SPECTRAL PROPERTIES OF OPERATORS THAT CHARACTERIZE $\ell_\infty^{(n)}$

B. L. CHALMERS AND B. SHEKHTMAN

Abstract. It is well known that the identity is an operator with the following property: if the operator, initially defined on an n-dimensional Banach space V, can be extended to any Banach space with norm 1, then V is isometric to $\ell_\infty^{(n)}$. We show that the set of all such operators consists precisely of those with spectrum lying in the unit circle. This result answers a question raised in [5] for complex spaces.

1. Introduction

Let V be an n-dimensional Banach space and let T be an operator mapping V into V. If X is another Banach space with $X \supset V$ we let

$$e(T, X) := \inf\{\|\tilde{T}\| : \ \tilde{T} : X \to V, \ \tilde{T}|_V = T\}$$

and

$$e(T) := \sup\{e(T, X) : X \supset V\}.$$

In particular, if $T = I_V$ is the identity on the space V, then $e(T, X) = \lambda(V, X)$ and $e(T) = \lambda(V)$, where $\lambda(V, X)$ and $\lambda(V)$ are the relative (to X) and absolute projection constants of the space V, respectively.

It is a classical result due to Nachbin [7] (cf [10]; see also [3]) that

$$e(I_V) = 1 \text{ iff } V \text{ is isometric to } \ell_\infty^{(n)}. \tag{1.1}$$

In this note we prove that the set of all such operators consists precisely of those with spectrum lying in the circle. Of course we cannot define an operator T other than (a scalar multiple of) the identity operator without specifying the space V. Hence this result has to be stated somewhat differently.

1991 *Mathematics Subject Classification.* Primary 46B20; Secondary 51M20.

Received: October 8, 1998.

We will formulate the result in terms of the action constants introduced in [2] and [4]. Let V be a fixed Banach space of dimension n, and let A be an $n \times n$ matrix. Let $A(V)$ be the set of all linear operators from V into V such that, for every $T \in A(V)$, there exists a basis in V with respect to which the matrix of the operator T is equal to A. (In this case we say that T corresponds to the matrix A and write $T \sim A$.) In this context we refer to A as an action. An *action constant* of A on V is defined to be

$$\lambda_A(V) := \inf\{e(T)/\|T\| : T \in A(V)\}.$$

With the help of this language we state the result in [5]:

Theorem ([5]). *Let A be an $n \times n$ matrix and the field be real. Then the implication*

$$(\lambda_A(V) = 1) \Rightarrow (V \simeq \ell_\infty^{(n)})$$

holds iff $A = I$.

An *unconditional* action constant of A on V is defined to be

$$\lambda_A^u(V) := \inf\{e(T) : T \in A(V)\}.$$

Further, let $\sigma(A)$ (the spectrum of A) = the set of all eigenvalues of A and let $\rho(A)$ (the spectral radius of A) = $\max\{|\lambda| : \lambda \in \sigma(A)\}$. If $T \sim A$, then $\sigma(T) = \sigma(A)$ and $\rho(T) = \rho(A)$. Also denote by $\mathbf{F}$ either the complex field $\mathbf{C}$ or the real field $\mathbf{R}$ and let $\mathbf{T}$ denote the unit circle (in the respective field), i.e., $\mathbf{T}=\{e^{i\theta} : \theta \in [0, 2\pi)\} \cap \mathbf{F}$. In [5] we made the following conjecture:

Conjecture. *Let A be an $n \times n$ matrix and let $\mathbf{F} = \mathbf{C}$. Then the implication*

$$(\lambda_A^u(V) = 1) \Rightarrow (V \simeq \ell_\infty^{(n)})$$

holds iff $\sigma(A) \subset \mathbf{T}$.

In this paper we prove the validity of this conjecture. In addition we prove that necessity part of the conjecture also holds in the real case.

2. Main results

In the following, if A is an $n \times n$ matrix, then $\{A : W \to W\} := \{T \sim A : W \to W\}$. The following theorem may be known. We give a proof of it because we could not find it in any of the usual references. It sets the stage for what follows and indicates the distinction between the real and complex case.

Theorem 1. *For any $n \times n$ matrix A with entries from the field $\mathbf{F}$ we have*

$$\rho(A) = \inf\{\|A : H \to H\| : H \textit{ is isometric to } \ell_2^{(n)}\} \tag{2.1}$$

$$= \inf\{\|A : V \to V\| : \ V \ an \ n - dimensional \ Banach \ space\}. \tag{2.2}$$

Moreover, the infimum in (2.1) and (2.2) is attained if and only if every eigenvalue $\lambda' \in \sigma(A)$ such that $|\lambda'| = \rho(A)$ is a zero of the minimal polynomial of A of multiplicity 1.

Proof in the Complex Case. First observe that, if λ is an eigenvalue of A, then there exists a vector $v \in \mathbf{C}^n$ such that $Av = \lambda v$. Hence

$$\|Av\|_V = |\lambda| \|v\|_V$$

and hence

$$\inf\{\|A : V \to V\| : \ V \ is \ n - dimensional\} \geq \max\{|\lambda| : \ \lambda \in \sigma(A)\}.$$

Now, for every $\epsilon > 0$, we wish to construct a Hilbert space H such that

$$\|A : \ H \to H\| \leq \rho(A) + \epsilon.$$

Equivalently, for every $\epsilon > 0$, we wish to construct a matrix B similar to A such that

$$\|B : \ \ell_2^{(n)} \to \ell_2^{(n)}\| \leq \rho(A) + \epsilon.$$

Let $\{\lambda_1, ..., \lambda_n\}$ be the spectrum of A, where some of the λ_j may be the same. Then, for an arbitrary $\eta > 0$, there exists $\eta_1, ..., \eta_n \in [0, \eta]$ such that A is similar to

$$B_\eta = \begin{bmatrix} \lambda_1 & \eta_1 & & & & & \\ & \lambda_2 & \eta_2 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & \lambda_{n-1} & \eta_{n-1} \\ & & & & & & \lambda_n \end{bmatrix}.$$

Comparing this matrix to

$$B_0 = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \lambda_{n-1} & \\ & & & & & & \lambda_n \end{bmatrix},$$

we have that there exists a constant $C > 0$ such that

$$\|(B_0 - B_\eta) : \ \ell_2^{(n)} \to \ell_2^{(n)}\| < C\eta.$$

Picking $\eta < \epsilon/C$, we have

$$\|B_\eta : \ \ell_2^{(n)} \to \ell_2^{(n)}\| < \|B_0 : \ \ell_2^{(n)} \to \ell_2^{(n)}\| + \epsilon$$

$$= \max\{|\lambda_j| : \ j = 1, ..., n\} + \epsilon.$$

Finally, we prove the "moreover" part of the theorem: If A is nilpotent, then the statement is obvious, since for any Banach space V

$$\|A : \ V \to V\| > 0.$$

Hence, without loss of generality we may assume that $\rho(A) = 1$. By way of contradiction, let $\lambda_0 \in \sigma(A)$ such that $|\lambda_0| = 1$ and λ_0 is a zero of the minimal polynomial of A of multiplicity strictly greater than 1. Then, using the Jordan form for A, we observe that there exist non-zero vectors v_1, $v_2 \in \mathbf{C}^n$ so that

$$Av_1 = \lambda_0 v_1 + v_2; \ Av_2 = \lambda_0 v_2.$$

If $\|A : \ V \to V\| = 1$, then $\|A^k : \ V \to V\| \leq 1$, $\forall k \in \mathbf{N}$. On the other hand

$$\|v_1\| \geq \|A^k v_1\| \geq \|\lambda_0^k v_1 + k\lambda_0^{k-1} v_2\| \geq k\|v_2\| - \|v_1\| \to \infty \ as \ k \to \infty. \ \blacksquare$$

Proof in the Real Case. The proof that, for every $\epsilon > 0$,

$$\epsilon + \rho(A) \geq \inf\{\|A : V \to V\| : \ V \ is \ isometric \ to \ \ell_2^{(n)}\}$$

is practically the same as in the complex case if the Jordan form

$$\begin{bmatrix} \lambda_1 & \eta_1 & & & & \\ & \lambda_2 & \eta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & \lambda_{n-1} & \eta_{n-1} \\ & & & & & \lambda_n \end{bmatrix}$$

is substituted for by the "real-block Jordan form"

$$\begin{bmatrix} \rho_1\Lambda_1 & \eta_1 I & & & & \\ & \rho_2\Lambda_2 & \eta_2 I & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & \rho_{n-1}\Lambda_{n-1} & \eta_{n-1} I \\ & & & & & \rho_n\Lambda_n \end{bmatrix},$$

where $\Lambda_j = \begin{bmatrix} \sin\theta_j & \cos\theta_j \\ -\cos\theta_j & \sin\theta_j \end{bmatrix}$ and I is the 2×2 identity matrix.

To prove that

$$\|A : V \to V\| \geq \rho(A) \tag{2.3}$$

we need a different trick, since A may have no real eigenvectors. Clearly (2.3) holds if A is nilpotent. Hence we can assume that $\rho(A) = 1$. If A has eigenvalues 1 or $^-1$ then the proof is done as in the previous case. If not then V can be written as a direct sum

$$V = V_1 \oplus V_2$$

where V_2 is a 2-dimensional Banach space and

$$A_{|V_2} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} =: A_1$$

for some α. It suffices to prove that

$$\|A_1 : V_2 \to V_2\| \geq 1.$$

Indeed, applying A_1 to the unit ball $B(V_2)$, we observe that the (Euclidean) area of $(A_1 B(V_2))$ is the same as the area of $B(V_2)$. Hence $[\partial(A_1 B(V_2))] \cap [\partial B(V_2)] \neq \emptyset$ and thus $\exists\, x \in S(V_2) :\ \|A_1 x\| = \|x\| = 1$. ■

Remark. It is clear from the proof of the theorem that (in the complex case) (2.1) can be replaced by

$$\inf\{\|A : V \to V\| :\ V \textit{ is isometric to } \ell_p^{(n)}\}$$

for every $p \in [1, \infty]$. In the real case it is not so. Indeed

Proposition 1. *Let* $\mathbf{F} = \mathbf{R}$ *and* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. *Then*

$$\inf\{\|A : V \to V\| :\ V \textit{ is isometric to } \ell_\infty^{(2)}\} = |\cos\theta| + |\sin\theta| > 1,$$

for $\theta \neq k\pi/2$, *although* $\rho(A) = 1$.

Proof. $\inf\{\|A : V \to V\| :\ V \textit{ is isometric to } \ell_\infty^{(2)}\} = \inf_S \|S^{-1}AS\|$, where the norm is the maximum of the absolute row sums and $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an arbitrary invertible 2×2 matrix. Further let $A_S = S^{-1}AS$, $\Delta = ad - bc$, $E = (ab+cd)/\Delta$, $P = (b^2+d^2)/\Delta$, $Q = (a^2+c^2)/\Delta$, $\delta = \cot\theta$, $\sigma = \sin\theta$, and note that $1 + E^2 = PQ$. Now, without loss, assume $0 < \theta < \pi/2$. Then

$$\frac{\|A_S\|}{\sigma} = \max\{|\delta + E| + |P|,\ |\delta - E| + |Q|\}.$$

Case 1. $\delta \geq |E| \geq 0$:

$$\frac{\|A_S\|}{\sigma} = \delta + \max\{E + |P|,\ -E + |Q|\}.$$

For E fixed, determine, by use of $|Q| = (1+E^2)/|P|$, that the two arguments of the max are equal to $\sqrt{1+2E^2}$ when $|P| = \sqrt{1+2E^2} - E$. On the other hand, if $|P| > \sqrt{1+2E^2} - E$ then obviously $E + |P| > \sqrt{1+2E^2}$, whereas if $|P| < \sqrt{1+2E^2} - E$ then $-E + |Q| = -E + (1+E^2)/|P| > -E + (1+E^2)/(\sqrt{1+2E^2} - E) = (1+2E^2 - E\sqrt{1+2E^2})/(\sqrt{1+2E^2} - E) = \sqrt{1+2E^2}$. We conclude that, for E fixed, $\inf \max\{E + |P|,\ -E + |Q|\} = \sqrt{1+2E^2}$. Thus, the inf (over all S) is achieved when $E = 0$, and thus when $|P| = |Q| = 1$, i.e., when $ab + cd = 0$ and $b^2 + d^2 = a^2 + c^2 = |ad - bc|$, and in particular when $d = a$ and $b = c = 0$.
Case 2. $|E| \geq \delta > 0$:
Assume first that $E > 0$. Then

$$\frac{\|A_S\|}{\sigma} = E + \max\{\delta + |P|,\ -\delta + |Q|\}.$$

Analogously as in Case 1, determine that the two arguments of the max are equal to $\sqrt{1+E^2+\delta^2}$ when $|P| = \sqrt{1+E^2+\delta^2} - \delta$. Thus, the inf (over all S) is achieved when $E = \delta$.
Assume finally that $E < 0$. Then

$$\frac{\|A_S\|}{\sigma} = -E + \max\{-\delta + |P|,\ \delta + |Q|\},$$

and the argument is completely symmetric and the conclusion is the same as in the case $E > 0$..

Note, however, that $\delta + \sqrt{1+2\delta^2} > \delta + 1$ and thus the inf over all S from both cases is achieved in Case 1. I.e., we find that the infimum is achieved for $S = I$ and thus

$$\inf \|A_S\| = \sigma(1+\delta) = \sin\theta + \cos\theta. \ \blacksquare$$

Theorem 2. *Let V be an n-dimensional Banach space and T be an operator on V such that $\sigma(T) \subset \mathbf{T}$ and $e(T) = 1$. Then V is isometric to $\ell_\infty^{(n)}$. Moreover, in this case, all the eigenvalues of T are simple roots of the minimal polynomial of T.*

Proof. Complex case: Since $e(T) = 1$ we have $\|T : V \to V\| = 1$ and by the "moreover" part of Theorem 1, T is diagonalizable; i.e., $\exists$ a basis in V with respect to which T can be represented as a diagonal matrix

$$T = \begin{bmatrix} e^{2\pi i\sigma_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & e^{2\pi i\sigma_n} \end{bmatrix}.$$

By Dirichlet's theorem (cf [9, p. 216]) we can find numbers $N, m_1, ..., m_n \in \mathbf{N}$ so that

$$\left|\sigma_j - \frac{m_j}{N}\right| < \frac{1}{N^{1+1/n}},$$

and this can be done for a sequence of $N \to \infty$. Then for the operators

$$T_N := \begin{bmatrix} e^{2\pi i \frac{m_1}{N}} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & e^{2\pi i \frac{m_n}{N}} \end{bmatrix} : \ V \to V,$$

there is a constant c_1 independent of N such that

$$\|T_N - T\| < \frac{c_1}{N^{1+1/n}},$$

and, since the extension constant is a continuous function, there exists a function $\phi(N)$ such that

$$e(T_N) \le e(T) + \phi(N) = 1 + \phi(N);\ \phi(N) \to 0 \ as \ N \to \infty.$$

Now let $\widetilde{T}$ extend T to X minimally. Then $T^{N-1} \odot \widetilde{T}$ extends $I_{|_V}$ with norm $\le \|T_N^{N-1}\| \|\widetilde{T}\|$, where of course $\|\widetilde{T}\| = e(T)$. But $T^{N-1} \odot \widetilde{T}$ is an extension of T_N^N, whence

$$e(T_N^N) \le e(T_N)\|T_N^{N-1}\| \le [1 + \phi(N)](1 + \frac{c_1}{N^2})^{N-1}.$$

Finally, since $T_N^N = I :\ V \to V$, we have

$$\lambda(V) \le (1 + \phi(N))(1 + \frac{c_1}{N^2})^{N-1} \to 1 \ as \ N \to \infty.$$

Hence $\lambda(V) = 1$ and $V \simeq \ell_\infty^{(n)}$.

Real case: The "real" case is again done in exactly the same way where the matrices

$$\begin{bmatrix} e^{2\pi i \sigma_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & e^{2\pi i \sigma_n} \end{bmatrix}$$

are replaced by matrices of the form

$$\begin{bmatrix} \Lambda_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \Lambda_k \end{bmatrix}, \quad \text{where } \Lambda_j = \begin{bmatrix} \cos(2\pi\sigma_j) & \sin(2\pi\sigma_j) \\ -\sin(2\pi\sigma_j) & \cos(2\pi\sigma_j) \end{bmatrix}$$

or Λ_j is a real number. But, furthermore, from Proposition 1 we see that in fact all the σ_j must be 0 or 1/2 and that in fact therefore all the matrices representing T must be diagonal matrices with entries ± 1. Hence T is an involution and the above argument works with $N = 2$. ∎

Corollary 1. *If $\sigma(A) \subset \mathbf{T}$ and $\lambda_A^u(V) = 1$, then $V \simeq \ell_\infty^{(n)}$.*

Proof. For every $\epsilon > 0$, let $T_\epsilon \sim A$ and $1 \leq e(T_\epsilon) \leq 1+\epsilon$. Then in particular $\|T_\epsilon\| \leq 1+\epsilon$. Hence by compactness $\exists\ \epsilon_1, ..., \epsilon_k, ...,\ \epsilon_k \to 0$, such that $T_{\epsilon_k} \to T,\ e(T_{\epsilon_k}) \to e(T) = 1$. Since eigenvalues are continuous functions of the operator, we have $\sigma(T) = \sigma(A) \subset \mathbf{T}$, and by the previous theorem we conclude that $V \simeq \ell_\infty^{(n)}$. ∎

We will now prove a converse to Theorem 2 in the case when the field $\mathbf{F} = \mathbf{C}$.

Theorem 3. *Let $\mathbf{F} = \mathbf{C}$ and let A be an $n \times n$ matrix such that $\rho(A) = 1$ and every eigenvalue of A of modulus one is a zero of the minimal polynomial of A of multiplicity 1. If $\sigma(A)$ is not a subset of the unit circle, then there exists a Banach space V and an operator $T : V \to V$ such that V is not isometric to $\ell_\infty^{(n)}$, $T \sim A$ and $e(T) = 1$.*

Proof. Let $\lambda_1, ..., \lambda_k \in \sigma(A) \cap \mathbf{T}$ and $\lambda_{k+1}, ..., \lambda_n \in \sigma(A) - \mathbf{T}$. If $n - k = 1$ then the proof is identical to ([5], Theorem 1, Case 1). If $n - k > 1$ then (for every $\epsilon > 0$) we can find a Banach space U with $\dim U = n - k$ such that

$$1 < d(U, \ell_\infty^{(n-k)}) < 1 + \epsilon,$$

where "d" denotes the Banach-Mazur distance (cf [10]). For every $\eta > 0$ there exists $\eta_{k+1}, ..., \eta_{n-1}$ so that $0 < \eta_j < \eta$ and A is similar to the matrix

$$\begin{bmatrix} A' & 0 \\ 0 & A'' \end{bmatrix},$$

where

$$A' = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \lambda_{k-1} & \\ & & & & & & \lambda_k \end{bmatrix},$$

and

$$A'' = \begin{bmatrix} \lambda_{k+1} & \eta_{k+1} & & & & \\ & \lambda_{k+2} & \eta_{k+2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & \lambda_{n-1} & \eta_{n-1} \\ & & & & & \lambda_n \end{bmatrix}.$$

Let $\gamma = \max\{|\lambda_{k+1}|, ..., |\lambda_n|\} < 1$. Then $\|A'' :\ U \to U\| \leq (\gamma + \eta)$ and hence

$$e(A'') \leq (\gamma + \eta)(1 + \epsilon).$$

Since $\gamma < 1$ we can pick η, ϵ so small that $e(A'') \le 1$. Let $V = \ell_\infty^{(k)} \oplus U$ where the direct sum is taken in the ℓ_∞-sense. Then $A = \begin{bmatrix} A' & 0 \\ 0 & A'' \end{bmatrix} : V \to V$, $e(A) = \max\{e(A'), e(A'')\} = 1$. V is not isometric to $\ell_\infty^{(n)}$, however, since U is not isometric to $\ell_\infty^{(n-k)}$ and yet the natural projection from $\ell_\infty^{(k)} \oplus U \to U$ is of norm one. ∎

The following generalization of Theorem 3 gives a converse to the corollary to Theorem 2 if $\mathbf{F} = \mathbf{C}$.

Theorem 4. *Let $\mathbf{F} = \mathbf{C}$ and A be an $n \times n$ matrix with $\rho(A) = 1$ and not all of whose eigenvalues lie on the unit circle. Then $\exists\, V$ such that V is not isometric to $\ell_\infty^{(n)}$ and yet $\lambda_A^u(V) = 1$.*

Proof. In order to prove this theorem we need to find a Banach space V ($\not\simeq \ell_\infty^{(n)}$) such that, for every $\delta > 0$, there is a matrix T_ϵ similar to A with the property that

$$e(T_\epsilon) < 1 + \delta.$$

The space V is the same as in the proof of Theorem 3. Likewise the proof is identical. This time for arbitrary $\eta > 0$ we write

$$A_\eta = \begin{bmatrix} A'_\eta & \\ & A''_\eta \end{bmatrix},$$

where

$$A'_\eta = \begin{bmatrix} \lambda_1 & \eta_1 & & & & \\ & \lambda_2 & \eta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & \lambda_{k-1} & \eta_{k-1} \\ & & & & & \lambda_k \end{bmatrix} : \ \ell_\infty^{(k)} \to \ell_\infty^{(k)};\ \eta_j < \delta$$

and A''_η is defined as in Theorem 3. $\|A'_\eta\| \le 1 + \eta$ and hence $e(A'_\eta) \le 1 + \delta$ and $e(A_\eta) \le \max\{e(A'_\eta),\ e(A''_\eta)\} \le 1 + \delta$. ∎

Note. After this paper was accepted for publication, M. I. Ostrovski ([8]), using methods similar to those in this paper, proved the following:

Theorem. *Let A be an $n \times n$ matrix and the field be real. Then the implication*

$$(\lambda_A^u(V) = 1) \Rightarrow (V \simeq l_\infty^{(n)})$$

holds iff A has the same spectrum as one of the $n!2^n$ isometries of $l_\infty^{(n)}$ ($n!$ permutations of the n standard basis elements each with arbitrary sign).

References

1. B. L. Chalmers and F. T. Metcalf, *A simple formula showing L^1 is a maximal overspace for two-dimensional real spaces*, Ann. Polon. Math. **56** (1992), 303–309.
2. B. L. Chalmers and K. C. Pan, *Finite-dimensional action constants* (to appear).
3. B. L. Chalmers and B. Shekhtman, *On the role of ℓ_∞ in approximation theory,* in *Approximation, Probability and Related Fields,* (G. Anastassiou and S. T. Rachev, eds.), (1994), Plenum.
4. B. L. Chalmers and B. Shekhtman, *Action constants,* in *Approximation, Probability and Related Fields,* (G. Anastassiou and S. T. Rachev, eds.), (1994), Plenum.
5. B. L. Chalmers and B. Shekhtman, *Actions that characterize $\ell_\infty^{(n)}$*, Linear Algebra Appl. **270** (1998), 155–169.
6. J. Lindenstrauss, *On the extension of operators with a finite-dimensional range*, Illinois J. Math. **8** (1964), 488–499.
7. L. Nachbin, *A theorem of the Hahn-Banach type for linear transformations*, Trans. Amer. Math. Soc. **68** (1950), 28–46.
8. M. I. Ostrovski, *Private communication.*
9. W. M. Schmidt, *Approximation to algebraic numbers*, Enseign. Math. **17** (1971), 188–248.
10. N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-dimensional Operator Ideals*, John Wiley and Sons, New York, 1989.

Department of Mathematics
University of California
Riverside, California 92521, USA

E-mail address: blc@math.ucr.edu

Department of Mathematics
University of South Florida
Tampa, Florida 33620-5700, USA

E-mail address: boris@tarski.math.usf.edu