REGULARIZED FUNCTIONAL CALCULI, SEMIGROUPS, AND COSINE FUNCTIONS FOR PSEUDODIFFERENTIAL OPERATORS

RALPH DELAUBENFELS AND YANSONG LEI

ABSTRACT. Let $iA_j(1 \leq j \leq n)$ be generators of commuting bounded strongly continuous groups, $A \equiv (A_1, A_2, ..., A_n)$. We show that, when f has sufficiently many polynomially bounded derivatives, then there exist $k, r > 0$ such that $f(A)$ has a $(1+|A|^2)^{-r}$-regularized $BC^k(f(\mathbf{R}^n))$ functional calculus. This immediately produces regularized semigroups and cosine functions with an explicit representation; in particular, when $f(\mathbf{R}^n) \subseteq \mathbf{R}$, then, for appropriate k, r, $t \mapsto (1-it)^{-k}e^{-itf(A)}(1+|A|^2)^{-r}$ is a Fourier-Stieltjes transform, and when $f(\mathbf{R}^n) \subseteq [0, \infty)$, then $t \mapsto (1+t)^{-k}e^{-tf(A)}(1+|A|^2)^{-r}$ is a Laplace-Stieltjes transform. With $A \equiv i(D_1, ..., D_n)$, $f(A)$ is a pseudodifferential operator on $L^p(\mathbf{R}^n)(1 \leq p < \infty)$ or $BUC(\mathbf{R}^n)$.

0. INTRODUCTION

In finite dimensions, the Jordan canonical form for matrices guarantees that, although a linear operator may not be diagonalizable, which is equivalent to having a $BC(\mathbf{C})$ functional calculus, it will be generalized scalar, that is, have a $BC^k(\mathbf{C})$ functional calculus, for some k; specifically, k may be chosen to be $n-1$, where n is the order of the largest Jordan block.

In infinite dimensions, even a bounded linear operator on a Hilbert space may fail to be generalized scalar; consider the left shift on ℓ^2.

Our favorite unbounded operators fail to be generalized scalar, on Banach spaces that are not Hilbert spaces. The operator $i\frac{d}{dx}$, on $L^2(\mathbf{R})$, is self-adjoint and thus has a $BC(\mathbf{R})$ functional calculus. However, on $L^p(\mathbf{R}), p \neq 2$, it does not have a $BC^m(\mathbf{R})$ functional calculus, for any nonnegative integer m; that is, it is not even generalized scalar (see [2, Lemma 5.3]).

1991 *Mathematics Subject Classification.* Primary 47A60; secondary 47D03, 47D06, 47D09, 47F05.

Key words and phrases. Regularized functional calculi, semigroups, cosine functions, pseudodifferential operators.

Received: May 27, 1996

Differential operators in more than one dimension may be even more poorly behaved. For any $n > 1$, there exist constant coefficient differential operators on $L^p(\mathbf{R}^n)$ that are not even decomposable, for any $p \neq 2$ ([1, Corollary 3.5]).

In this paper, we show that constant coefficient differential operators $p(D)$, on $L^p(\mathbf{R}^n)(1 \leq p < \infty)$ or $BUC(\mathbf{R}^n)$, have a $(1 + \triangle)^{-r}$-regularized $BC^k(p(\mathbf{R}^n))$ functional calculus, for appropriate numbers r and k, where $\triangle$ is the Laplacian, p is a polynomial. This means that, for any $g \in BC^k(p(\mathbf{R}^n))$, $g(p(D))(1 + \triangle)^{-r}$ is a bounded operator. More generally, if $iA_1, ..., iA_n$ generate commuting bounded strongly continuous groups, $A \equiv (A_1, ..., A_n)$ and f has sufficiently many polynomially bounded derivatives, then $f(A)$ has a $(1 + |A|^2)^{-r}$-regularized $BC^k(f(\mathbf{R}^n))$ functional calculus (Theorem 2.17). See [8] for regularized $BC^k(\mathbf{R})$ functional calculi for generators of polynomially bounded groups.

As an immediate corollary, when $f(\mathbf{R}^n)$ is contained in a left half-plane, it follows that $f(A)$ generates a $(1+|A|^2)^{-r}$-regularized semigroup, with the intuitively natural representation

$$W(t) \equiv \left[(z \mapsto e^{tz})(f(A))\right](1 + |A|^2)^{-r} \ \ (t \geq 0).$$

Identically, when $f(\mathbf{R}^n)$ is contained in a left half-line, then $f(A)$ generates a $(1 + |A|^2)^{-r}$-regularized cosine function

$$S(t) \equiv \left[(z \mapsto \cosh(t\sqrt{z}))(f(A))\right](1 + |A|^2)^{-r} \ \ (t \in \mathbf{R}).$$

The existence of these regularized semigroups and cosine functions is known (see [10], [15], [16], [4, Chapter XIII], [3], [12], [13]); we offer our approach as a simple, intuitive, constructive and unified corollary of our regularized functional calculus.

For example, on $L^p(\mathbf{R}^n)(1 \leq p < \infty)$, we may simultaneously deal with the Schrödinger equation (ill-posed for $p \neq 2$) and the wave equation (ill-posed for $p \neq 2, n > 1$), by constructing a regularized $BC^k((-\infty, 0])$ functional calculus for the Laplacian.

In Section I we give some preliminary material relating regularized functional calculi to regularized semigroups and cosine functions. Our main results are in Section II. Section III has the particular case of pseudodifferential operators on the usual function spaces $BUC(\mathbf{R}^n)$ or $L^p(\mathbf{R}^n)$ $(1 \leq p < \infty)$. See [7] for regularized functional calculi for the Schrödinger operator with potential, on such spaces.

All operators are linear, on a Banach space, X. We will write $\mathcal{D}(B)$ for the domain of the operator B, $\rho(B)$ for its resolvent set, $Im(B)$ for the image of B. We will denote by $B(X)$ the space of all bounded operators from X into itself. Throughout this paper, $C \in B(X)$ is injective, and commutes with B; that is, $CB \subseteq BC$. When B generates a strongly continuous semigroup, we will denote that semigroup by $\{e^{tB}\}_{t \geq 0}$; see [9] or [14] for material on strongly continuous semigroups and their applications.

1. Regularized functional calculi, regularized semigroups and regularized cosine functions

We show in this section how a regularized functional calculus produces intuitively natural constructions of regularized semigroups and regularized cosine functions. Growth estimates also follow automatically.

Definition 1.1. The complex number λ is in $\rho_C(B)$, the *C-resolvent* of B, if $(\lambda - B)$ is injective and $Im(C) \subseteq Im(\lambda - B)$.

Definition 1.2. Denote by $B_C(X)$ the space of all operators G such that $GC \in B(X)$, with norm

$$\|G\|_{B_C(X)} \equiv \|GC\|.$$

Definition 1.3. Suppose $\mathcal{F}$ is a Banach algebra of complex-valued functions, defined on a subset of the complex plane such that $f_0(z) \equiv 1 \in \mathcal{F}$. A *$C$-regularized $\mathcal{F}$ functional calculus* for B is a continuous linear map $f \mapsto f(B)$, from $\mathcal{F}$ into $B_C(X)$, such that

(1) $f(B)g(B)C = [(fg)(B)]\, C$, for all $f, g \in \mathcal{F}$;

(2) $g(B)BC \subseteq Bg(B)C = (f_1 g)(B)C$, whenever both g and $f_1 g \in \mathcal{F}$, where $f_1(z) \equiv z$; and

(3) $f_0(B)C = C$.

Remark 1.4. When $\mathcal{F}$ contains f_0 and $g_\lambda(z) \equiv (\lambda - z)^{-1}$, for some complex λ, then (1), (2) and (3) of Definition 1.3 are equivalent to (1), (2$'$) and (3), where (2$'$) is the following:

(2$'$) $\lambda \in \rho_C(B)$ and $[g_\lambda(B)]\, C = (\lambda - B)^{-1}C$, whenever $g_\lambda \in \mathcal{F}$.

See [6] and [8] for some basic results on regularized functional calculi. Note that an I-regularized $\mathcal{F}$ functional calculus is a $\mathcal{F}$ functional calculus.

Definition 1.5. A *C-regularized semigroup generated by B* is a strongly continuous family $\{W(t)\}_{t \geq 0} \subseteq B(X)$ such that

(1) $W(0) = C$;

(2) $W(t)W(s) = CW(t+s)$, for all $s, t \geq 0$; and

(3) $Bx = C^{-1}\left[\lim_{t \to 0} \frac{1}{t}(W(t)x - Cx)\right]$, with maximal domain.

See [4] and the references therein, for basic material on regularized semigroups and their relationship to the abstract Cauchy problem.

Definition 1.6. A *C-regularized cosine function generated by B* is a strongly continuous family $\{S(t)\}_{t \in \mathbf{R}} \subseteq B(X)$ such that

(1) $S(0) = C$,

(2) $S(t+s)C + S(t-s)C = 2S(t)S(s)$, for all $s, t \in \mathbf{R}$; and

(3) $Bx = \left[(\frac{d}{dt})^2 S(t)x|_{t=0}\right]$, with maximal domain.

A regularized cosine function deals with ill-posed second-order abstract Cauchy problems just as regularized semigroups deal with ill-posed first-order abstract Cauchy problems.

Proposition 1.7. *Suppose $\omega \in \mathbf{R}$, B has a C-regularized $BC^k(\{z \,|\, Re(z) \leq \omega\})$ functional calculus, and $C\left(\mathcal{D}(B)\right)$ is dense. Then $C^{-1}BC$ generates a C-regularized semigroup $\{W(t)\}_{t\geq 0}$ given by*

$$W(t) = \left[\left(z \mapsto e^{tz}\right)(B)\right] C \ \ (t \geq 0).$$

$\|W(t)\|$ is $O((1+t)^k e^{\omega t})$.

Proof. Define, for $t \geq 0$, $j = 0, 1, 2$,

$$W_j(t) \equiv \left[(z \mapsto (1+\omega - z)^{-j} e^{tz})(B)\right] C^{j+1} = \left((1+\omega - B)^{-1} C\right)^j W_0(t).$$

Since $t \mapsto (1+\omega-z)^{-1}e^{tz}$ is continuous, as a map from $[0,\infty)$ into $BC^k(\{z \,|\, Re(z) \leq \omega\})$, and B has a C-regularized $BC^k(\{z \,|\, Re(z) \leq \omega\})$ functional calculus, it follows that $t \mapsto W_1(t)$ is a continuous function from $[0,\infty)$ into $B(X)$. Thus, for $x \in C((\mathcal{D}(B))$, $t \mapsto W_0(t)x = W_1(t)(1+\omega-B)C^{-1}x$ is continuous from $[0,\infty)$ into X; since $\|W_0(t)\|$ is bounded for t in bounded intervals, and $C((\mathcal{D}(B))$ is dense, the same is true for all $x \in X$; that is, $\{W_0(t)\}_{t\geq 0}$ is strongly continuous. The algebraic properties of a regularized semigroup, for $\{W_j(t)\}_{t\geq 0}$, follow from the definition of a C-regularized functional calculus. Thus, for $j = 0, 1, 2$, $\{W_j(t)\}_{t\geq 0}$ is a $(1+\omega-B)^{-j}C^{j+1}$-regularized semigroup.

A calculation shows that $t \mapsto (z \mapsto (1+\omega-z)^{-2}e^{tz})$ is continuously differentiable, as a map from $[0,\infty)$ into $BC^k(\{z \,|\, Re(z) \leq \omega\})$, with

$$\frac{d}{dt}(z \mapsto (1+\omega-z)^{-2}e^{tz}) = (z \mapsto z(1+\omega-z)^{-2}e^{tz}),$$

thus, since B has a C-regularized $BC^k(\{z \,|\, Re(z) \leq \omega\})$ functional calculus, it follows that $t \mapsto W_2(t)$ is a differentiable function from $[0,\infty)$ into $B(X)$, with

$$\frac{d}{dt}W_2(t) = BW_2(t) \ \ \forall t \geq 0.$$

This implies that $\{W_2(t)\}_{t\geq 0}$ is generated by an extension of B; since $\rho_C(B)$ is nonempty, $C^{-1}BC$ is the generator ([4, Corollary 3.12]). By [4, Proposition 3.10], B is also the generator of $\{W_0(t)\}_{t\geq 0}$.

The growth condition on $\|W_0(t)\}$ follows from the fact that

$$\|z \mapsto e^{tz}\|_{BC^k(\{z \,|\, Re(z)\leq\omega\})} \ \text{ is } \ O((1+t)^k e^{\omega t}). \ \blacksquare$$

Replacing $z \mapsto e^{tz}$ with $z \mapsto \cosh(t\sqrt{z})$, in the proof above, gives us the following.

Proposition 1.8. *Suppose $\omega \geq 0$, B has a C-regularized $BC^k((-\infty, \omega])$ functional calculus and $\mathcal{D}(B)$ is dense. Then $C^{-1}BC$ generates a C-regularized cosine function $\{S(t)\}_{t\in\mathbf{R}}$ given by*

$$S(t) = \left[(z \mapsto \cosh(t\sqrt{z}))(B)\right] C \quad (t \in \mathbf{R}).$$

$\|S(t)\|$ is $O((1+t^2)^k e^{t\sqrt{\omega}})$.

When the half-plane in Proposition 1.7 is replaced by the real line ($[0, \infty)$), we get a nice representation of the regularized semigroup, as a Fourier-Stieltjes (Laplace-Stieltjes) transform.

Lemma 1.9. *Suppose $\{W(t)\}_{t\geq 0}$ is an exponentially bounded C-regularized semigroup generated by B. Then*

$$\lim_{\lambda\to\infty} \lambda(\lambda - B)^{-1} W(t)x = W(t)x, \;\; \forall x \in X, t \geq 0.$$

Proof. There exists a Banach space Z, continuously embedded between $Im(C)$ and X, such that $B|_Z$ generates a strongly continuous semigroup, and $W(t) = e^{tB|_Z}C$ ([4, Chapter V]). This implies that, for any $z \in Z$, $\lambda(\lambda - B|_Z)^{-1}z$ converges to z in Z, as $\lambda \to \infty$. Since the norm in Z is stronger than the norm in X, and $W(t)x \in Z$, for all $x \in X, t \geq 0$, the result follows. ∎

Proposition 1.10.

(1) *If B has a C-regularized $BC^k(\mathbf{R})$ functional calculus, then $-iC^{-1}BC$ generates a C-regularized group $\{W(t)\}_{t\in\mathbf{R}}$ such that, for all $x \in X, x^* \in X^*$, the map $t \mapsto (1-it)^{-k}\langle W(t)x, x^*\rangle$ is a Fourier-Stieltjes transform of a complex-valued measure of bounded variation.*

(2) *If B has a C-regularized $BC^k([0,\infty))$ functional calculus, then $-C^{-1}BC$ generates a C-regularized semigroup $\{W(t)\}_{t\geq 0}$ such that, for all $x \in X, x^* \in X^*$, the map $t \mapsto (1+t)^{-k}\langle W(t)x, x^*\rangle$ is a Laplace-Stieltjes transform of a complex-valued measure of bounded variation.*

Proof. We will prove (1); it will be clear how the proof would be modified for (2).

It follows from Proposition 1.7 that $-iC^{-1}BC$ generates a C-regularized group $\{W(t)\}_{t\in\mathbf{R}}$, given by $W(t) \equiv \left[(z \mapsto e^{-itz})(B)\right] C$. Fix $x \in X, x^* \in X^*$. Since

$$f \mapsto \left\langle \left[((1+D)^{-k}f)(B)\right] Cx, x^* \right\rangle$$

defines a bounded linear functional on $C_0(\mathbf{R})$, there exists a complex-valued measure of bounded variation, μ, such that

$$\left\langle \left[((1+D)^{-k}f)(B)\right] Cx, x^* \right\rangle = \int_{\mathbf{R}} f(s)\, d\mu(s), \;\; \forall f \in C_0(\mathbf{R});$$

choosing $f_\lambda(s) \equiv \lambda(\lambda - is)^{-1}e^{-its}$ gives us, by Lemma 1.9 and dominated convergence, for any $t \geq 0$,

$$\begin{aligned}(1-it)^{-k}\langle W(t)x, x^*\rangle &= \lim_{\lambda\to\infty}(1-it)^{-k}\left\langle \lambda(\lambda - iB)^{-1}W(t)x, x^*\right\rangle \\ &= \lim_{\lambda\to\infty}(1-it)^{-k}\langle [f_\lambda(B)]\, Cx, x^*\rangle \\ &= \lim_{\lambda\to\infty}(1-it)^{-k}\int_{\mathbf{R}}(1+D)^k f_\lambda(s)\, d\mu(s) \\ &= \int_{\mathbf{R}} e^{-its}\, d\mu(s). \blacksquare\end{aligned}$$

2. Functional calculus on function spaces with polynomial growth conditions

Throughout this section, $iA_1, iA_2, ..., iA_n$ are generators of commuting bounded strongly continuous groups $\{e^{itA_j}\}_{t\in\mathbf{R}}(1 \leq j \leq n)$, $A \equiv (A_1, A_2, ..., A_n)$.

We will use some standard terminology. We will write $x = (x_1, x_2, ..., x_n)$, for a vector in $\mathbf{R}^n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ for a vector in $(\mathbf{N}\cup\{0\})^n$, $x^\alpha \equiv x_1^{\alpha_1}..., x_n^{\alpha_n}, |x|^2 \equiv \sum_{k=1}^n |x_k|^2, |\alpha| \equiv \sum_{k=1}^n \alpha_k$; see, for example, [9, Chapter 2.3].

Let F be the Fourier transform, FL^1 be the set of all inverse Fourier transforms of L^1 functions; that is,

$$FL^1 \equiv \{f \in C_0(\mathbf{R}^n) \mid Ff \in L^1(\mathbf{R}^n)\}. \tag{2.1}$$

Define, for $f \in FL^1$, a bounded operator $f(A)$ by:

$$f(A) \equiv (2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n} e^{i(x\cdot A)}Ff(x)dx. \tag{2.2}$$

We define the operator $-|A|^2$ as the generator of the strongly continuous semigroup $\{(z \mapsto e^{-t|z|^2})(A)\}_{t\geq 0}$.

Lemma 2.3.

(a) $(fg)(A) = f(A)g(A) \quad \forall f, g \in FL^1$.

(b) *There is $M < \infty$ such that*

$$\|f(A)\| \leq M\|f\|_{FL^1} \quad \forall f \in FL^1.$$

(c) *For all $r > 0$, $z \mapsto (1+|z|^2)^{-r} \in FL^1$, with*

$$(1+|A|^2)^{-r} = \left(z \mapsto (1+|z|^2)^{-r}\right)(A).$$

(d) *(Bernstein's Theorem) If $k > \dfrac{n}{2}, k \in \mathbf{N}$, then $H^k(\mathbf{R}^n) \hookrightarrow FL^1$ and there exists $M > 0$ such that*

$$\|u\|_{FL^1} \leq M\|u\|_{L^2}^{1-\frac{n}{2k}}\sum_{|\alpha|=k}\|D^\alpha u\|_{L^2}^{\frac{n}{2k}} \quad \forall u \in H^k(\mathbf{R}^n).$$

Assertions (a) and (b) are straightforward to verify, and (d) is well-known. For (c), we need the following.

Lemma 2.4 ([5, Lemma 2.2]). *If A has a $\mathcal{F}$ functional calculus, and $t \mapsto k_t \in C([a,b], \mathcal{F})$, then*

$$\int_a^b k_t(A)\,dt = \left(z \mapsto \int_a^b k_t(z)\,dt\right)(A).$$

Proof of Lemma 2.3(c). First, note that, since

$$\|F(z \mapsto e^{-t|z|^2})\|_{L^1(\mathbf{R})} = \|F(z \mapsto e^{-|z|^2})\|_{L^1(\mathbf{R})}, \quad \forall t > 0,$$

it follows that

$$\left(z \mapsto \frac{1}{\Gamma(r)} \int_{\frac{1}{n}}^n t^{r-1} e^{-t} e^{-t|z|^2}\,dt\right) \to \left(z \mapsto \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t} e^{-t|z|^2}\,dt\right),$$

as $n \to \infty$, in FL^1.

Thus we may apply Lemma 2.4 as follows.

$$\begin{aligned}
(1+|A|^2)^{-r} &= \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t} e^{-t|A|^2}\,dt \\
&= \lim_{n\to\infty} \frac{1}{\Gamma(r)} \int_{\frac{1}{n}}^n t^{r-1} e^{-t} \left[\left(z \mapsto e^{-t|z|^2}\right)(A)\right]\,dt \\
&= \lim_{n\to\infty} \left(z \mapsto \frac{1}{\Gamma(r)} \int_{\frac{1}{n}}^n t^{r-1} e^{-t} e^{-t|z|^2}\,dt\right)(A) \\
&= \left(z \mapsto \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t} e^{-t|z|^2}\,dt\right)(A) \\
&= \left(z \mapsto (1+|z|^2)^{-r}\right)(A). \blacksquare
\end{aligned}$$

Definition 2.5. For $l \geq -1, k \in \mathbf{N} \bigcup \{0\}$, define:

$$B(l,k) \equiv \{f \in C^k(\mathbf{R}^n) \mid \sum_{|\alpha| \leq k} \|(1+|x|)^{-l|\alpha|} D^\alpha f\|_\infty < \infty\} \tag{2.6}$$

with $\|f\|_{B(l,k)} = \sum_{|\alpha|\leq k} \|(1+|x|)^{-l|\alpha|} D^\alpha f\|_\infty$.

It is easy to check that $B(l,k)$ is a Banach algebra, and $B(0,k) = BC^k(\mathbf{R}^n)$.

Theorem 2.7. *Let* $k = [\frac{n}{2}] + 1$. *Then*

(1) *A has a* $(1+|A|^2)^{-\frac{l+1}{2}s}$*-regularized* $B(l,k)$ *functional calculus, whenever* $s > \frac{n}{2}$.

(2) *If* $f(t,\cdot)$ *is a family of functions in* $B(l,k)$ *with a parameter* $t \geq 0$ *satisfying:*

$$|D_x^\alpha f(t,x)| \leq M_1(t) M_2(t)^{|\alpha|} \cdot (1+|x|)^{l|\alpha|} \quad \forall t \geq 0, x \in \mathbf{R}^n,$$

where $M_2(t) \geq 1$, *then there exists a constant* M *so that*

$$\|(x \mapsto (1+|x|^2)^{-\frac{1+l}{2}s} f(t,x))(A)\| \leq M M_1(t) M_2(t)^{\frac{n}{2}} \quad \forall t \geq 0.$$

Proof. (1) According to Lemma 2.3 (b), it is sufficient to prove that $x \to (1+|x|^2)^{-\frac{1+l}{2}s} f(x) \in FL^1$ and there exists $M(s) \geq 0$ such that:

$$\|(1+|x|^2)^{-\frac{l+1}{2}s} f(x)\|_{FL^1} \leq M(s) \|f\|_{B(l,k)}$$

whenever $s > \frac{n}{2}$, for all $f \in B(l,k)$.

Let $f \in B(l,k)$. Then

$$|D^\alpha f(x)| \leq \|f\|_{B(l,k)} \cdot (1+|x|)^{l|\alpha|}, \quad \forall |\alpha| \leq k. \tag{2.8}$$

Denote $g(x) \equiv (1+|x|^2)^{-\frac{l+1}{2}s} f(x)$. By Leibniz's formula,

$$D^\alpha g(x) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} D^\beta f \cdot D^\gamma [(1+|x|^2)^{-\frac{l+1}{2}s}].$$

So

(2.9)

$$\begin{aligned} |D^\alpha g(x)| &\leq M \|f\|_{B(l,k)} \sum_{\beta+\gamma=\alpha} (1+|x|)^{l|\beta|} (1+|x|)^{-(l+1)s-|\gamma|} \\ &\leq M \|f\|_{B(l,k)} (1+|x|)^{l|\alpha|-(l+1)s}. \end{aligned}$$

Now we are going to follow a proof similar to the proof in [13, Lemma 2.2]. By [11, Lemma 2.3], there exists a $\psi \in C_c^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}\psi \subset \{x \in \mathbf{R}^n; 2^{-1} < |x| < 2\}$ and $\sum_{-\infty}^{\infty} \psi(2^{-m}x) = 1 \quad \forall x \in \mathbf{R}^n \setminus \{0\}$. Let $\phi \in C_c^\infty(\mathbf{R}^n)$ be such that $\phi(x) = 1$ when $|x| \leq 1$ and $\phi(x) = 0$ when $|x| \geq 2$. Then we have

$$\begin{aligned} g(x) &= g(x)\cdot\phi(x) + g(x)\cdot(1-\phi(x)) \sum_{-\infty}^{\infty} \psi(2^{-m}x) \\ &= g(x)\cdot\phi(x) + g(x)\cdot(1-\phi(x)) \sum_{0}^{\infty} \psi(2^{-m}x) \\ &= g(x)\cdot\phi(x) + g(x)\cdot(1-\phi(x))\psi(x) + g(x)\cdot(1-\phi(x))\psi(2^{-1}x) \\ &+ \sum_{2}^{\infty} g(x)\cdot\psi(2^{-m}x) = g(x)\cdot\mu(x) + \sum_{m=2}^{\infty} g_m(x) \end{aligned}$$

where $\mu(x) \in C_c^\infty(\mathbf{R}^n)$, $g_m(x) = g(x)\psi(2^{-m}x)$.

Since $\mu(x) \in C_c^\infty(\mathbf{R}^n)$, it is easy to check that $g(x) \cdot \mu(x) \in FL^1$ and

$$\|g(x)\mu(x)\|_{FL^1} \leq M\|f\|_{B(l,k)}. \tag{2.10}$$

Using Leibniz's formula, we have

$$D^\alpha g_m(x) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} 2^{-m|\gamma|} D^\beta g(x)(D^\gamma\psi)(2^{-m}x).$$

So,

$$|D^\alpha g_m(x)| \leq M\|f\|_{B(l,k)} \cdot 2^{m(l|\alpha|-(l+1)s)} \cdot 1_{\{2^{m-1}\leq|x|\leq 2^{m+1}\}}(x) \tag{2.11}$$

where $1_{\{2^{m-1}\leq|x|\leq 2^{m+1}\}}(x)$ is the characteristic function. Therefore

$$\|D^\alpha g_m(x)\|_{L^2} \leq M\|f\|_{B(l,k)} \cdot 2^{m(l|\alpha|-(l+1)s+\frac{n}{2})} \quad \forall|\alpha| \leq k. \tag{2.12}$$

Using (2.12) when $|\alpha| = k$ and $\alpha = 0$, it follows from Bernstein's theorem that $g_m \in FL^1$ and:

$$\begin{aligned}\|g_m\|_{FL^1} &\leq M\|g_m\|_{L^2}^{1-\frac{n}{2k}} \sum_{|\alpha|=k} \|D^\alpha g_m\|_{L^2}^{\frac{n}{2k}} \\ &\leq M\|f\|_{B(l,k)} \cdot 2^{m(l+1)(\frac{n}{2}-s)}.\end{aligned}$$

Therefore, when $s > \dfrac{n}{2}$,

$$\sum_{m=2}^{\infty} \|g_m\|_{FL^1} \leq M\|f\|_{B(l,k)}. \tag{2.13}$$

Combining (2.10) and (2.13) concludes the proof of (1).

(2) Following exactly the same proof as in (1), replacing $f(x)$ with $f(t,x)$ we can show that $f(t,\cdot) \in FL^1$ and

$$\|(1+|x|^2)^{-\frac{l+1}{2}s} f(t,x)\|_{FL^1} \leq MM_1(t)M_2(t)^{\frac{n}{2}}.$$

Then Lemma 2.3 (b) concludes the proof. ∎

Remark 2.14. *When $l = 0$, Theorem 2.7 is [4, Proposition 12.3].*

Definition 2.15. *If there exists m so that $z \mapsto \frac{f(z)}{(1+|z|^2)^m} \in FL^1$, then*

$$f(A) \equiv (1+|A|^2)^m \left[(z \mapsto \frac{f(z)}{(1+|z|^2)^m})(A)\right].$$

Note that, by Theorem 2.7, Definition 2.15 applies to any f with $[\frac{n}{2}]+1$ polynomially bounded derivatives.

Lemma 2.16. *Suppose f is as in Definition 2.15. Then*

(a) $\mathcal{D}(f(A))$ *is dense; and*

(b) $(1+|A|^2)^r f(A)(1+|A|^2)^{-r} = f(A)$, *for all $r > 0$.*

Proof. (a) follows from the fact that $\mathcal{D}(|A|^{2m}) \subseteq \mathcal{D}(f(A))$.

Assertion (b) follows from the fact that $(1+|A|^2)^{-r} = \big(z \mapsto (1+|z|^2)^{-r}\big)(A)$ commutes with $g(A)$, for all $g \in FL^1$:

$$\begin{aligned}
&(1+|A|^2)^r f(A)(1+|A|^2)^{-r}\\
&\equiv (1+|A|^2)^r(1+|A|^2)^m \left[(z \mapsto \frac{f(z)}{(1+|z|^2)^m})(A)\right](1+|A|^2)^{-r}\\
&= (1+|A|^2)^{r+m}(1+|A|^2)^{-r} \left[(z \mapsto \frac{f(z)}{(1+|z|^2)^m})(A)\right]\\
&= (1+|A|^2)^m \left[(z \mapsto \frac{f(z)}{(1+|z|^2)^m})(A)\right] \equiv f(A). \blacksquare
\end{aligned}$$

Note that, by (b) of Lemma 2.16 and Lemma 2.3(c), the definition of $f(A)$ is independent of m.

Theorem 2.17. *Suppose that $k = [\frac{n}{2}] + 1$, $f \in C^k(\mathbf{R}^n)$ and, for some $\mu \geq -1$, $M \geq 0$,*

$$|D^\alpha f(x)| \leq M(1+|x|)^{\mu|\alpha|}, \ \ \forall x \in \mathbf{R}^n, 1 \leq |\alpha| \leq k.$$

Then for all $s > \frac{n}{2}$, $f(A)$ has a $(1+|A|^2)^{-\frac{\mu+1}{2}s}$-regularized $BC^k(f(\mathbf{R}^n))$ functional calculus.

Proof. According to Theorem 2.7(1), we must first show that $g \circ f$ is in $B(\mu, k)$, for all $g \in BC^k(f(\mathbf{R}^n))$ and there exists $M \geq 0$ such that

$$\|g \circ f\|_{B(\mu,k)} \leq M\|g\|_{BC^k(f(\mathbf{R}^n))}, \ \ \forall g \in BC^k(f(\mathbf{R}^n)). \tag{2.18}$$

By induction on $|\alpha|$, for any $x \in \mathbf{R}^n, 1 \leq |\alpha| \leq k$,

$$D^\alpha(g \circ f)(x) = \sum_{1 \leq |\beta| \leq |\alpha|} (D^\beta g)(f(x))A_\beta(x),$$

where A_β has the form

$$A_\beta = \prod_{j=1}^{\beta_{j,\alpha}} D^{\alpha_{j,\beta}} f, \ \ \sum_j |\alpha_{j,\beta}| = |\alpha|.$$

The growth conditions on $D^\alpha f$ now imply that, for any $x \in \mathbf{R}^n, 1 \le |\alpha| \le k$,

$$|D^\alpha(g\circ f)(x)| \le \sum_{1\le|\beta|\le|\alpha|} |(D^\beta g)(f(x))| \prod_{j=1}^{\beta_{j,\alpha}} M(1+|x|)^{\mu|\alpha_{j,\beta}|}$$
$$\le \left(\sum_{1\le|\beta|\le|\alpha|} M^{\beta_{j,\alpha}}\right) \|g\|_{BC^k(f(\mathbf{R}^n))}(1+|x|)^{\mu|\alpha|},$$

so that

$$\|(g\circ f)\|_{B(\mu,k)} \le \|(g\circ f)\|_{BC(\mathbf{R}^n)} + \sum_{1\le|\alpha|\le k}\left(\sum_{1\le|\beta|\le|\alpha|} M^{\beta_{j,\alpha}}\right)\|g\|_{BC^k(f(\mathbf{R}^n))},$$

as desired.

Let $B \equiv f(A), C \equiv (1+|A|^2)^{-r},\ r \equiv \frac{(\mu+1)s}{2}$. Theorem 2.7 and (2.18) imply that

$$g(B) \equiv (g\circ f)(A) \equiv (1+|A|^2)^r \left[(z \mapsto \frac{g(f(z))}{(1+|z|^2)^r})(A)\right]$$

(see Definition 2.15) defines a continuous linear map from $BC^k(f(\mathbf{R}^n))$ into $B_C(X)$.

By Lemma 2.3(a), $g \mapsto g(B)$ satisfies (1) of Definition 1.3.

Suppose now that both g and gf_1 (see Definition 1.3(2)) are in $BC^k(f(\mathbf{R}^n))$. Then for m sufficiently large,

$$\begin{aligned}
&g(B)BC\\
&= (1+|A|^2)^r\left[(z\mapsto \frac{g(f(z))}{(1+|z|^2)^r})(A)\right](1+|A|^2)^m\\
&\quad \left[(z\mapsto \frac{f(z)}{(1+|z|^2)^m})(A)\right](1+|A|^2)^{-r}\\
&\subseteq (1+|A|^2)^{r+m}\left[(z\mapsto \frac{g(f(z))}{(1+|z|^2)^r})(A)\right]\left[(z\mapsto \frac{f(z)}{(1+|z|^2)^m})(A)\right](1+|A|^2)^{-r}\\
&= (1+|A|^2)^{r+m}\left[(z\mapsto \frac{f(z)}{(1+|z|^2)^m})(A)\right](1+|A|^2)^{-r}\left[(z\mapsto \frac{g(f(z))}{(1+|z|^2)^r})(A)\right]\\
&= (1+|A|^2)^{r+m}\left[(z\mapsto \frac{f(z)}{(1+|z|^2)^{r+m}})(A)\right]\left[(z\mapsto \frac{g(f(z))}{(1+|z|^2)^r})(A)\right]\\
&= Bg(B)C.
\end{aligned}$$

Also, from the last two lines,

$$\begin{aligned}
Bg(B)C &= (1+|A|^2)^{r+m}\left[(z\mapsto \frac{f(z)g(f(z))}{(1+|z|^2)^{2r+m}})(A)\right]\\
&= (z\mapsto \frac{(f_1g)(f(z))}{(1+|z|^2)^r})(A)\\
&\equiv [(f_1g)(B)]\,C.
\end{aligned}$$

Thus $g \mapsto g(B)$ satisfies (2) of Definition 1.3.

Finally,

$$f_0(B) \equiv (f_0 \circ f)(A) = f_0(A) \equiv (1+|A|^2)^r \left(z \mapsto (1+|z|^2)^{-r}\right)(A) = I,$$

by Lemma 2.3(c), so that $g \mapsto g(B)$ satisfies (3) of Definition 1.3. This concludes the proof. ■

Corollary 2.19. *Suppose p is a polynomial of degree N. Then for all $s > \frac{n}{2}$, $p(A)$ has a $(1+|A|^2)^{-\frac{N}{2}s}$-regularized $BC^k(p(\mathbf{R}^n))$ functional calculus.*

Note that, if f is as in Theorem 2.17 and $f(\mathbf{R}^n) \subseteq \{z \mid Rez \le \omega\}$, then it follows immediately from Theorem 2.17, Proposition 1.7 and Lemma 2.16 that

$$W(t) \equiv \left[(z \mapsto e^{tz})(f(A))\right](1+|A|^2)^{-\frac{\mu+1}{2}s},$$

for $t \ge 0$, defines a $(1+|A|^2)^{-\frac{\mu+1}{2}s}$-regularized semigroup generated by $f(A)$, with $\|W(t)\| = O((1+t)^k e^{\omega t})$.

By applying Theorem 2.7(2), we may improve the growth condition on $\{W(t)\}_{t\ge 0}$, by replacing k with $\frac{n}{2}$.

Corollary 2.20. *Suppose that $\mu \ge -1$, ω is a real number, f is as in Theorem 2.17 and*

$$Re(f(x)) \le \omega, \ \ \forall x \in \mathbf{R}^n.$$

Then, for all $s > \dfrac{n}{2}$, $f(A)$ generates a norm continuous $(1+|A|^2)^{-\frac{\mu+1}{2}s}$-regularized semigroup $\{W(t)\}_{t\ge 0}$ satisfying, for some constant M,

$$\|W(t)\| \le M(1+t)^{\frac{n}{2}} e^{\omega t} \quad \forall t \ge 0.$$

Proof. By Theorem 2.17, $f(A)$ has a C-regularized $BC^k(\{z \mid Re(f(z)) \le \omega\})$ functional calculus, where $C \equiv (1+|A|^2)^{-\frac{\mu+1}{2}s}$. For $t \ge 0$, let

$$W(t) \equiv \left[(z \mapsto e^{tz})(f(A))\right] C = \left[(z \mapsto e^{tf(z)})(A)\right] C.$$

By Proposition 1.7 and Lemma 2.16, $\{W(t)\}_{t\ge 0}$ is a C-regularized semigroup generated by $f(A)$.

By induction on $|\alpha|$, as in the proof of Theorem 2.17,

$$|D^\alpha e^{tf(x)}| \le (1+t)^{|\alpha|} e^{\omega t}(1+|x|)^{\mu|\alpha|}$$

for $1 \le |\alpha| \le k$. Thus by Theorem 2.7(2), the growth condition on $W(t)$ follows. ■

Remark 2.21. Corollary 2.20 generalizes [12, Theorem 4.2]; note that, as in Corollary 2.19, if p is a polynomial of degree N, then we may choose $\mu = N-1$, in Corollary 2.20. A similar result, except for a weaker growth estimate of the regularized semigroup, is in [4, Theorem 12.11].

Remark 2.22. For f as in Corollary 2.20, we may also define a semigroup of unbounded operators

$$\{e^{tf(A)}\}_{t\geq 0} \equiv \{(z \mapsto e^{tf(z)})(A)\}_{t\geq 0}$$

directly with Definition 2.15. By Theorem 2.17, for each $t \geq 0$, $e^{tf(A)}$ has a regularized $BC^k(\{z \,|\, |z| \leq e^{t\omega}\})$ functional calculus.

Remark 2.23. Without the condition on the range of f, in Corollary 2.20, if f is as in Theorem 2.17, then it follows from Theorem 2.17 that there exists an injective operator C, with dense range, such that $f(A)$ generates a C-regularized semigroup. Choose $g(z) \equiv e^{-|z|^2}$; then we may choose $C \equiv g(f(A))(1+|A|^2)^{-\frac{\mu+1}{2}s}$, for $s > \frac{n}{2}$. The C-regularized semigroup is constructed from the regularized functional calculus:

$$W(t) \equiv \left[(z \mapsto e^{tz}e^{-|z|^2})(f(A))\right](1+|A|^2)^{-\frac{\mu+1}{2}s} \;\; (t \geq 0).$$

In fact, such a regularized semigroup can also be constructed without the polynomial growth conditions on f, using Theorem 2.1; see [4, Definition 12.10], where $f(A)$ is *defined* as the generator of the regularized semigroup $\{(z \mapsto e^{tf(z)}g(z))(A)\}_{t\geq 0}$, for appropriate g.

The proof of Corollary 2.20, with $z \mapsto e^{tf(z)}$ replaced by $\cosh(t\sqrt{f(z)})$, gives us the following.

Corollary 2.24. *Suppose f is as in Theorem 2.17, $\omega \geq 0$ and $f(\mathbf{R}^n) \subseteq (-\infty, \omega]$. Then, for all $s > \dfrac{n}{2}$, $f(A)$ generates a $(1+|A|^2)^{-\frac{\mu+1}{2}s}$-regularized cosine function $\{S(t)\}_{t\in\mathbf{R}}$ satisfying, for some constant M,*

$$\|S(t)\| \leq M(1+|t|)^n e^{t\sqrt{\omega}}, \;\; \forall t \in \mathbf{R}.$$

Remark 2.25. See [16] for cosine functions generated by $p(A)$, where p is a polynomial.

Finally, Theorem 2.17 and Proposition 1.10 immediately give us the following two corollaries.

Corollary 2.26. *Suppose f is as in Theorem 2.17 and $f(\mathbf{R}^n) \subseteq \mathbf{R}$. Then, for all $s > \dfrac{n}{2}$, $i(f(A))$ generates a norm-continuous $(1+|A|^2)^{-\frac{\mu+1}{2}s}$-regularized group $\{W(t)\}_{t\geq\mathbf{R}}$ such that, for all $x \in X, x^* \in X^*$, the map*

$$t \mapsto (1-it)^{-k}\langle W(t)x, x^*\rangle$$

is a Fourier-Stieltjes transform of a complex-valued measure of bounded variation.

Corollary 2.27. *Suppose f is as in Theorem 2.17 and $f(\mathbf{R}^n) \subseteq [0,\infty)$. Then, for all $s > \frac{n}{2}$, $-f(A)$ generates a norm continuous $(1+|A|^2)^{-\frac{\mu+1}{2}s}$-regularized semigroup $\{W(t)\}_{t\geq 0}$ such that, for all $x \in X, x^* \in X^*$, the map*

$$t \mapsto (1+t)^{-k} \langle W(t)x, x^* \rangle$$

is a Laplace-Stieltjes transform of a complex-valued measure of bounded variation.

3. Differential operators

In this section we consider the corresponding results for differential operators on the usual function spaces $L^p(\mathbf{R}^n)(1 \leq p < \infty), C_0(\mathbf{R}^n)$ or $BUC(\mathbf{R}^n)$. Noting that, for each j $(1 \leq j \leq n)$, $iD_j \equiv \frac{\partial}{\partial x_j}$ is the generator of the translation group with respect to the j-th space variable enables us to immediately apply Section II to pseudo-differential operators of the form $f(D)$, for f as in Theorem 2.17. The results in $L^p(\mathbf{R}^n)$, for $1 < p < \infty$, can be improved, by applying the Riesz-Thorin convexity theorem to the proof of Theorem 2.7, as in the proof of [13, Lemma 2.2], allowing us to replace $s > \frac{n}{2}$ with $s > n|\frac{1}{2} - \frac{1}{p}|$. We will merely list these corresponding results here.

Note that, in Theorem 3.1, if $f(D)$ is replaced by a constant coefficient differential operator $p(D)$, where p is a polynomial of degree N, the $(\mu+1)$ may be replaced by N, as in Corollary 2.19.

In the following, assume $\ell, \mu \geq -1$.

Theorem 3.1. *Let X be $L^p(\mathbf{R}^n)(1 \leq p < \infty), C_0(\mathbf{R}^n)$ or $BUC(\mathbf{R}^n)$. Let $n_X = n|\frac{1}{2} - \frac{1}{p}|$ when $X = L^p(\mathbf{R}^n)(1 < p < \infty)$, otherwise $n_X = \frac{n}{2}$. Let $k = [\frac{n}{2}] + 1, iD \equiv (\frac{\partial}{\partial x_1} ..., \frac{\partial}{\partial x_n})$. Then*

1. *D has a $(1-\triangle)^{-\frac{\ell+1}{2}s}$-regularized $B(\ell,k)$ functional calculus, whenever $s > n_X$.*
2. *Suppose that f is as in Theorem 2.17. Then $f(D)$ has a $(1-\triangle)^{-\frac{\mu+1}{2}s}$-regularized $BC^k(f(\mathbf{R}^n))$ functional calculus for all $s > n_X$.*
3. *If, in addition to the assumptions in (2), f satisfies $Ref \leq \omega$ for some $\omega \in \mathbf{R}$, then for all $s > n_X, f(D)$ generates a norm-continuous $(1-\triangle)^{-\frac{\mu+1}{2}s}$-regularized semigroup $\{W(t)\}_{t\geq 0}$ satisfying, for some constant M,*

$$\|W(t)\| \leq M(1+t)^{n_X} e^{\omega t} \quad \forall t \geq 0.$$

4. *If, in addition to the assumptions in (2), $f(\mathbf{R}^n) \subseteq \mathbf{R}$, then for all $s > n_X$, $i(f(D))$ generates a norm-continuous $(1-\triangle)^{-\frac{\mu+1}{2}s}$-regularized*

group $\{W(t)\}_{t\in\mathbf{R}}$ *such that, for all* $x \in X, x^* \in X^*$, *the map*

$$t \mapsto (1-it)^{-k} \langle W(t)x, x^* \rangle$$

is a Fourier-Stieltjes transform of a complex-valued measure of bounded variation.

(5) *If, in addition to the assumptions in* (2), $f(\mathbf{R}^n) \subseteq [0,\infty)$, *then for all* $s > n_X$, $-(f(D))$ *generates a norm-continuous* $(1-\triangle)^{-\frac{\mu+1}{2}s}$-*regularized semigroup* $\{W(t)\}_{t\geq 0}$ *such that, for all* $x \in X, x^* \in X^*$, *the map*

$$t \mapsto (1+t)^{-k} \langle W(t)x, x^* \rangle$$

is a Laplace-Stieltjes transform of a complex-valued measure of bounded variation.

(6) *If, in addition to the assumptions in* (2), $f(\mathbf{R}^n) \subseteq (-\infty,\omega]$ $(\omega \geq 0)$, *then, for all* $s > n_X$, $f(A)$ *generates a* $(1+|A|^2)^{-\frac{\mu+1}{2}s}$-*regularized cosine function* $\{S(t)\}_{t\in\mathbf{R}}$ *satisfying, for some constant* M,

$$\|S(t)\| \leq M(1+t^2)^{n_X} e^{t\sqrt{\omega}}, \ \ \forall t \in \mathbf{R}.$$

Remark 3.2. Theorem 3.1 (3) generalizes [13, Theorem 2.3], where f is required to be a polynomial.

Open Question 3.3. Can the smoothness (the k in BC^k, of (2)–(5) of Theorem 3.1) be interpolated, as the regularizing is, for $X = L^p(\mathbf{R}^n), 1 < p < \infty$? Since, for f as in Theorem 2.17, $f(A)$ has a $BC(f(\mathbf{R}^n))$ functional calculus on $L^2(\mathbf{R}^n)$, this sounds plausible.

Example 3.4. By Theorem 3.1, for $s > n_X$, $\triangle$, on $X \equiv BUC(\mathbf{R}^n)$ or $L^p(\mathbf{R}^n)$ $(1 \leq p < \infty)$, has a $(1-\triangle)^{-s}$-regularized $BC^k((-\infty,0])$ functional calculus. This implies that $\triangle$ generates a $(1-\triangle)^{-s}$-regularized cosine function that is $O((1+t^2)^{n_X})$ and a $(1-\triangle)^{-s}$-regularized semigroup $\{W(t)\}_{t\geq 0}$, such that, for all $x \in X, x^* \in X^*$, the map

$$t \mapsto (1+t)^{-k} \langle W(t)x, x^* \rangle$$

is a Laplace-Stieltjes transform of a complex-valued measure of bounded variation. Also $i\triangle$ generates a $(1-\triangle)^{-s}$-regularized group $\{S(t)\}_{t\in\mathbf{R}}$, such that, for all $x \in X, x^* \in X^*$, the map

$$t \mapsto (1-it)^{-k} \langle S(t)x, x^* \rangle$$

is a Fourier-Stieltjes transform of a complex-valued measure of bounded variation.

The regularized semigroup generated by $\triangle$ ($i\triangle$) provides a representation of solutions of the heat (Schrödinger) equation, in X, with initial data in $\mathcal{D}(\triangle^s)$. The regularized cosine function provides solutions of the wave equation. Note that $i\triangle$ fails to generate a strongly continuous semigroup unless $X = L^2(\mathbf{R}^n)$, and for $n > 1$, $\triangle$ fails to generate a cosine function unless $X = L^2(\mathbf{R}^n)$. ∎

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Scientia Research Institute
P.O. Box 988
Athens, Ohio 45701, USA

E-mail address: 72260.2403@compuserve.com

Mathematics Department
Ohio University
Athens, Ohio 45701, USA

E-mail address: ylei@oucsace.cs.ohiou.edu