x_i - x_i^{(q)}\big\|,\ \big\|y_i - y_i^{(q)}\big\| \le \delta, \quad i = 0, \dots, n. \tag{5.11}$$

Define $\{u_i^{(q)}\}_{i=0}^n \subset X$ by

$$u_0^{(q)} = x_0^{(q)}, \quad u_n^{(q)} = x_n^{(q)}, \quad u_i^{(q)} = u_i, \quad i = 1, \dots, n-1. \tag{5.12}$$

Since the pair of sequences $\big(\{x_i^{(q)}\}_{i=0}^n, \{y_i^{(q)}\}_{i=0}^n\big)$ is (f)-optimal it follows from (5.12) that

$$\sum_{i=0}^{n-1} f\Big(x_i^{(q)}, x_{i+1}^{(q)}, y_i^{(q)}, y_{i+1}^{(q)}\Big) \le \sum_{i=0}^{n-1} f\Big(u_i^{(q)}, u_{i+1}^{(q)}, y_i^{(q)}, y_{i+1}^{(q)}\Big). \tag{5.13}$$

By the definition of δ (see (5.10)), (5.11), (5.12), and (5.7) for $i = 0, \dots, n-1$,

$$\begin{gathered}\Big| f\Big(x_i^{(q)}, x_{i+1}^{(q)}, y_i^{(q)}, y_{i+1}^{(q)}\Big) - f\big(x_i, x_{i+1}, y_i, y_{i+1}\big)\Big| \le (8n)^{-1}\epsilon, \\ \Big| f\Big(u_i^{(q)}, u_{i+1}^{(q)}, y_i^{(q)}, y_{i+1}^{(q)}\Big) - f\big(u_i, u_{i+1}, y_i, y_{i+1}\big)\Big| \le (8n)^{-1}\epsilon.\end{gathered} \tag{5.14}$$

It follows from these relations and (5.9) that

$$\sum_{i=0}^{n-1} f\Big(x_i^{(q)}, x_{i+1}^{(q)}, y_i^{(q)}, y_{i+1}^{(q)}\Big) - \sum_{i=0}^{n-1} f\Big(u_i^{(q)}, u_{i+1}^{(q)}, y_i^{(q)}, y_{i+1}^{(q)}\Big) > \epsilon. \tag{5.15}$$

This is contradictory to (5.13). The obtained contradiction proves that (5.8) is valid. Analogously we can show that for each $\{u_i\}_{i=0}^n \subset Y$ satisfying $u_0 = y_0,\ u_n = y_n$, the following relation holds:

$$\sum_{i=0}^{n-1} f\left(x_i, x_{i+1}, y_i, y_{i+1}\right) \geq \sum_{i=0}^{n-1} f\left(x_i, x_{i+1}, u_i, u_{i+1}\right). \tag{5.16}$$

This completes the proof of the proposition. □

PROPOSITION 5.3. *Let $f \in \mathfrak{M}$ and let $x \in X$, $y \in Y$. Then there exists an (f)-minimal pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ such that $x_0 = x$, $y_0 = y$.*

Proof. By Proposition 4.2, for each integer $n \geq 2$ there exists an (f)-optimal pair of sequences $\{x_i^{(n)}\}_{i=0}^{n} \subset X$, $\{y_i^{(n)}\}_{i=0}^{n} \subset Y$ such that $x_0^{(n)} = x$, $y_0^{(n)} = y$. There exist a pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ and a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer $i \geq 0$

$$x_i^{(n_k)} \longrightarrow x_i, \quad y_i^{(n_k)} \longrightarrow y_i \quad \text{as } k \longrightarrow \infty. \tag{5.17}$$

It follows from Proposition 5.2 that the pair of sequences $\{x_i\}_{i=0}^{\infty}$, $\{y_i\}_{i=0}^{\infty}$ is (f)-minimal. The proposition is proved. □

6. Preliminary lemmas for Theorem 2.1

Let $f \in \mathfrak{M}$. There exist $x_f \in X$, $y_f \in Y$ such that

$$\sup_{y \in Y} f\left(x_f, x_f, y, y\right) = f\left(x_f, x_f, y_f, y_f\right) = \inf_{x \in X} f\left(x, x, y_f, y_f\right). \tag{6.1}$$

Let $r \in (0,1)$. Define $f_r : X^2 \times Y^2 \to \mathbb{R}^1$ by

$$f_r\left(x_1, x_2, y_1, y_2\right) = f\left(x_1, x_2, y_1, y_2\right) + r\|x_1 - x_f\| - r\|y_1 - y_f\|, \quad x_1, x_2 \in X, \ y_1, y_2 \in Y. \tag{6.2}$$

Clearly $f_r \in \mathfrak{M}$,

$$\sup_{y \in Y} f_r\left(x_f, x_f, y, y\right) = f_r\left(x_f, x_f, y_f, y_f\right) = \inf_{x \in X} f_r\left(x, x, y_f, y_f\right). \tag{6.3}$$

LEMMA 6.1. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta \in (0, \epsilon)$ such that for each integer $n \geq 2$ and each (f_r, δ)-good pair of sequences $\{x_i\}_{i=0}^{n} \subset X$, $\{y_i\}_{i=0}^{n} \subset Y$ satisfying*

$$x_n, x_0 = x_f, \qquad y_n, y_0 = y_f, \tag{6.4}$$

the following relations hold:

$$\|x_i - x_f\|, \ \|y_i - y_f\| \leq \epsilon, \quad i = 0, \ldots, n. \tag{6.5}$$

Proof. Choose a number

$$\delta \in \left(0, 8^{-1} r\epsilon\right). \tag{6.6}$$

Assume that an integer $n \geq 2$, $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ is an (f_r, δ)-good pair of sequences and (6.4) is valid. Set

$$\xi_1, \xi_2 = x_f, \quad \xi_3, \xi_4 = y_f, \quad \xi = \left(\xi_1, \xi_2, \xi_3, \xi_4\right). \tag{6.7}$$

Consider the sets $\Lambda_X(\xi, n)$, $\Lambda_Y(\xi, n)$ and the functions $(f_r)^{(\xi,n)}$, $f^{(\xi,n)}$ (see (3.4), (3.5), and (3.6)). It follows from (6.1) and Proposition 5.1 that

$$\begin{aligned} \sup&\left\{\sum_{i=0}^{n-1} f\left(x_f, x_f, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^n \in \Lambda_Y(\xi, n)\right\} \\ &= nf\left(x_f, x_f, y_f, y_f\right) \\ &= \inf\left\{\sum_{i=0}^{n-1} f\left(p_i, p_{i+1}, y_f, y_f\right) : \{p_i\}_{i=0}^n \in \Lambda_X(\xi, n)\right\}. \end{aligned} \tag{6.8}$$

Equation (6.8) and Proposition 4.1 imply that

$$\begin{aligned} \sup&\left\{\sum_{i=0}^{n-1} f\left(x_f, x_f, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^n \in \Lambda_Y(\xi, n)\right\} \\ &= \inf\left\{\sup\left\{\sum_{i=0}^{n-1} f\left(p_i, p_{i+1}, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^n \in \Lambda_Y(\xi, n)\right\} : \{p_i\}_{i=0}^n \in \Lambda_X(\xi, n)\right\}, \end{aligned} \tag{6.9}$$

$$\begin{aligned} \inf&\left\{\sum_{i=0}^{n-1} f\left(p_i, p_{i+1}, y_f, y_f\right) : \{p_i\}_{i=0}^n \in \Lambda_X(\xi, n)\right\} \\ &= \sup\left\{\inf\left\{\sum_{i=0}^{n-1} f\left(p_i, p_{i+1}, u_i, u_{i+1}\right) : \{p_i\}_{i=0}^n \in \Lambda_X(\xi, n)\right\} : \{u_i\}_{i=0}^n \in \Lambda_Y(\xi, n)\right\}. \end{aligned} \tag{6.10}$$

It follows from (6.3) and Proposition 5.1 that

$$\begin{aligned} \sup&\left\{\sum_{i=0}^{n-1} f_r\left(x_f, x_f, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^n \in \Lambda_Y(\xi, n)\right\} \\ &= nf_r\left(x_f, x_f, y_f, y_f\right) \\ &= \inf\left\{\sum_{i=0}^{n-1} f_r\left(p_i, p_{i+1}, y_f, y_f\right) : \{p_i\}_{i=0}^n \in \Lambda_X(\xi, n)\right\}. \end{aligned} \tag{6.11}$$

Equation (6.11) and Proposition 4.1 imply that

$$\sup\left\{\sum_{i=0}^{n-1} f_r\left(x_f, x_f, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n)\right\}$$
$$= \inf\left\{\sup\left\{\sum_{i=0}^{n-1} f_r\left(p_i, p_{i+1}, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n)\right\} : \{p_i\}_{i=0}^{n} \in \Lambda_X(\xi, n)\right\}, \tag{6.12}$$

$$\inf\left\{\sum_{i=0}^{n-1} f_r\left(p_i, p_{i+1}, y_f, y_f\right) : \{p_i\}_{i=0}^{n} \in \Lambda_X(\xi, n)\right\}$$
$$= \sup\left\{\inf\left\{\sum_{i=0}^{n-1} f_r\left(p_i, p_{i+1}, u_i, u_{i+1}\right) : \{p_i\}_{i=0}^{n} \in \Lambda_X(\xi, n)\right\} : \{u_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n)\right\}. \tag{6.13}$$

By (6.4) and (6.7)

$$\{x_i\}_{i=0}^{n} \in \Lambda_X(\xi, n), \qquad \{y_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n). \tag{6.14}$$

Since $(\{x_i\}_{i=0}^{n}, \{y_i\}_{i=0}^{n})$ is an (f_r, δ)-good pair of sequences, we conclude that

$$\begin{aligned}\sup&\left\{\sum_{i=0}^{n-1} f_r\left(x_i, x_{i+1}, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n)\right\} - \delta \\ &\leq \sum_{i=0}^{n-1} f_r\left(x_i, x_{i+1}, y_i, y_{i+1}\right) \\ &\leq \inf\left\{\sum_{i=0}^{n-1} f_r\left(p_i, p_{i+1}, y_i, y_{i+1}\right) : \{p_i\}_{i=0}^{n} \in \Lambda_X(\xi, n)\right\} + \delta.\end{aligned} \tag{6.15}$$

It follows from Proposition 4.4, (6.12), (6.13), and (6.15) that

$$\begin{aligned}\sup&\left\{\sum_{i=0}^{n-1} f_r\left(x_i, x_{i+1}, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n)\right\} \\ &\leq \sup\left\{\sum_{i=0}^{n-1} f_r\left(x_f, x_f, u_i, u_{i+1}\right) : \{u_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n)\right\} + 2\delta,\end{aligned} \tag{6.16}$$

$$\begin{aligned}\inf&\left\{\sum_{i=0}^{n-1} f_r\left(p_i, p_{i+1}, y_i, y_{i+1}\right) : \{p_i\}_{i=0}^{n} \in \Lambda_X(\xi, n)\right\} \\ &\geq \inf\left\{\sum_{i=0}^{n-1} f_r\left(p_i, p_{i+1}, y_f, y_f\right) : \{p_i\}_{i=0}^{n} \in \Lambda_X(\xi, n)\right\} - 2\delta.\end{aligned} \tag{6.17}$$

By (6.2), (6.8), (6.11), and (6.16)

$$\begin{aligned} nf(x_f,x_f,y_f,y_f) &= nf_r(x_f,x_f,y_f,y_f) \\ &\geq \sup\left\{\sum_{i=0}^{n-1} f_r(x_i,x_{i+1},u_i,u_{i+1}) : \{u_i\}_{i=0}^n \in \Lambda_Y(\xi,n)\right\} - 2\delta \\ &\geq -2\delta + \sum_{i=0}^{n-1} f_r(x_i,x_{i+1},y_f,y_f) \\ &= -2\delta + r\sum_{i=0}^{n-1}\|x_i - x_f\| + \sum_{i=0}^{n-1} f(x_i,x_{i+1},y_f,y_f) \\ &\geq -2\delta + r\sum_{i=0}^{n-1}\|x_i - x_f\| + nf(x_f,x_f,y_f,y_f). \end{aligned} \tag{6.18}$$

By (6.2), (6.8), (6.11), and (6.17)

$$\begin{aligned} nf(x_f,x_f,y_f,y_f) &= nf_r(x_f,x_f,y_f,y_f) \\ &\leq \inf\left\{\sum_{i=0}^{n-1} f_r(p_i,p_{i+1},y_i,y_{i+1}) : \{p_i\}_{i=0}^n \in \Lambda_X(\xi,n)\right\} + 2\delta \\ &\leq 2\delta + \sum_{i=0}^{n-1} f_r(x_f,x_f,y_i,y_{i+1}) \\ &= 2\delta - r\sum_{i=0}^{n-1}\|y_i - y_f\| + \sum_{i=0}^{n-1} f(x_f,x_f,y_i,y_{i+1}) \\ &\leq 2\delta - r\sum_{i=0}^{n-1}\|y_i - y_f\| + nf(x_f,x_f,y_f,y_f). \end{aligned} \tag{6.19}$$

Equations (6.6), (6.18), and (6.19) imply that for $i = 1,\ldots,n-1$

$$\|x_i - x_f\| \leq r^{-1}(2\delta) < \epsilon, \qquad \|y_i - y_f\| \leq 2\delta r^{-1} < \epsilon. \tag{6.20}$$

This completes the proof of the lemma. □

Choose a number

$$D_0 \geq \sup\left\{\left|f_r(x_1,x_2,y_1,y_2)\right| : x_1,x_2 \in X,\ y_1,y_2 \in Y\right\}. \tag{6.21}$$

We can easily prove the following lemma.

LEMMA 6.2. *Let $n \geq 2$ be an integer, M be a positive number, and let $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ be an (f_r, M)-good pair of sequences. Then the pair of sequences $\{\bar{x}_i\}_{i=0}^n \subset X, \{\bar{y}_i\}_{i=0}^n \subset Y$ defined by*

$$\bar{x}_i = x_i, \quad \bar{y}_i = y_i, \quad i = 1, \dots, n-1, \qquad \bar{x}_0, \bar{x}_n = x_f, \qquad \bar{y}_0, \bar{y}_n = y_f \tag{6.22}$$

is $(f_r, M+8D_0)$-good.

By using the uniform continuity of the function $f_r : X \times X \times Y \times Y$ we can easily prove the following lemma.

LEMMA 6.3. *Let ϵ be a positive number. There exists a number $\delta > 0$ such that for each integer $n \geq 2$ and each sequences $\{x_i\}_{i=0}^n, \{\bar{x}_i\}_{i=0}^n \subset X, \{y_i\}_{i=0}^n, \{\bar{y}_i\}_{i=0}^n \subset Y$ which satisfy*

$$\|\bar{x}_j - x_j\|, \ \|\bar{y}_j - y_j\| \leq \delta, \quad j = 0, n, \quad x_j = \bar{x}_j, \quad y_j = \bar{y}_j, \quad j = 1, \dots, n-1, \tag{6.23}$$

the following relation holds:

$$\left| \sum_{i=0}^{n-1} \left[f_r\left(x_i, x_{i+1}, y_i, y_{i+1}\right) - f_r\left(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}\right) \right] \right| \leq \epsilon. \tag{6.24}$$

Lemma 6.3 implies the following result.

LEMMA 6.4. *Assume that $\epsilon > 0$. Then there exists a number $\delta > 0$ such that for each integer $n \geq 2$, each (f_r, ϵ)-good pair of sequences $\{x_i\}_{i=0}^n \subset X, \{y_i\}_{i=0}^n \subset Y$ and each pair of sequences $\{\bar{x}_i\}_{i=0}^n \subset X$, $\{\bar{y}_i\}_{i=0}^n \subset Y$ the following assertion holds.*

If (6.23) is valid, then the pair of sequences $(\{\bar{x}_i\}_{i=0}^n, \{\bar{y}_i\}_{i=0}^n)$ is $(f_r, 2\epsilon)$-good.

Lemmas 6.4 and 6.1 imply the following.

LEMMA 6.5. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta \in (0, \epsilon)$ such that for each integer $n \geq 2$ and each (f_r, δ)-good pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ which satisfies $\|x_j - x_f\|, \|y_j - y_f\| \leq \delta$, $j = 0, n$, the following relations hold: $\|x_i - x_f\|$, $\|y_i - y_f\| \leq \epsilon$, $i = 0, \dots, n$.*

Denote by $\mathrm{Card}(E)$ the cardinality of a set E.

LEMMA 6.6. *Let M be a positive number and let $\epsilon \in (0,1)$. Then there exists an integer $n_0 \geq 4$ such that for each (f_r, M)-good pair of sequences $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ which satisfies*

$$x_0, x_{n_0} = x_f, \qquad y_0, y_{n_0} = y_f, \tag{6.25}$$

there is $j \in \{1,\dots,n_0-1\}$ *for which*

$$\|x_j - x_f\|,\ \|y_j - y_f\| \le \epsilon. \tag{6.26}$$

Proof. Choose a natural number

$$n_0 > 8 + 8(r\epsilon)^{-1} M. \tag{6.27}$$

Set

$$\xi_1, \xi_2 = x_f, \qquad \xi_3, \xi_4 = y_f, \qquad \xi = \{\xi_i\}_{i=1}^4. \tag{6.28}$$

Assume that $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ is an (f_r, M)-good pair of sequences and (6.25) holds. It follows from Proposition 4.4 that

$$\begin{aligned} &\sup\left\{\sum_{i=0}^{n_0-1} f_r\big(x_i, x_{i+1}, u_i, u_{i+1}\big) : \{u_i\}_{i=0}^{n_0} \in \Lambda_Y\big(\xi, n_0\big)\right\} \\ &\le \inf\left\{\sup\left\{\sum_{i=0}^{n_0-1} f_r\big(p_i, p_{i+1}, u_i, u_{i+1}\big) : \{u_i\}_{i=0}^{n_0} \in \Lambda_Y\big(\xi, n_0\big)\right\} : \{p_i\}_{i=0}^{n_0} \in \Lambda_X\big(\xi, n_0\big)\right\} + 2M, \end{aligned} \tag{6.29}$$

$$\begin{aligned} &\inf\left\{\sum_{i=0}^{n_0-1} f_r\big(p_i, p_{i+1}, y_i, y_{i+1}\big) : \{p_i\}_{i=0}^{n_0} \in \Lambda_X\big(\xi, n_0\big)\right\} \\ &\ge \sup\left\{\inf\left\{\sum_{i=0}^{n_0-1} f_r\big(p_i, p_{i+1}, u_i, u_{i+1}\big) : \{p_i\}_{i=0}^{n_0} \in \Lambda_X\big(\xi, n_0\big)\right\} : \{u_i\}_{i=0}^{n_0} \in \Lambda_Y\big(\xi, n_0\big)\right\} - 2M. \end{aligned} \tag{6.30}$$

By Proposition 5.1, (6.3), and Propositions 4.1, 4.2

$$\begin{aligned} &\inf\left\{\sup\left\{\sum_{i=0}^{n_0-1} f_r\big(p_i, p_{i+1}, u_i, u_{i+1}\big) : \{u_i\}_{i=0}^{n_0} \in \Lambda_Y\big(\xi, n_0\big)\right\} : \{p_i\}_{i=0}^{n_0} \in \Lambda_X\big(\xi, n_0\big)\right\} \\ &\quad = \sup\left\{\inf\left\{\sum_{i=0}^{n_0-1} f_r\big(p_i, p_{i+1}, u_i, u_{i+1}\big) : \{p_i\}_{i=0}^{n_0} \in \Lambda_X\big(\xi, n_0\big)\right\} : \{u_i\}_{i=0}^{n_0} \in \Lambda_Y\big(\xi, n_0\big)\right\} \\ &\quad = n_0 f_r\big(x_f, x_f, y_f, y_f\big). \end{aligned} \tag{6.31}$$

Equations (6.2), (6.29), (6.30), and (6.31) imply that

$$
\begin{aligned}
n_0 f\big(x_f, x_f, y_f, y_f\big) &= n_0 f_r\big(x_f, x_f, y_f, y_f\big) \\
&\geq -2M + \sup\left\{\sum_{i=0}^{n_0-1} f_r\big(x_i, x_{i+1}, u_i, u_{i+1}\big) : \{u_i\}_{i=0}^{n_0} \in \Lambda_Y\big(\xi, n_0\big)\right\} \\
&\geq -2M + \sum_{i=0}^{n_0-1} f_r\big(x_i, x_{i+1}, y_f, y_f\big) \\
&= -2M + \sum_{i=0}^{n_0-1} f\big(x_i, x_{i+1}, y_f, y_f\big) + r\sum_{i=0}^{n_0-1} \|x_i - x_f\|,
\end{aligned}
\tag{6.32}
$$

$$
\begin{aligned}
n_0 f\big(x_f, x_f, y_f, y_f\big) &= n_0 f_r\big(x_f, x_f, y_f, y_f\big) \\
&\leq 2M + \inf\left\{\sum_{i=0}^{n_0-1} f_r\big(p_i, p_{i+1}, y_i, y_{i+1}\big) : \{p_i\}_{i=0}^{n_0} \in \Lambda_X\big(\xi, n_0\big)\right\} \\
&\leq 2M + \sum_{i=0}^{n_0-1} f_r\big(x_f, x_f, y_i, y_{i+1}\big) \\
&= 2M + \sum_{i=0}^{n_0-1} f\big(x_f, x_f, y_i, y_{i+1}\big) - r\sum_{i=0}^{n_0-1} \|y_i - y_f\|.
\end{aligned}
\tag{6.33}
$$

It follows from (6.1) and Proposition 5.1 that

$$
\sum_{i=0}^{n_0-1} f\big(x_i, x_{i+1}, y_f, y_f\big) \geq n_0 f\big(x_f, x_f, y_f, y_f\big) \geq \sum_{i=0}^{n_0-1} f\big(x_f, x_f, y_i, y_{i+1}\big). \tag{6.34}
$$

Together with (6.32) and (6.33) this implies that

$$
\begin{gathered}
n_0 f\big(x_f, x_f, y_f, y_f\big) \geq -2M + n_0 f\big(x_f, x_f, y_f, y_f\big) + r\sum_{i=0}^{n_0-1} \|x_i - x_f\|, \\
n_0 f\big(x_f, x_f, y_f, y_f\big) \leq 2M + n_0 f\big(x_f, x_f, y_f, y_f\big) - r\sum_{i=0}^{n_0-1} \|y_i - y_f\|, \\
r\sum_{i=0}^{n_0-1} \|x_i - x_f\| \leq 2M, \qquad r\sum_{i=0}^{n_0-1} \|y_i - y_f\| \leq 2M.
\end{gathered}
\tag{6.35}
$$

By (6.25), (6.27), and (6.35)

$$\begin{aligned}
\epsilon\mathrm{Card}\big\{i\in\{1,\dots,n_0-1\}:\|x_i-x_f\|\ge\epsilon\big\}&\le 2Mr^{-1},\\
\epsilon\mathrm{Card}\big\{i\in\{1,\dots,n_0-1\}:\|y_i-y_f\|\ge\epsilon\big\}&\le 2Mr^{-1},\\
\mathrm{Card}\big\{i\in\{1,\dots,n_0-1\}:\|x_i-x_f\|<\epsilon,\ \|y_i-y_f\|<\epsilon\big\}&\ge n_0-1-4M(\epsilon r)^{-1}>6.
\end{aligned}\tag{6.36}$$

This completes the proof of the lemma. □

Lemmas 6.2 and 6.6 imply the following.

Lemma 6.7. *Let $\epsilon\in(0,1)$, $M\in(0,\infty)$. Then there exists an integer $n_0\ge 4$ such that for each (f_r,M)-good pair of sequences $\{x_i\}_{i=0}^{n_0}\subset X$, $\{y_i\}_{i=0}^{n_0}\subset Y$ there is $j\in\{1,\dots,n_0-1\}$ for which $\|x_f-x_j\|,\|y_f-y_j\|\le\epsilon$.*

Lemma 6.8. *Let $\epsilon\in(0,1)$, $M\in(0,\infty)$. Then there exists an integer $n_0\ge 4$ and a neighborhood U of f_r in $\mathfrak{M}$ such that for each $g\in U$ and each (g,M)-good pair of sequences $\{x_i\}_{i=0}^{n_0}\subset X$, $\{y_i\}_{i=0}^{n_0}\subset Y$ there is $j\in\{1,\dots,n_0-1\}$ for which*

$$\|x_f-x_j\|,\|y_f-y_j\|\le\epsilon.\tag{6.37}$$

Proof. By Lemma 6.7 there is an integer $n_0\ge 4$ such that for each $(f_r,M+8)$-good pair of sequences $\{x_i\}_{i=0}^{n_0}\subset X$, $\{y_i\}_{i=0}^{n_0}\subset Y$ there is $j\in\{1,\dots,n_0-1\}$ for which (6.37) is valid. Set

$$U=\big\{g\in\mathfrak{M}:\rho\big(f_r,g\big)\le\big(16n_0\big)^{-1}\big\}.\tag{6.38}$$

Assume that $g\in U$ and $\{x_i\}_{i=0}^{n_0}\subset X$, $\{y_i\}_{i=0}^{n_0}\subset Y$ is a (g,M)-good pair of sequences. By (6.38) the pair of sequences $\{x_i\}_{i=0}^{n_0}$, $\{y_i\}_{i=0}^{n_0}$ is $(f_r,M+8)$-good. It follows from the definition of n_0 that there exists $j\in\{1,\dots,n_0-1\}$ for which (6.37) is valid. The lemma is proved. □

Lemma 6.9. *Let $\epsilon\in(0,1)$. Then there exist a neighborhood U of f_r in $\mathfrak{M}$, a number $\delta\in(0,\epsilon)$, and an integer $n_1\ge 4$ such that for each $g\in U$, each integer $n\ge 2n_1$, and each (g,δ)-good pair of sequences $\{x_i\}_{i=0}^{n}\subset X$, $\{y_i\}_{i=0}^{n}\subset Y$ the relation*

$$\|x_i-x_f\|,\|y_i-y_f\|\le\epsilon\tag{6.39}$$

holds for all $i\in[n_1,n-n_1]$. Moreover, if $\|x_0-x_f\|,\|y_0-y_f\|\le\delta$, then (6.39) holds for all $i\in[0,n-n_1]$, and if $\|x_n-x_f\|,\|y_n-y_f\|\le\delta$, then (6.39) is valid for all $i\in[n_1,n]$.

Proof. By Lemma 6.5 there exists $\delta_0\in(0,\epsilon)$ such that for each integer $n\ge 2$ and each (f_r,δ_0)-good pair of sequences $\{x_i\}_{i=0}^{n}\subset X$, $\{y_i\}_{i=0}^{n}\subset Y$ satisfying

$$\|x_j-x_f\|,\|y_j-y_f\|\le\delta_0,\quad j=0,n,\tag{6.40}$$

the relation (6.39) is valid for $i=0,\dots,n$. By Lemma 6.8 there exists an integer $n_0\ge 4$ and a neighborhood U_0 of f_r in $\mathfrak{M}$ such that for each $g\in U_0$ and each $(g,8)$-good pair of sequences $\{x_i\}_{i=0}^{n_0} \subset X, \{y_i\}_{i=0}^{n_0} \subset Y$ there is $j \in \{1, \dots, n_0 - 1\}$ for which

$$\|x_j - x_f\|, \|y_j - y_f\| \le \delta_0. \tag{6.41}$$

Fix an integer

$$n_1 \ge 4n_0 \tag{6.42}$$

and a number

$$\delta \in \big(0, 4^{-1}\delta_0\big). \tag{6.43}$$

Define

$$U = U_0 \cap \Big\{g \in \mathfrak{M} : \rho\big(g, f_r\big) \le 16^{-1}\delta n_1^{-1}\Big\}. \tag{6.44}$$

Assume that $g \in U$, an integer $n \ge 2n_1$, and $\{x_i\}_{i=0}^{n} \subset X,\ \{y_i\}_{i=0}^{n} \subset Y$ is a (g, δ)-good pair of sequences. It follows from (6.42), (6.43), and the definition of n_0, U_0 that there exists a sequence of integers $\{t_i\}_{i=1}^{k} \subset [0, n]$ such that

$$\begin{aligned} t_1 \le n_0, \quad & t_{i+1} - t_i \in \big[n_0, 3n_0\big], \quad i = 1, \dots, k-1, \\ n - t_k \le n_0, \quad & \|x_{t_i} - x_f\|, \|y_{t_i} - y_f\| \le \delta_0, \quad i = 1, \dots, k, \end{aligned} \tag{6.45}$$

and, moreover, if $\|x_0 - x_f\|, \|y_0 - y_f\| \le \delta$, then $t_1 = 0$, and if $\|x_n - x_f\|, \|y_n - y_f\| \le \delta$, then $t_k = n$. Clearly $k \ge 2$. Fix $q \in \{1, \dots, k-1\}$. To complete the proof of the lemma it is sufficient to show that for each integer $i \in [t_q, t_{q+1}]$ the relation (6.39) holds.

Define sequences $\{x_i^{(q)}\}_{i=0}^{t_{q+1}-t_q} \subset X,\ \{y_i^{(q)}\}_{i=0}^{t_{q+1}-t_q} \subset Y$ by

$$x_i^{(q)} = x_{i+t_q}, \quad y_i^{(q)} = y_{i+t_q}, \quad i \in \big[0, t_{q+1} - t_q\big]. \tag{6.46}$$

It is easy to see that $\{x_i^{(q)}\}_{i=0}^{t_{q+1}-t_q},\ \{y_i^{(q)}\}_{i=0}^{t_{q+1}-t_q}$ is a (g, δ)-good pair of sequences. Together with (6.43), (6.44), and (6.45) this implies that the pair of sequences $\{x_i^{(q)}\}_{i=0}^{t_{q+1}-t_q}$, $\{y_i^{(q)}\}_{i=0}^{t_{q+1}-t_q}$ is (f_r, δ_0)-good.

It follows from (6.43), (6.45), (6.46), and the definition of δ_0 (see (6.40)) that

$$\big\|x_i^{(q)} - x_f\big\|, \big\|y_i^{(q)} - y_f\big\| \le \epsilon, \quad i = 0, \dots, t_{q+1} - t_q. \tag{6.47}$$

Together with (6.46) this implies that $\|x_i - x_f\|, \|y_i - y_f\| \le \epsilon,\ i = t_q, \dots, t_{q+1}$. This completes the proof of the lemma. □

7. Preliminary lemmas for Theorem 2.2

For each metric space K denote by $C(K)$ the space of all continuous functions on K with the topology of uniform convergence ($\|\phi\| = \sup\{|\phi(z)| : z \in K\},\ \phi \in C(K)$).

Let $f \in \mathfrak{M}$. There exist $x_f \in X,\ y_f \in Y$ such that

$$\sup_{y \in Y} f\big(x_f, x_f, y, y\big) = f\big(x_f, x_f, y_f, y_f\big) = \inf_{x \in X} f\big(x, x, y_f, y_f\big) \tag{7.1}$$

(see equation (6.1)).

Let $r \in (0,1)$. Define $f_r : X \times X \times Y \times Y \to \mathbb{R}^1$ by

$$f_r\big(x_1,x_2,y_1,y_2\big) = f\big(x_1,x_2,y_1,y_2\big) + r\|x_1 - x_f\| - r\|y_1 - y_f\|, \quad x_1,x_2 \in X,\ y_1,y_2 \in Y \tag{7.2}$$

(see equation (6.2)). Clearly $f_r \in \mathfrak{M}$. Define functions $f_r^{(X)} : X \times X \to \mathbb{R}^1$, $f_r^{(Y)} : Y \times Y \to \mathbb{R}^1$ by

$$f_r^{(X)}\big(x_1,x_2\big) = f_r\big(x_1,x_2,y_f,y_f\big), \quad x_1,x_2 \in X, \tag{7.3}$$

$$f_r^{(Y)}\big(y_1,y_2\big) = f_r\big(x_f,x_f,y_1,y_2\big), \quad y_1,y_2 \in Y. \tag{7.4}$$

Lemma 7.1. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta \in (0,\epsilon)$ for which the following assertion holds.*

Assume that an integer $n \geq 2$,

$$\{x_i\}_{i=0}^n \subset X, \quad x_0, x_n = x_f \tag{7.5}$$

and for each $\{z_i\}_{i=0}^n \subset X$ satisfying

$$z_0 = x_0, \qquad z_n = x_n, \tag{7.6}$$

the relation

$$\sum_{i=0}^{n-1} f_r^{(X)}\big(x_i,x_{i+1}\big) \leq \sum_{i=0}^{n-1} f_r^{(X)}\big(z_i,z_{i+1}\big) + \delta \tag{7.7}$$

holds. Then

$$\|x_i - x_f\| \leq \epsilon, \quad i = 0,\dots,n. \tag{7.8}$$

Proof. Choose a number

$$\delta \in \big(0, 8^{-1} r\epsilon\big). \tag{7.9}$$

Assume that an integer $n \geq 2$, $\{x_i\}_{i=0}^n \subset X$, (7.5) is valid and for each sequence $\{z_i\}_{i=0}^n \subset X$ satisfying (7.6), the relation (7.7) holds. This implies that

$$\sum_{i=0}^{n-1} f_r\big(x_i,x_{i+1},y_f,y_f\big) \leq n f_r\big(x_f,x_f,y_f,y_f\big) + \delta = n f\big(x_f,x_f,y_f,y_f\big) + \delta. \tag{7.10}$$

It follows from (7.1), (7.2), and (7.5) that

$$\begin{aligned}
\sum_{i=0}^{n-1} f_r\big(x_i,x_{i+1},y_f,y_f\big) &= r\sum_{i=0}^{n-1}\|x_i - x_f\| + \sum_{i=0}^{n-1} f\big(x_i,x_{i+1},y_f,y_f\big)\\
&\geq r\sum_{i=0}^{n-1}\|x_i - x_f\| + nf\left(n^{-1}\sum_{i=0}^{n-1}x_i, n^{-1}\sum_{i=0}^{n-1}x_i, y_f, y_f\right)\\
&\geq r\sum_{i=0}^{n-1}\|x_i - x_f\| + nf\big(x_f,x_f,y_f,y_f\big).
\end{aligned} \tag{7.11}$$

Together with (7.9) and (7.10) this implies that for each $i \in \{0,\dots,n-1\}$, the relation $\|x_i - x_f\| \le r^{-1}\delta < \epsilon$ is true. This completes the proof of the lemma. $\square$

Definition 7.2. Let $g \in C(X \times X)$, $n \ge 1$ be an integer and let $M \in [0,\infty)$. A sequence $\{\bar{x}_i\}_{i=0}^n \subset X$ is called (g,X,M)-good if for each sequence $\{x_i\}_{i=0}^n \subset X$ satisfying $x_0 = \bar{x}_0,\ x_n = \bar{x}_n$ the relation $M + \sum_{i=0}^{n-1} g(x_i, x_{i+1}) \ge \sum_{i=0}^{n-1} g(\bar{x}_i, \bar{x}_{i+1})$ is valid.

Definition 7.3. Let $g \in C(Y \times Y)$, $n \ge 1$ be an integer and let $M \in [0,\infty)$. A sequence $\{\bar{y}_i\}_{i=0}^n \subset Y$ is called (g,Y,M)-good if for each sequence $\{y_i\}_{i=0}^n \subset Y$ satisfying $y_0 = \bar{y}_0$, $y_n = \bar{y}_n$ the relation $\sum_{i=0}^{n-1} g(y_i, y_{i+1}) \le M + \sum_{i=0}^{n-1} g(\bar{y}_i, \bar{y}_{i+1})$ is valid.

Definition 7.4. Let $n_1 \ge 0,\ n_2 > n_1$ be integers, and let $\{g_i\}_{i=n_1}^{n_2-1} \subset C(X \times X)$, $M \in [0,\infty)$. A sequence $\{\bar{x}_i\}_{i=n_1}^{n_2} \subset X$ is called $(\{g_i\}_{i=n_1}^{n_2-1}, X, M)$-good if for each sequence $\{x_i\}_{i=n_1}^{n_2} \subset X$ satisfying $x_{n_1} = \bar{x}_{n_1},\ x_{n_2} = \bar{x}_{n_2}$

$$M + \sum_{i=n_1}^{n_2-1} g_i\big(x_i, x_{i+1}\big) \ge \sum_{i=n_1}^{n_2-1} g_i\big(\bar{x}_i, \bar{x}_{i+1}\big). \tag{7.12}$$

Definition 7.5. Let $n_1 \ge 0,\ n_2 > n_1$ be integers, and let $\{g_i\}_{i=n_1}^{n_2-1} \subset C(Y \times Y)$, $M \in [0,\infty)$. A sequence $\{\bar{y}_i\}_{i=n_1}^{n_2} \subset Y$ is called $(\{g_i\}_{i=n_1}^{n_2-1}, Y, M)$-good if for each sequence $\{y_i\}_{i=n_1}^{n_2} \subset Y$ satisfying $y_{n_1} = \bar{y}_{n_1},\ y_{n_2} = \bar{y}_{n_2}$

$$\sum_{i=n_1}^{n_2-1} g_i\big(y_i, y_{i+1}\big) \le \sum_{i=n_1}^{n_2-1} g_i\big(\bar{y}_i, \bar{y}_{i+1}\big) + M. \tag{7.13}$$

Analogously to Lemma 7.1 we can establish the following.

LEMMA 7.6. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta \in (0,\epsilon)$ such that for each integer $n \ge 2$ and each $(f_r^{(Y)}, Y, \delta)$-good sequence $\{y_i\}_{i=0}^n \subset Y$ satisfying $y_0, y_n = y_f$ the following relation holds: $\|y_i - y_f\| \le \epsilon,\ i = 0,\dots,n$.*

By using Lemmas 6.3 and 7.1 we can easily deduce the following lemma.

LEMMA 7.7. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta > 0$ such that for each integer $n \ge 2$ and each $(f_r^{(X)}, X, \delta)$-good sequence $\{x_i\}_{i=0}^n \subset X$ satisfying $\|x_0 - x_f\|$, $\|x_n - x_f\| \le \delta$ the following relation holds: $\|x_i - x_f\| \le \epsilon,\ i = 0,\dots,n$.*

By using Lemmas 6.3 and 7.6 we can easily deduce the following lemma.

LEMMA 7.8. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta > 0$ such that for each integer $n \ge 2$ and each $(f_r^{(Y)}, Y, \delta)$-good sequence $\{y_i\}_{i=0}^n \subset Y$ satisfying $\|y_0 - y_f\|$, $\|y_n - y_f\| \le \delta$ the following relation holds: $\|y_i - y_f\| \le \epsilon,\ i = 0,\dots,n$.*

LEMMA 7.9. *Let $\epsilon \in (0,1)$ and let M be a positive number. Then there exists an integer $n_0 \geq 4$ such that for each $(f_r^{(X)}, X, M)$-good sequence $\{x_i\}_{i=0}^{n_0} \subset X$ satisfying*

$$x_0 = x_f, \qquad x_{n_0} = x_f \tag{7.14}$$

there is $j \in \{1, \ldots, n_0 - 1\}$ for which

$$\|x_j - x_f\| \leq \epsilon. \tag{7.15}$$

Proof. Choose a natural number

$$n_0 > 8 + 8M(r\epsilon)^{-1}. \tag{7.16}$$

Assume that $\{x_i\}_{i=0}^{n_0} \subset X$ is an $(f_r^{(X)}, X, M)$-good sequence and (7.14) is valid. It is easy to see that

$$\begin{aligned} M + n_0 f\big(x_f, x_f, y_f, y_f\big) &= n_0 f_r\big(x_f, x_f, y_f, y_f\big) + M \\ &\geq \sum_{i=0}^{n_0-1} f_r\big(x_i, x_{i+1}, y_f, y_f\big) \\ &= r\sum_{i=0}^{n_0-1} \|x_i - x_f\| + \sum_{i=0}^{n_0-1} f\big(x_i, x_{i+1}, y_f, y_f\big) \\ &\geq r\sum_{i=0}^{n_0-1} \|x_i - x_f\| + n_0 f\left(n_0^{-1}\sum_{i=0}^{n_0-1} x_i, n_0^{-1}\sum_{i=0}^{n_0-1} x_i, y_f, y_f\right) \\ &\geq r\sum_{i=0}^{n_0-1} \|x_i - x_f\| + n_0 f\big(x_f, x_f, y_f, y_f\big). \end{aligned} \tag{7.17}$$

Together with (7.16) this implies that there is $j \in \{1, \ldots, n_0 - 1\}$ for which (7.15) is valid. This completes the proof of the lemma. □

Analogously to Lemma 7.9 we can establish the following lemma.

LEMMA 7.10. *Let $\epsilon \in (0,1)$ and let M be a positive number. Then there exists an integer $n_0 \geq 4$ such that for each $(f_r^{(Y)}, Y, M)$-good sequence $\{y_i\}_{i=0}^{n_0} \subset Y$ satisfying $y_0 = y_f$, $y_{n_0} = y_f$ there is $j \in \{1, \ldots, n_0 - 1\}$ for which $\|y_j - y_f\| \leq \epsilon$.*

Choose a number $D_0 \geq \sup\{|f_r(x_1, x_2, y_1, y_2)| : x_1, x_2 \in X,\ y_1, y_2 \in Y\}$. We can easily prove the following lemma.

LEMMA 7.11. (1) *Assume that $n \geq 2$ is an integer, M is a positive number, a sequence $\{x_i\}_{i=0}^{n} \subset X$ is $(f_r^{(X)}, X, M)$-good and $\bar{x}_0 = x_f$, $\bar{x}_n = x_f$, $\bar{x}_i = x_i$, $i = 1, \ldots, n-1$. Then the sequence $\{\bar{x}_i\}_{i=0}^{n}$ is $(f_r^{(X)}, X, M + 8D_0)$-good.*

(2) *Assume that $n \geq 2$ is an integer, M is a positive number, a sequence $\{y_i\}_{i=0}^{n} \subset Y$ is $(f_r^{(Y)}, Y, M)$-good and $\bar{y}_0 = y_f,\ \bar{y}_n = y_f,\ \bar{y}_i = y_i,\ i = 1, \dots, n-1$. Then the sequence $\{\bar{y}_i\}_{i=0}^{n}$ is $(f_r^{(Y)}, Y, M + 8D_0)$-good.*

Lemmas 7.9, 7.10, and 7.11 imply the following two results.

LEMMA 7.12. *Let $\epsilon \in (0,1)$ and let M be a positive number. Then there exists an integer $n_0 \geq 4$ such that for each $(f_r^{(X)}, X, M)$-good sequence $\{x_i\}_{i=0}^{n_0} \subset X$ there is $j \in \{1, \dots, n_0 - 1\}$ for which $\|x_j - x_f\| \leq \epsilon$.*

LEMMA 7.13. *Let $\epsilon \in (0,1)$ and let M be a positive number. Then there exists an integer $n_0 \geq 4$ such that for each $(f_r^{(Y)}, Y, M)$-good sequence $\{y_i\}_{i=0}^{n_0} \subset Y$ there is $j \in \{1, \dots, n_0 - 1\}$ for which $\|y_j - y_f\| \leq \epsilon$.*

By using Lemmas 7.12 and 7.13, analogously to the proof of Lemma 7.12, we can establish the following two results.

LEMMA 7.14. *Let $\epsilon \in (0,1)$, $M \in (0,\infty)$. Then there exists an integer $n_0 \geq 4$ and a neighborhood U of $f_r^{(X)}$ in $C(X \times X)$ such that for each $\{g_i\}_{i=0}^{n_0-1} \subset U$ and each $(\{g_i\}_{i=0}^{n_0-1}, X, M)$-good sequence $\{x_i\}_{i=0}^{n_0} \subset X$ there is $j \in \{1, \dots, n_0 - 1\}$ for which $\|x_f - x_j\| \leq \epsilon$.*

LEMMA 7.15. *Let $\epsilon \in (0,1)$, $M \in (0,\infty)$. Then there exists an integer $n_0 \geq 4$ and a neighborhood U of $f_r^{(Y)}$ in $C(Y \times Y)$ such that for each $\{g_i\}_{i=0}^{n_0-1} \subset U$ and each $(\{g_i\}_{i=0}^{n_0-1}, Y, M)$-good sequence $\{y_i\}_{i=0}^{n_0} \subset Y$ there is $j \in \{1, \dots, n_0 - 1\}$ for which $\|y_f - y_j\| \leq \epsilon$.*

LEMMA 7.16. *Let $\epsilon \in (0,1)$. Then there exist a neighborhood U of $f_r^{(X)}$ in $C(X \times X)$, a number $\delta \in (0,\epsilon)$, and an integer $n_1 \geq 4$ such that for each integer $n \geq 2n_1$, each $\{g_i\}_{i=0}^{n-1} \subset U$ and each $(\{g_i\}_{i=0}^{n-1}, X, \delta)$-good sequence $\{x_i\}_{i=0}^{n} \subset X$ the relation*

$$\|x_i - x_f\| \leq \epsilon \tag{7.18}$$

holds for all integers $i \in [n_1, n - n_1]$. Moreover, if $\|x_0 - x_f\| \leq \delta$, then (7.18) holds for all integers $i \in [0, n - n_1]$, and if $\|x_n - x_f\| \leq \delta$, then (7.18) is valid for all integers $i \in [n_1, n]$.

Proof. By Lemma 7.7 there exists $\delta_0 \in (0,\epsilon)$ such that for each integer $n \geq 2$ and each $(f_r^{(X)}, X, \delta_0)$-good sequence $\{x_i\}_{i=0}^{n} \subset X$ satisfying $\|x_0 - x_f\|, \|x_n - x_f\| \leq \delta_0$, the relation (7.18) is valid for $i = 0, \dots, n$. By Lemma 7.14 there exist an integer $n_0 \geq 4$ and a neighborhood U_0 of $f_r^{(X)}$ in $C(X \times X)$ such that for each $\{g_i\}_{i=0}^{n_0-1} \subset U_0$ and each $(\{g_i\}_{i=0}^{n_0-1}, X, 8)$-good sequence $\{x_i\}_{i=0}^{n} \subset X$ there is $j \in \{1, \dots, n_0 - 1\}$ for which $\|x_j - x_f\| \leq \delta_0$.

Choose an integer $n_1 \geq 4n_0$ and a number $\delta \in (0, 4^{-1}\delta_0)$. Define

$$U = U_0 \cap \{g \in C(X \times X) : \|g - f_r^{(X)}\| \leq (16n_1)^{-1}\delta\}. \tag{7.19}$$

Assume that an integer $n \geq 2n_1$, $\{g_i\}_{i=0}^{n-1} \subset U$ and a sequence $\{x_i\}_{i=0}^{n} \subset X$ is $(\{g_i\}_{i=0}^{n-1}, X, \delta)$-good. Arguing as in the proof of Lemma 6.9, we can show that (7.18) is valid for all integers $i \in [n_1, n - n_1]$ and, moreover, if $\|x_0 - x_f\| \leq \delta$, then (7.18), holds for all integers $i \in [0, n - n_1]$, and if $\|x_n - x_f\| \leq \delta$, then (7.18) is valid for all integers $i \in [n_1, n]$. The lemma is thus proved. □

Analogously to Lemma 7.16 we can prove the following lemma.

LEMMA 7.17. *Let $\epsilon \in (0, 1)$. Then there exist a neighborhood U of $f_r^{(Y)}$ in $C(Y \times Y)$, a number $\delta \in (0, \epsilon)$, and an integer $n_1 \geq 4$ such that for each integer $n \geq 2n_1$, each $\{g_i\}_{i=0}^{n-1} \subset U$, and each $(\{g_i\}_{i=0}^{n-1}, Y, \delta)$-good sequence $\{y_i\}_{i=0}^{n} \subset Y$ the relation*

$$\|y_i - y_f\| \leq \epsilon \tag{7.20}$$

holds for all integers $i \in [n_1, n - n_1]$. Moreover, if $\|y_0 - y_f\| \leq \delta$, then (7.20) holds for all integers $i \in [0, n - n_1]$, and if $\|y_n - y_f\| \leq \delta$, then (7.20) is valid for all integers $i \in [n_1, n]$.

8. Proofs of Theorems 2.1 and 2.2

We use the notation from Sections 1, 2, 3, 4, 5, 6, and 7.

Let $f \in \mathfrak{M}$. There exists a pair $(x_f, y_f) \in X \times Y$ such that (6.1) holds. Let $r \in (0, 1)$ and let $i \geq 1$ be an integer. Consider the function $f_r : X \times X \times Y \times Y$ defined by (6.2). Clearly all lemmas from Sections 6 and 7 are valid for f_r.

By Lemma 7.16 there exist a number

$$\gamma_1(f, r, i) \in (0, 2^{-i}), \tag{8.1}$$

a number

$$\delta_1(f, r, i) \in (0, 2^{-i}), \tag{8.2}$$

and an integer $n_1(f, r, i) \geq 4$ such that the following property holds:

(a) for each integer $n \geq 2n_1(f, r, i)$, each $\{g_j\}_{j=0}^{n-1} \subset C(X \times X)$ satisfying

$$\|g_j - f_r^{(X)}\| \leq \gamma_1(f, r, i), \quad j = 0, \ldots, n-1, \tag{8.3}$$

and each $(\{g_j\}_{j=0}^{n-1}, X, \delta_1(f, r, i))$-good sequence $\{x_j\}_{j=0}^{n} \subset X$ the following relation holds:

$$\|x_j - x_f\| \leq 2^{-i}, \quad j \in [n_1(f, r, i), n - n_1(f, r, i)]. \tag{8.4}$$

By Lemma 7.17 there exist numbers

$$\delta_2(f, r, i),\ \gamma_2(f, r, i) \in (0, 2^{-i}), \tag{8.5}$$

and an integer $n_2(f,r,i) \ge 4$ such that the following property holds:

(b) for each integer $n \ge 2n_2(f,r,i)$, each $\{g_j\}_{j=0}^{n-1} \subset C(Y \times Y)$ satisfying

$$\|g_j - f_r^{(Y)}\| \le \gamma_2(f,r,i), \quad j = 0,\dots,n-1 \tag{8.6}$$

and each $(\{g_j\}_{j=0}^{n-1}, Y, \delta_2(f,r,i))$-good sequence $\{y_j\}_{j=0}^{n} \subset Y$ the following relation holds:

$$\|y_j - y_f\| \le 2^{-i}, \quad j \in \big[n_2(f,r,i), n - n_2(f,r,i)\big]. \tag{8.7}$$

Set

$$\begin{aligned} n_3(f,r,i) &= n_1(f,r,i) + n_2(f,r,i),\\ \delta_3(f,r,i) &= \min\{\delta_1(f,r,i), \delta_2(f,r,i)\},\\ \gamma_3(f,r,i) &= \min\{\gamma_1(f,r,i), \gamma_2(f,r,i)\}. \end{aligned} \tag{8.8}$$

It follows from the uniform continuity of the function f_r that there exists a number

$$\delta_4(f,r,i) \in \big(0, \delta_3(f,r,i)\big) \tag{8.9}$$

such that for each $x_1, x_2, \bar{x}_1, \bar{x}_2 \in X,\ y_1, y_2, \bar{y}_1, \bar{y}_2 \in Y$ satisfying

$$\|x_j - \bar{x}_j\|,\ \|y_j - \bar{y}_j\| \le \delta_4(f,r,i), \quad j = 1,2, \tag{8.10}$$

the following relation holds:

$$\big|f_r\big(x_1,x_2,y_1,y_2\big) - f_r\big(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2\big)\big| \le 16^{-1}\gamma_3(f,r,i). \tag{8.11}$$

By Lemma 6.9 there exist numbers

$$\gamma_4(f,r,i) \in \big(0, 16^{-1}\gamma_3(f,r,i)\big), \qquad \delta_5(f,r,i) \in \big(0, 8^{-1}\delta_4(f,r,i)\big) \tag{8.12}$$

and an integer $n_4(f,r,i) \ge 4$ such that the following property holds:

(c) for each $g \in \mathfrak{M}$ satisfying $\rho(g, f_r) \le \gamma_4(f,r,i)$, each integer $n \ge 2n_4(f,r,i)$, and each $(g, \delta_5(f,r,i))$-good pair of sequences

$$\{x_j\}_{j=0}^{n} \subset X, \qquad \{y_j\}_{j=0}^{n} \subset Y, \tag{8.13}$$

the relation

$$\|x_j - x_f\|, \|y_j - y_f\| \le 8^{-1}\delta_4(f,r,i) \tag{8.14}$$

holds for all $j \in [n_4(f,r,i), n - n_4(f,r,i)]$; moreover, if

$$\|x_0 - x_f\|, \|y_0 - y_f\| \le \delta_5(f,r,i), \tag{8.15}$$

then (8.14) holds for all integers $j \in [0, n - n_4(f,r,i)]$, and if

$$\|x_n - x_f\|, \|y_n - y_f\| \le \delta_5(f,r,i), \tag{8.16}$$

then (8.14) is valid for all integers $j \in [n_4(f,r,i), n]$.

By Lemma 6.9 there exist numbers

$$\gamma(f,r,i) \in \left(0, 8^{-1}\gamma_4(f,r,i)\right), \qquad \delta(f,r,i) \in \left(0, 8^{-1}\delta_5(f,r,i)\right), \tag{8.17}$$

and an integer $n_5(f,r,i) \geq 4$ such that the following property holds:

(d) for each $g \in \mathfrak{M}$ satisfying $\rho(g, f_r) \leq \gamma(f,r,i)$, each integer $n \geq 2n_5(f,r,i)$, and each $(g, \delta(f,r,i))$-good pair of sequences

$$\left\{x_j\right\}_{j=0}^{n} \subset X, \qquad \left\{y_j\right\}_{j=0}^{n} \subset Y, \tag{8.18}$$

the inequality

$$\|x_j - x_f\|, \|y_j - y_f\| \leq 8^{-1}\delta_5(f,r,i) \tag{8.19}$$

holds for all $j \in [n_5(f,r,i), n - n_5(f,r,i)]$.

Set

$$U(f,r,i) = \left\{g \in \mathfrak{M} : \rho\left(g, f_r\right) < \gamma(f,r,i)\right\}. \tag{8.20}$$

Define

$$\mathfrak{F} = \cap_{k=1}^{\infty} \cup \left\{U(f,r,i) : f \in \mathfrak{M},\ r \in (0,1),\ i = k, k+1, \dots\right\}. \tag{8.21}$$

It is easy to see that $\mathfrak{F}$ is a countable intersection of open everywhere dense sets in $\mathfrak{M}$.

Proof of Theorem 2.1. Let $h \in \mathfrak{F}$. There exists a pair $(x_1, y_1) \in X \times Y$ such that

$$\sup_{y \in Y} h\left(x_1, x_1, y, y\right) = h\left(x_1, x_1, y_1, y_1\right) = \inf_{x \in X} h\left(x, x, y_1, y_1\right) \tag{8.22}$$

(see (3.1) and (3.2)).

Assume that $(x_2, y_2) \in X \times Y$ and

$$\sup_{y \in Y} h\left(x_2, x_2, y, y\right) = h\left(x_2, x_2, y_2, y_2\right) = \inf_{x \in X} h\left(x, x, y_2, y_2\right). \tag{8.23}$$

We show that

$$x_2 = x_1, \qquad y_2 = y_1. \tag{8.24}$$

Define sequences $\{x_j^{(1)}\}_{j=0}^{\infty}, \{x_j^{(2)}\}_{j=0}^{\infty} \subset X$, $\{y_j^{(1)}\}_{j=0}^{\infty}, \{y_j^{(2)}\}_{j=0}^{\infty} \subset Y$ by

$$x_j^{(1)} = x_1, \quad x_j^{(2)} = x_2, \quad y_j^{(1)} = y_1, \quad y_j^{(2)} = y_2, \quad j = 0, 1, \dots. \tag{8.25}$$

It follows from (8.22), (8.25), and Proposition 5.1 that the pairs of sequences

$$\left(\left\{x_j^{(1)}\right\}_{j=0}^{\infty}, \left\{y_j^{(1)}\right\}_{j=0}^{\infty}\right), \qquad \left(\left\{x_j^{(2)}\right\}_{j=0}^{\infty}, \left\{y_j^{(2)}\right\}_{j=0}^{\infty}\right) \tag{8.26}$$

are (h)-minimal. Let $\epsilon \in (0,1)$. Choose a natural number k such that

$$2^{-k} < 64^{-1}\epsilon. \tag{8.27}$$

There exist $f \in \mathfrak{M}$, $r \in (0,1)$, and an integer $i \geq k$ such that

$$h \in U(f,r,i). \tag{8.28}$$

Since the pairs of sequences $(\{x_j^{(1)}\}_{j=0}^\infty, \{y_j^{(1)}\}_{j=0}^\infty)$, $(\{x_j^{(2)}\}_{j=0}^\infty, \{y_j^{(2)}\}_{j=0}^\infty)$ are (h)-minimal, it follows from (8.25), (8.27), (8.28), property (d), and (8.20) that

$$\begin{gathered}\|x_1 - x_f\|, \|x_2 - x_f\|, \|y_1 - y_f\|, \|y_2 - y_f\| \le 8^{-1}\delta_5(f,r,i) < 2^{-i} < \epsilon, \\ \|x_1 - x_2\|, \|y_1 - y_2\| \le 2\epsilon.\end{gathered} \tag{8.29}$$

Since ϵ is an arbitrary number in the interval $(0,1)$, we conclude that (8.24) is valid. Therefore, we have shown that there exists a unique pair $(x_h, y_h) \in X \times Y$ such that

$$\sup_{y\in Y} h\big(x_h, x_h, y, y\big) = h\big(x_h, x_h, y_h, y_h\big) = \inf_{x\in X} h\big(x, x, y_h, y_h\big). \tag{8.30}$$

Let $\epsilon > 0$. Choose a natural number k for which (8.27) holds. There exist $f \in \mathfrak{M}$, $r \in (0,1)$ and an integer $i \ge k$ for which (8.28) is valid. Consider the sequences $\{x_j^{(h)}\}_{j=0}^\infty \subset X$, $\{y_j^{(h)}\}_{j=0}^\infty \subset Y$ defined by

$$x_j^{(h)} = x_h, \quad y_j^{(h)} = y_h, \quad j = 0, 1, \dots. \tag{8.31}$$

It was shown above that the pair of sequences $\{x_j^{(h)}\}_{j=0}^\infty$, $\{y_j^{(h)}\}_{j=0}^\infty$ is (h)-minimal. It follows from (8.27), (8.31), (8.20), and property (d) that

$$\|x_h - x_f\|, \|y_h - y_f\| \le 8^{-1}\delta_5(f,r,i). \tag{8.32}$$

Assume that $g \in U(f,r,i)$, an integer $n \ge 2n_4(f,r,i)$, and $\{x_j\}_{j=0}^n \subset X$, $\{y_j\}_{j=0}^n \subset Y$ is a $(g, \delta_5(f,r,i))$-good pair of sequences. It follows from property (c), (8.17), (8.20), and (8.32) that the following properties hold:

(i) $\|x_j - x_f\|, \|y_j - y_f\| \le 8^{-1}\delta_4(f,r,i)$, and $\|x_j - x_h\|, \|y_j - y_h\| \le \epsilon$ for all integers $j \in [n_4(f,r,i), n - n_4(f,r,i)]$;

(ii) if $\|x_0 - x_f\|, \|y_0 - y_f\| \le \delta_5(f,r,i)$, then $\|x_j - x_f\|, \|y_j - y_f\| \le 8^{-1}\delta_4(f,r,i)$ for all integers $j \in [0, n - n_4(f,r,i)]$;

(iii) if $\|x_n - x_f\|, \|y_n - y_f\| \le \delta_5(f,r,i)$, then $\|x_j - x_f\|, \|y_j - y_f\| \le 8^{-1}\delta_4(f,r,i)$ for all integers $j \in [n_4(f,r,i), n]$.

Together with (8.32) this implies that the following properties hold:

(i) if $\|x_0 - x_h\|, \|y_0 - y_h\| \le 2^{-1}\delta_5(f,r,i)$, then $\|x_j - x_f\|, \|y_j - y_f\| \le 8^{-1}\delta_4(f,r,i)$ for all integers $j \in [0, n - n_4(f,r,i)]$;

(ii) if $\|x_n - x_h\|, \|y_n - y_h\| \le 2^{-1}\delta_5(f,r,i)$, then $\|x_j - x_f\|, \|y_j - y_f\| \le 8^{-1}\delta_4(f,r,i)$ is valid for all integers $j \in [n_4(f,r,i), n]$.

This completes the proof of the theorem. □

Proof of Theorem 2.2. Let $h \in \mathfrak{F}$, $z \in X$, $\xi \in Y$. By Theorem 2.1 there exists a unique pair $(x_h, y_h) \in X \times Y$ such that

$$\sup_{y\in Y} h\big(x_h, x_h, y, y\big) = h\big(x_h, x_h, y_h, y_h\big) = \inf_{x\in X} h\big(x, x, y_h, y_h\big). \tag{8.33}$$

By Proposition 5.3 there is an (h)-minimal pair of sequences $\{\bar{x}_j\}_{j=0}^\infty \subset X$, $\{\bar{y}_j\}_{j=0}^\infty \subset Y$ for which

$$\bar{x}_0 = z, \qquad \bar{y}_0 = \xi. \tag{8.34}$$

We show that the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (h)-overtaking optimal. Theorem 2.1 implies that

$$\bar{x}_j \longrightarrow x_h, \quad \bar{y}_j \longrightarrow y_h \quad \text{as } j \longrightarrow \infty. \tag{8.35}$$

Let $\{x_i\}_{i=0}^{\infty} \subset X$ and $x_0 = z$. We show that

$$\limsup_{T\to\infty} \left[\sum_{j=0}^{T-1} h\big(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}\big) - \sum_{j=0}^{T-1} h\big(x_j, x_{j+1}, \bar{y}_j, \bar{y}_{j+1}\big) \right] \le 0. \tag{8.36}$$

Assume the contrary. Then there exists a number $\Gamma_0 > 0$ and a strictly increasing sequence of natural numbers $\{T_k\}_{k=1}^{\infty}$ such that for all integers $k \ge 1$

$$\sum_{j=0}^{T_k-1} h\big(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}\big) - \sum_{j=0}^{T_k-1} h\big(x_j, x_{j+1}, \bar{y}_j, \bar{y}_{j+1}\big) \ge \Gamma_0. \tag{8.37}$$

We show that

$$x_j \longrightarrow x_h \quad \text{as } j \longrightarrow \infty. \tag{8.38}$$

For $j = 0, 1, \ldots$ define a function $g_j : X \times X \to \mathbb{R}^1$ by

$$g_j\big(u_1, u_2\big) = h\big(u_1, u_2, \bar{y}_j, \bar{y}_{j+1}\big), \quad u_1, u_1 \in X. \tag{8.39}$$

Clearly $g_j \in C(X \times X)$, $j = 0, 1, \ldots$. Let $\epsilon > 0$. Choose a natural number q such that

$$2^{-q} < 64^{-1}\epsilon. \tag{8.40}$$

There exist $f \in \mathfrak{M}$, $r \in (0,1)$, and an integer $p \ge q$ such that

$$h \in U(f, r, p). \tag{8.41}$$

Since the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (h)-minimal, it follows from the definition of $U(f,r,p)$ (see (8.20)), (8.41), and property (d) that for all integers $j \ge n_5(f,r,p)$

$$\|\bar{x}_j - x_f\|, \|\bar{y}_j - y_f\| \le 8^{-1}\delta_5(f,r,p). \tag{8.42}$$

By (8.33), Proposition 5.1, (8.41), and property (d)

$$\|x_h - x_f\|, \|y_h - y_f\| \le 8^{-1}\delta_5(f,r,p). \tag{8.43}$$

Since the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (h)-minimal there exists a constant $c_0 > 0$ such that for each integer $T \ge 1$

$$\sum_{j=0}^{T-1} h\big(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}\big) \le \inf\left\{ \sum_{j=0}^{T-1} h\big(u_j, u_{j+1}, \bar{y}_j, \bar{y}_{j+1}\big) : \{u_j\}_{j=0}^{T} \subset X,\ u_0 = z \right\} + c_0. \tag{8.44}$$

Equations (8.44), (8.39), and (8.37) imply that the following property holds:

(e) for each $\Delta > 0$ there exists an integer $j(\Delta) \geq 1$ such that for each pair of integers $n_1 \geq j(\Delta),\ n_2 > n_1$ the sequence $\{x_j\}_{j=n_1}^{n_2}$ is $(\{g_j\}_{j=n_1}^{n_2-1}, X, \Delta)$-good.

Consider the function $f_r^{(X)} : X \times X \to \mathbb{R}^1$ defined by (7.3). For $j = 0, 1, \dots$ define a function $\bar{g}_j : X \times X \to \mathbb{R}^1$ by

$$\bar{g}_j\big(u_1, u_2\big) = f_r\big(u_1, u_2, \bar{y}_j, \bar{y}_{j+1}\big), \quad u_1, u_2 \in X. \tag{8.45}$$

It follows from (7.3), (8.12), (8.42), (8.45), and the definition of $\delta_4(f, r, p)$ (see (8.9), (8.11)) that for all integers $j \geq n_5(f, r, p)$

$$\big\|\bar{g}_j - f_r^{(X)}\big\| \leq 16^{-1}\gamma_3(f, r, p). \tag{8.46}$$

By (8.12), (8.17), (8.20), (8.39), (8.41), (8.45), and (8.46) for all integers $j \geq n_5(f, r, p)$

$$\big\|g_j - f_r^{(X)}\big\| \leq 16^{-1}\gamma_3(f, r, p) + \gamma(f, r, p) < \gamma_3(f, r, p). \tag{8.47}$$

It follows from (8.47), properties (e) and (a), and (8.8) that there exists an integer $m_0 \geq 1$ such that $\|x_j - x_f\| \leq 2^{-p}$ for all integers $j \geq m_0$. Together with (8.40) and (8.43) this implies that for all integers $j \geq m_0$, the relation $\|x_j - x_h\| \leq 2^{-p} + 2^{-p} < \epsilon$ is true. Since ϵ is an arbitrary positive number, we conclude that

$$\lim_{j\to\infty} x_j = x_h. \tag{8.48}$$

There exists a number $\epsilon_0 > 0$ such that for each $z_1, z_2, \bar{z}_1, \bar{z}_2 \in X$ and each $\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2 \in Y$ which satisfy

$$\big\|z_j - \bar{z}_j\big\|, \big\|\xi_j - \bar{\xi}_j\big\| \leq 2\epsilon_0, \quad j = 1, 2, \tag{8.49}$$

the following relation holds:

$$\big|h\big(z_1, z_2, \xi_1, \xi_2\big) - h\big(\bar{z}_1, \bar{z}_2, \bar{\xi}_1, \bar{\xi}_2\big)\big| \leq 8^{-1}\Gamma_0. \tag{8.50}$$

By (8.35) and (8.48) there exists an integer $j_0 \geq 8$ such that for all integers $j \geq j_0$

$$\|x_j - x_h\| \leq 2^{-1}\epsilon_0, \qquad \|\bar{x}_j - x_h\| \leq 2^{-1}\epsilon_0. \tag{8.51}$$

There exists an integer $s \geq 1$ such that

$$T_s > j_0. \tag{8.52}$$

Define a sequence $\{x_j^*\}_{j=0}^{s} \subset X$ by

$$x_j^* = x_j, \quad j = 0, \dots, T_s - 1, \quad x_{T_s}^* = \bar{x}_{T_s}. \tag{8.53}$$

Since the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (h)-minimal, we conclude that by (8.53)

$$\sum_{j=0}^{T_s-1} h\big(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}\big) - \sum_{j=0}^{T_s-1} h\big(x_j^*, x_{j+1}^*, \bar{y}_j, \bar{y}_{j+1}\big) \leq 0. \tag{8.54}$$

On the other hand, it follows from (8.37), (8.51), (8.52), (8.53), and the definition of ϵ_0 (see (8.49),(8.50)) that

$$\begin{aligned}
&\sum_{j=0}^{T_s-1} h\big(\bar{x}_j,\bar{x}_{j+1},\bar{y}_j,\bar{y}_{j+1}\big)-\sum_{j=0}^{T_s-1} h\big(x_j^*,x_{j+1}^*,\bar{y}_j,\bar{y}_{j+1}\big)\\
&\quad=\sum_{j=0}^{T_s-1} h\big(\bar{x}_j,\bar{x}_{j+1},\bar{y}_j,\bar{y}_{j+1}\big)-\sum_{j=0}^{T_s-1} h\big(x_j,x_{j+1},\bar{y}_j,\bar{y}_{j+1}\big)\\
&\qquad+h\big(x_{T_s-1},x_{T_s},\bar{y}_{T_s-1},\bar{y}_{T_s}\big)-h\big(x_{T_s-1}^*,x_{T_s}^*,\bar{y}_{T_s-1},\bar{y}_{T_s}\big)\\
&\quad\ge \Gamma_0+h\big(x_{T_s-1},x_{T_s},\bar{y}_{T_s-1},\bar{y}_{T_s}\big)-h\big(x_{T_s-1},\bar{x}_{T_s},\bar{y}_{T_s-1},\bar{y}_{T_s}\big)\\
&\quad\ge \Gamma_0-8^{-1}\Gamma_0.
\end{aligned}$$

This is contradictory to (8.54). The obtained contradiction proves that (8.36) holds.

Analogously we can show that for each sequence $\{y_j\}_{j=0}^{\infty}\subset Y$ satisfying $y_0=\xi$

$$\limsup_{T\to\infty}\left[\sum_{j=0}^{T-1} h\big(\bar{x}_j,\bar{x}_{j+1},y_j,y_{j+1}\big)-\sum_{j=0}^{T-1} h\big(\bar{x}_j,\bar{x}_{j+1},\bar{y}_j,\bar{y}_{j+1}\big)\right]\le 0. \tag{8.55}$$

This implies that the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty},\{\bar{y}_j\}_{j=0}^{\infty})$ is (h)-overtaking optimal. This completes the proof of the theorem. □

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