CRITICAL GROUPS OF CRITICAL POINTS PRODUCED BY LOCAL LINKING WITH APPLICATIONS

KANISHKA PERERA

Abstract. We prove the existence of nontrivial critical points with nontrivial critical groups for functionals with a local linking at 0. Applications to elliptic boundary value problems are given.

1. Introduction

Let F be a real C^1 function defined on a Banach space X. We say that F has a local linking near the origin if X has a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_1 < \infty$, $F(0) = 0$, and, for some $r > 0$,

$$\begin{cases} F(u) \leq 0 & \text{for } u \in X_1,\ \|u\| \leq r, \\ F(u) > 0 & \text{for } u \in X_2,\ 0 < \|u\| \leq r. \end{cases} \tag{1}$$

Then it is clear that 0 is a critical point of F.

The notion of local linking was introduced by Li and Liu [7], [8], who proved the existence of nontrivial critical points under various assumptions on the behavior of F at infinity. These results were recently generalized by Brézis and Nirenberg [3], Li and Willem [9], and several other authors.

In infinite dimensional Morse theory (see Chang [5] or Mawhin and Willem [11]), the local behavior of F near an isolated critical point u_0, $F(u_0) = c$, is described by the sequence of critical groups

$$C_q(F, u_0) = H_q(F_c \cap U, (F_c \cap U) \backslash \{u_0\}) \quad q \in \mathbb{Z}$$

where F_c is the sublevel set $\{u \in X : F(u) \leq c\}$, U is a neighborhood of u_0 such that u_0 is the only critical point of F in $F_c \cap U$, and $H_*(\cdot, \cdot)$ denote the singular relative homology groups.

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It was proved in Liu [10] that if F has a local linking near the origin, $\dim X_1 = j$, and 0 is an isolated critical point of F, then $C_j(F, 0) \neq 0$. In the present paper we use this fact to obtain a nontrivial critical point u with either $C_{j+1}(F, u) \neq 0$ or $C_{j-1}(F, u) \neq 0$. When X is a Hilbert space and F is C^2, this yields Morse index estimates for u via the Shifting theorem.

When X is a Hilbert space and dF is Lipschitz in a neighborhood of the origin, we extend the result of Liu [10] to the case where F satisfies the "relaxed" local linking condition

$$\begin{cases} F(u) \leq 0 & \text{for } u \in X_1, \ \|u\| \leq r, \\ \\ F(u) \geq 0 & \text{for } u \in X_2, \ \|u\| \leq r \end{cases} \tag{2}$$

(see Brézis and Nirenberg [3] and Li and Willem [9]), and thus obtain a nontrivial critical point with a nontrivial critical group in this case also.

We apply our abstract result to elliptic boundary value problems, including an equation asymptotically linear at $-\infty$ and superlinear at $+\infty$, and prove new multiplicity results.

2. Abstract Result

Throughout this section we assume that F satisfies the Palais-Smale compactness condition (PS) and has only isolated critical values, with each critical value corresponding to a finite number of critical points.

Theorem 2.1. *Suppose that there is a critical point u_0 of F, $F(u_0) = c$, with $C_j(F, u_0) \neq 0$ for some $j \geq 0$ and regular values a, b of F, $a < c < b$, such that $H_j(F_b, F_a) = 0$. Then F has a critical point u with either*

$$c < F(u) < b \quad \textit{and} \quad C_{j+1}(F, u) \neq 0, \quad \textit{or}$$

$$a < F(u) < c \quad \textit{and} \quad C_{j-1}(F, u) \neq 0.$$

Proof of Theorem 2.1 makes use of the following topological lemma:

Lemma 2.2. *If $B' \subset B \subset A \subset A'$ are topological spaces such that $H_j(A, B) \neq 0$ and $H_j(A', B') = 0$, then either*

$$H_{j+1}(A', A) \neq 0 \textit{ or } H_{j-1}(B, B') \neq 0.$$

Proof. Suppose that $H_{j+1}(A', A) = 0$. Since $H_j(A', B')$ is also trivial, it follows from the following portion of the exact sequence of the triple (A', A, B') that $H_j(A, B') = 0$:

$$H_{j+1}(A', A) \xrightarrow{\partial_*} H_j(A, B') \xrightarrow{i_*} H_j(A', B')$$

Since $H_j(A, B) \neq 0$, now it follows from the following portion of the exact sequence of the triple (A, B, B') that $H_{j-1}(B, B') \neq 0$:

$$H_j(A, B') \xrightarrow{j_*} H_j(A, B) \xrightarrow{\partial_*} H_{j-1}(B, B') \ \square$$

Proof of Theorem 2.1. Take ϵ, $0 < \epsilon < \min\{c-a, b-c\}$ such that c is the only critical value of F in $[c-\epsilon, c+\epsilon]$. Then, since $C_j(F, u_0) \neq 0$, it follows from Chapter I, Theorem 4.2 of Chang [5] that $H_j(F_{c+\epsilon}, F_{c-\epsilon}) \neq 0$. Since $H_j(F_b, F_a) = 0$, by Lemma 2.2, either $H_{j+1}(F_b, F_{c+\epsilon}) \neq 0$ or $H_{j-1}(F_{c-\epsilon}, F_a) \neq 0$, and the conclusion follows from Chapter I, Theorem 4.3 and Corollary 4.1 of Chang [5]. ∎

As mentioned before, if F has a local linking near the origin, $\dim X_1 = j$, then $C_j(F, 0) \neq 0$ (see Liu [10]), and hence the following corollary is immediate from Theorem 2.1:

Corollary 2.3. *Suppose F has a local linking near the origin,* $\dim X_1 = j$*. Assume also that there are regular values a, b of F, $a < 0 < b$, such that $H_j(F_b, F_a) = 0$. Then F has a critical point u with either*

$$0 < F(u) < b \quad \textit{and} \quad C_{j+1}(F, u) \neq 0, \quad \textit{or}$$
$$a < F(u) < 0 \quad \textit{and} \quad C_{j-1}(F, u) \neq 0.$$

If X is a Hilbert space, F is C^2, and u is a critical point of F, we denote by $m(u)$ the Morse index of u and by $m^*(u) = m(u) + \dim\ker d^2F(u)$ the large Morse index of u. We recall that if u is nondegenerate and $C_q(F, u) \neq 0$, then $m(u) = q$ (see Chapter I, Theorem 4.1 of Chang [5]). Let us also recall that it follows from the Shifting theorem (Chapter I, Theorem 5.4 of Chang [5]) that if u is degenerate, 0 is an isolated point of the spectrum of $d^2F(u)$, and $C_q(F, u) \neq 0$, then $m(u) \leq q \leq m^*(u)$. Hence we have the following corollary:

Corollary 2.4. *Let X be a Hilbert space and F be C^2 in Theorem 2.1. Assume that for every degenerate critical point u of F, 0 is an isolated point of the spectrum of $d^2F(u)$. Then F has a critical point u with either*

$$c < F(u) < b \quad \textit{and} \quad m(u) \leq j+1 \leq m^*(u), \quad \textit{or}$$
$$a < F(u) < c \quad \textit{and} \quad m(u) \leq j-1 \leq m^*(u).$$

Remark 2.5. In particular, Corollary 2.4 yields a critical point $u \neq u_0$ with $m(u) \leq j+1$ and $j-1 \leq m^*(u)$. Benci and Fortunato [2] have proved this fact for the special case where u_0 is a nondegenerate critical point with Morse index j, but without assuming that the critical points of F are isolated. Their proof is based on a generalized Morse theory due to Benci and Giannoni [1]. However, Corollary 2.4 says, in addition, that u is at a level different from $F(u_0)$.

If X is a Hilbert space and dF is Lipschitz in a neighborhood of the origin, we can relax the local linking condition as in (2). This follows from the following extension of the result of Liu [10] (see also Theorem 5.6 of Kryszewski and Szulkin [6]):

Theorem 2.6. *Let X be a Hilbert space and dF be Lipschitz in a neighborhood of the origin. Suppose that F satisfies the local linking condition (2),* $\dim X_1 = j$*. Then $C_j(F, 0) \neq 0$.*

Our proof of Theorem 2.6 uses the following "deformation" lemma:

Lemma 2.7. *Under the assumptions of Theorem 2.6 there exist a closed ball B centered at the origin and a homeomorphism h of X onto X such that*

1. 0 *is the only critical point of F in $h(B)$,*
2. $h|_{B \cap X_1} = id_{B \cap X_1}$,
3. $F(u) > 0$ *for $u \in h(B \cap X_2 \backslash \{0\})$.*

Proof. Take open balls B', B'' centered at the origin, with $\overline{B'} \subset B''$, such that 0 is the only critical point of F in B' and dF is Lipschitz in B'', and let $B \subset B'$ be a closed ball centered at the origin with radius $\leq r$ (in (2)). Since B and $(B')^c$ are disjoint closed sets there is a locally Lipschitz nonnegative function $g \leq 1$ satisfying

$$g = \begin{cases} 1 & \text{on } B \\ 0 & \text{outside } B'. \end{cases}$$

Consider the vector field

$$V(u) = g(u)\, \|Pu\|\, dF(u)$$

where P is the orthogonal projection onto X_2. Clearly V is locally Lipshitz and bounded on X. Consider the flow $\eta(t) = \eta(t, u)$ defined by

$$\frac{d\eta}{dt} = V(\eta), \;\; \eta|_{t=0} = u.$$

Clearly, η is defined for $t \in [0, 1]$. Let $h = \eta(1, \cdot)$. Since $h|_{(B')^c} = \mathrm{id}_{(B')^c}$ and h is one-to-one, $h(B) \subset B'$ and *1* follows. For $u \in B \cap X_2 \backslash \{0\}$,

$$F(h(u)) = F(u) + \int_0^1 g(\eta(t))\, \|P\eta(t)\|\, \|dF(\eta(t))\|^2\, dt > 0$$

since $F(u) \geq 0$ and $g(u)\, \|Pu\|\, \|dF(u)\|^2 > 0$. ∎

Proof of Theorem 2.6. By *1* of Lemma 2.7, $C_j(F, 0) = H_j(F_0 \cap h(B), F_0 \cap h(B) \backslash \{0\})$.

By the local linking condition (2) and *2* and *3* of Lemma 2.7, $\partial B \cap X_1 \subset F_0 \cap h(B) \backslash \{0\} \subset h(B \backslash X_2)$ and $B \cap X_1 \subset F_0 \cap h(B)$. Since $h|_{\partial B \cap X_1} = \mathrm{id}_{\partial B \cap X_1}$, the inclusion $\partial B \cap X_1 \hookrightarrow h(B \backslash X_2)$ can also be written as the composition of the inclusion $\partial B \cap X_1 \overset{i'}{\hookrightarrow} B \backslash X_2$ and the restriction of h to $B \backslash X_2$. Hence we have the following commutative diagram induced by inclusions and h:

$$\begin{array}{ccccc} H_{j-1}(B \backslash X_2) & \xleftarrow{\;i'_*\;} & H_{j-1}(\partial B \cap X_1) & \longrightarrow & H_{j-1}(B \cap X_1) \\ {\scriptstyle h_*}\downarrow & & {\scriptstyle i''_*}\downarrow & & \downarrow \\ H_{j-1}(h(B \backslash X_2)) & \longleftarrow & H_{j-1}(F_0 \cap h(B) \backslash \{0\}) & \xrightarrow[\;i_*\;]{} & H_{j-1}(F_0 \cap h(B)) \end{array}$$

Since $\partial B \cap X_1$ is a strong deformation retract of $B \backslash X_2$ and h is a homeomorphism, i'_* and h_* are isomorphisms and hence i''_* is a monomorphism.

Since $\operatorname{rank} H_{j-1}(B \cap X_1) < \operatorname{rank} H_{j-1}(\partial B \cap X_1)$, then it follows that i_* is not a monomorphism.

Now it follows from the following portion of the exact sequence of the pair $(F_0 \cap h(B), F_0 \cap h(B) \backslash \{0\})$ that $C_j(F, 0) = H_j(F_0 \cap h(B), F_0 \cap h(B) \backslash \{0\}) \neq 0$:

$$C_j(F,0) \xrightarrow{\partial_*} H_{j-1}(F_0 \cap h(B) \backslash \{0\}) \xrightarrow{i_*} H_{j-1}(F_0 \cap h(B)) \qquad \blacksquare$$

3. Elliptic Boundary Value Problems

Consider the problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & g(u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^{\ltimes}$ with smooth boundary $\partial\Omega$ and $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

(g_1): $|g(u)| \leq C\,(1 + |u|^{p-1})$ with $2 < p < \frac{2n}{n-2}$, for some $C > 0$,
(g_2): $g(0) = 0 = g(a)$ for some $a > 0$,
(g_3): there are constants $\mu > 2$ and $A > 0$ such that

$$0 < \mu\, G(u) \leq u\, g(u) \quad \text{for } |u| \geq A,$$

where $G(u) := \int_0^u g(t)\, dt$.

Let $\lambda = g'(0)$ and let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $-\Delta$ with Dirichlet boundary condition.

Theorem 3.1. *Assume that g satisfies $(g_1) - (g_3)$ and one of the following conditions:*

1. $\lambda_j < \lambda < \lambda_{j+1}$,
2. $\lambda_j = \lambda < \lambda_{j+1}$ *and, for some* $\delta > 0$,

$$G(u) \geq \frac{1}{2}\, \lambda\, u^2 \quad \textit{for } |u| \leq \delta,$$

3. $\lambda_j < \lambda = \lambda_{j+1}$ *and, for some* $\delta > 0$,

$$G(u) \leq \frac{1}{2}\, \lambda\, u^2 \quad \textit{for } |u| \leq \delta.$$

If $j \geq 3$, problem (3) has at least four nontrivial solutions.

Proof. Solutions of (3) are the critical points of the C^2 functional

$$F(u) = \int_\Omega \frac{1}{2}\, |\nabla u|^2 - G(u)$$

defined on $X = H_0^1(\Omega)$. It is well known that F satisfies (PS).

By a standard argument involving a cut-off technique and the strong maximum principle, F has a local minimizer u_0 with $0 < u_0 < a$,

$$\operatorname{rank} C_q(F, u_0) = \delta_{q0}.$$

Since $\lim_{t\to\infty} F(\pm\, t\phi_1) = -\infty$, where $\phi_1 > 0$ is the first Dirichlet eigenfunction of $-\Delta$, then F also has two mountain pass points $u_1^\pm$ with $u_1^- < u_0 < u_1^+$,

$$\operatorname{rank} C_q(F, u_1^\pm) = \delta_{q1}$$

(see the proof of Theorem B in Chang, Li, and Liu [4]).

Let X_1 be the j-dimensional space spanned by the eigenfunctions corresponding to $\lambda_1, \cdots, \lambda_j$ and let X_2 be its orthogonal complement in X. Then F has a local linking near the origin with respect to the decomposition $X = X_1 \oplus X_2$ (see the proof of Theorem 4 in Li and Willem [9]) and hence

$$C_j(F, 0) \neq 0.$$

Also, for $\alpha < 0$ and $|\alpha|$ sufficiently large,

$$H_q(X, F_\alpha) = 0 \quad \forall q \in \mathbb{Z}$$

(see Lemma 3.2 of Wang [13]). Therefore, by Theorem 2.1, F has a nontrivial critical point u_j with either

$$C_{j+1}(F, u_j) \neq 0 \text{ or } C_{j-1}(F, u_j) \neq 0.$$

Since $j \geq 3$, a comparison of the critical groups shows that u_0, $u_1^\pm$, u_j are distinct nontrivial critical points of F. ■

Next we consider the following asymmetric problem of the Ambrosetti-Prodi type

$$\text{(4)} \qquad \begin{cases} -\Delta u + a(x)\, u &= g(x,u) \quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where $a \in L^\infty(\Omega)$ and $g \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies

(g_1): $|g(x,u)| \leq C\,(1 + |u|^{p-1})$ with $2 < p < \frac{2n}{n-2}$, for some $C > 0$,

(g_2): $g(x,0) = g_u(x,0) = 0$,

(g_3): $\lim_{u\to-\infty} \frac{g(x,u)}{u} < \lambda_1$, uniformly in $\overline{\Omega}$,

(g_4): $\overline{\lim}_{u\to-\infty} \Big(G(x,u) - \frac{1}{2}\, u\, g(x,u)\Big) < +\infty$, uniformly in $\overline{\Omega}$,

(g_5): there are $\mu > 2$ and $A > 0$ such that

$$0 < \mu\, G(x,u) \leq u\, g(x,u) \quad \text{for } u \geq A,$$

where $G(x,u) := \int_0^u g(x,t)\, dt$.

Here $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ denote the eigenvalues of $-\Delta + a$ with Dirichlet boundary condition.

Theorem 3.2. *Assume that g satisfies $(g_1) - (g_5)$ and one of the following conditions:*

1. $\lambda_j < 0 < \lambda_{j+1}$,
2. $\lambda_j = 0 < \lambda_{j+1}$ *and, for some* $\delta > 0$,
$$G(x,u) \geq 0 \quad \textit{for } |u| \leq \delta,$$
3. $\lambda_j < 0 = \lambda_{j+1}$ *and, for some* $\delta > 0$,
$$G(x,u) \leq 0 \quad \textit{for } |u| \leq \delta.$$

If $j \geq 3$, problem (4) has at least three nontrivial solutions.

We seek critical points of

$$F(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + a(x)\, u^2\right) - G(x,u)$$

on $X = H_0^1(\Omega)$.

Lemma 3.3. *If g satisfies (g_1), $(g_3) - (g_5)$, then, for $\alpha < 0$ and $|\alpha|$ sufficiently large,*

$$H_q(X, F_\alpha) = 0 \quad \forall q \in \mathbb{Z}.$$

Proof. Let $\tilde{X} = C_0^1(\overline{\Omega})$ and $\tilde{F} = F|_{\tilde{X}}$. By elliptic regularity, F and $\tilde{F}$ have the same critical set. If F does not have any critical values in (α, α'), then F_α (respectively $\tilde{F}_\alpha$) is a strong deformation retract of $\{u \in X : F(u) < \alpha'\}$ (respectively $\{u \in \tilde{X} : \tilde{F}(u) < \alpha'\}$) (see Chapter I, Theorem 3.2 and Chapter III, Theorem 1.1 of Chang [5]). Since $\tilde{X}$ is dense in X, by a theorem of Palais [12],

$$H_q(X, \{F < \alpha'\}) \cong H_q(\tilde{X}, \{\tilde{F} < \alpha'\}).$$

Therefore it suffices to prove that, for $\alpha < 0$ and $|\alpha|$ large,

$$H_q(\tilde{X}, \tilde{F}_\alpha) = 0 \quad \forall q \in \mathbb{Z}.$$

Let $S^\infty = \left\{u \in \tilde{X} : \|u\|_X = 1\right\}$ be the unit sphere in $\tilde{X}$ and let $S_+^\infty = \{u \in S^\infty : u > 0 \text{ somewhere}\}$, which is a relatively open subset of S^∞, contractible to $\{\phi_1\}$ via $(t,u) \longmapsto \frac{(1-t)\,u + t\,\phi_1}{\|(1-t)\,u + t\,\phi_1\|}$ $t \in [0,1]$. We shall show that $\tilde{F}_\alpha$ is homotopy equivalent to S_+^∞ for $\alpha < 0$ and $|\alpha|$ large.

By (g_3) and (g_5),

$$-C\,(1+u^2) \le G(x,u) \le \frac{1}{2}\,\lambda_1\, u^2 + C \quad \text{for } u \le A,$$

$$G(x,u) \ge C\, u^\mu \quad \text{for } u \ge A,$$

where C denotes (possibly different) positive constants. Thus for $u \in S_+^\infty$,

$$\begin{aligned} \tilde{F}(tu) &= \frac{1}{2}\left(1 + \int_\Omega a u^2\right) t^2 - \int_\Omega G(x,tu) \\ &\le C\left(1 + t^2 - t^\mu \int_{tu \ge A} u^\mu\right) \end{aligned}$$

and it follows that

$$\lim_{t \to \infty} \tilde{F}(tu) = -\infty.$$

On the other hand, in $N = \left\{u \in \tilde{X} : u \le 0 \text{ everywhere}\right\}$, the nonpositive cone in $\tilde{X}$,

$$\tilde{F}(u) \ge \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + a(x)\, u^2 - \lambda_1\, u^2\right) - C \ge -C.$$

By (g_4) and (g_5),

$$\gamma := \sup_{\overline{\Omega} \times \mathbb{R}} \left(G(x,u) - \frac{1}{2}\, u\, g(x,u)\right) < +\infty.$$

Thus for $u \in S_+^\infty$ and $t > 0$,

$$\begin{aligned} \frac{d}{dt}\,\tilde{F}(tu) &= \Big(1 + \int_\Omega a u^2\Big)\, t - \int_\Omega u\, g(x, tu) \\ &= \frac{2}{t}\left\{\tilde{F}(tu) + \int_\Omega G(x, tu) - \frac{1}{2}\, tu\, g(x, tu)\right\} \\ &\le \frac{2}{t}\left\{\tilde{F}(tu) + \gamma\,|\Omega|\right\} < 0 \end{aligned}$$

if $\tilde{F}(tu) < -\gamma\,|\Omega|$.

Fix $\alpha < \min\left\{\inf_N \tilde{F},\ -\gamma\,|\Omega|,\ \inf_{\|u\|<1} \tilde{F}\right\}$. Then it follows that for each $u \in S_+^\infty$ there exists a unique $T(u) \ge 1$ such that

$$\tilde{F}(tu) \begin{cases} > \alpha & \text{for } 0 \le t < T(u) \\ = \alpha & \text{for } t = T(u) \\ < \alpha & \text{for } t > T(u), \end{cases}$$

and

$$\tilde{F}_\alpha = \left\{tu : u \in S_+^\infty,\ t \ge T(u)\right\}.$$

By the implicit function theorem, $T \in C(S_+^\infty, [1, \infty))$. Hence

$$\eta(s, tu) = \begin{cases} (1 - s)\, tu + s\, T(u)\, u & \text{if } 1 \le t < T(u) \\ tu & \text{if } t \ge T(u) \end{cases}$$

defines a strong deformation retraction of $\left\{tu : u \in S_+^\infty,\ t \ge 1\right\} \simeq S_+^\infty$ onto $\tilde{F}_\alpha$. ∎

Proof of Theorem 3.2. Since $F(-t\phi_1) < 0$ for $t > 0$ sufficiently small, by standard arguments, F has a local minimizer u_0 with $u_0 < 0$,

$$\operatorname{rank} C_q(F, u_0) = \delta_{q0}.$$

Since $\lim_{t\to\infty} F(t\phi_1) = -\infty$, then F also has a mountain pass point u_1,

$$\operatorname{rank} C_q(F, u_1) = \delta_{q1}.$$

As in the proof of Theorem 3.1,

$$C_j(F, 0) \ne 0,$$

so, using Lemma 3.3, F also has a nontrivial critical point u_j with either

$$C_{j+1}(F, u_j) \ne 0 \text{ or } C_{j-1}(F, u_j) \ne 0.$$

Since $j \ge 3$, u_0, u_1, u_j are distinct nontrivial solutions of (4). ∎

Finally we give an application of Theorem 2.1 to the problem

$$\begin{cases} -\Delta u + a(x)\, u = \lambda\, g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

where $a \in L^\infty(\Omega)$ and $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

(g_1)**:** $\overline{\lim}_{|u|\to\infty} \frac{g(u)}{u} < 0$,
(g_2)**:** $g(0) = g'(0) = 0$.

Theorem 3.4. *Assume that g satisfies (g_1), (g_2), and one of the following conditions:*

1. $\lambda_j < 0 < \lambda_{j+1}$,
2. $\lambda_j = 0 < \lambda_{j+1}$ *and, for some* $\delta > 0$,
$$G(u) \geq 0 \quad \text{for } |u| \leq \delta,$$
3. $\lambda_j < 0 = \lambda_{j+1}$ *and, for some* $\delta > 0$,
$$G(u) \leq 0 \quad \text{for } |u| \leq \delta.$$

If $j \geq 3$, problem (5) has at least four nontrivial solutions for every λ sufficiently large.

Example 3.5. $g(u) = \pm\,|u|\,u - u^3$

Remark 3.6. See Brézis and Nirenberg [3] and Li and Willem [9] for at least two nontrivial solutions.

Proof of Theorem 3.4. Since, for λ sufficiently large, there is an a priori estimate for the solutions of (5) by the maximum principle, we may also assume that $g(u) = bu$ with $b < 0$, for $|u|$ large. Then the functional

$$F(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + au^2\right) - \lambda\, G(u)$$

is well defined on $X = H_0^1(\Omega)$, and bounded below and satisfies (PS) for λ large.

Since $F(\pm\, t\phi_1) < 0$ for $t > 0$ sufficiently small, F has two local minimizers $u_0^\pm$ with $u_0^- < 0 < u_0^+$,

$$\operatorname{rank} C_q(F, u_0^\pm) = \delta_{q0}.$$

Then F also has a mountain pass point u_1,

$$\operatorname{rank} C_q(F, u_1) = \delta_{q1}.$$

As before,

$$C_j(F, 0) \neq 0,$$

and, for $\alpha < \inf F$,

$$\operatorname{rank} H_q(X, F_\alpha) = \delta_{q0},$$

so F has a (fourth) nontrivial critical point u_j with either

$$C_{j+1}(F, u_j) \neq 0 \text{ or } C_{j-1}(F, u_j) \neq 0. \blacksquare$$

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Department of Mathematics
University of California Irvine
Irvine, CA 92697-3875, USA

E-mail: kperera@math.uci.edu