GLOBAL ATTRACTORS FOR TWO-PHASE STEFAN PROBLEMS IN ONE-DIMENSIONAL SPACE

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ABSTRACT. In this paper we consider one-dimensional two-phase Stefan problems for a class of parabolic equations with nonlinear heat source terms and with nonlinear flux conditions on the fixed boundary. Here, both time-dependent and time-independent source terms and boundary conditions are treated. We investigate the large time behavior of solutions to our problems by using the theory for dynamical systems. First, we show the existence of a global attractor $\mathcal{A}$ of autonomous Stefan problem. The main purpose in the present paper is to prove that the set $\mathcal{A}$ attracts all solutions of non-autonomous Stefan problems as time tends to infinity under the assumption that time-dependent data converge to time-independent ones as time goes to infinity.

1. INTRODUCTION

Let us consider a two-phase Stefan problem $SP = SP(\rho; a; b_0^t, b_1^t; \beta, g, f_0, f_1, u_0, \ell_0)$ described as follows: Find a function $u = u(t,x)$ on $Q(T) = (0,T) \times (0,1)$, $0 < T < \infty$, and a curve $x = \ell(t)$, $0 < \ell < 1$, on $[0,T]$ satisfying

$$\rho(u)_t - a(u_x)_x + \xi + g(u) = \begin{cases} f_0 & \text{in } Q_\ell^{(0)}(T), \\ f_1 & \text{in } Q_\ell^{(1)}(T), \end{cases} \tag{1.1}$$

$$Q_\ell^{(0)}(T) = \{(t,x); 0 < t < T, 0 < x < \ell(t)\},$$

$$Q_\ell^{(1)}(T) = \{(t,x); 0 < t < T, \ell(t) < x < 1\},$$

1991 *Mathematics Subject Classification.* Primary 35K22; secondary 65R35, 35B35.
Key words and phrases. Global attractor, Stefan problem, non-autonomous problem.
Received: May 20, 1997.

$$\xi(t,x) \in \beta(u(t,x)) \quad \text{for a.e. } (t,x) \in Q(T),$$

(1.2) $$u(t,\ell(t)) = 0 \quad \text{for } 0 \le t \le T,$$

(1.3) $$\ell'(t) = -a(u_x)(t,\ell(t)-) + a(u_x)(t,\ell(t)+) \text{ for a.e.} t \in [0,T],$$

(1.4) $$a(u_x)(t,0+) \in \partial b_0^t(u(t,0)) \quad \text{for a.e. } t \in [0,T],$$

(1.5) $$-a(u_x)(t,1-) \in \partial b_1^t(u(t,1)) \quad \text{for a.e. } t \in [0,T],$$

(1.6) $$u(0,x) = u_0(x) \quad \text{for } x \in [0,1],$$

(1.7) $$\ell(0) = \ell_0,$$

where $\rho : R \to R$ and $a : R \to R$ are continuous increasing functions; β is a maximal monotone graph in $R \times R$; $g : R \to R$ is a Lipschitz continuous function; $f_i (i = 0,1)$ is a given function on $(0,\infty) \times (0,1)$; $b_i^t (i = 0,1)$ is a proper l.s.c. convex function on R for each $t \ge 0$ and ∂b_i^t denotes its subdifferential in R; u_0 is a given initial function and ℓ_0 is a number with $0 < \ell_0 < 1$.

In this paper, we treat a class of nonlinear parabolic equations of the form (1.1), which includes as a typical example,

$$c_i u_t - (|u_x|^{p-2} u_x)_x + \sigma(u) + g(u) \ni f_i, \quad i = 0,1,$$

for positive constants c_0, c_1 and $2 \le p < \infty$, where

$$\sigma(r) = \begin{cases} 1 & \text{for } r > 1, \\ [0,1] & \text{for } r = 1, \\ 0 & \text{for } -1 < r < 1, \\ [-1,0] & \text{for } r = -1, \\ -1 & \text{for } r < -1, \end{cases}$$

and

$$g(r) = \begin{cases} (r-1) & \text{for } r > 1, \\ r(r+1)(r-1) & \text{for } -1 \le r \le 1, \\ (r+1) & \text{for } r < -1. \end{cases}$$

Also, it should be noticed that boundary condition (1.4) and (1.5) represent various linear or nonlinear boundary conditions (see [1, Section 5] and Remark 2.1 in this paper).

Aiki and Kenmochi already established uniqueness, local existence in time and behavior of solutions for our problem SP(cf. [7, 1, 2]). In case $\rho(r) = a(r) = r$, $\beta \equiv 0$ and $f_0 \equiv f_1 \equiv 0$ with the boundary condition, $u(i) = c_i$ for $i = 0,1$ where c_i is some constant, the problem SP is completely solved by Mimura, Yamada and Yotsutani in [10, 11, 12]. They showed that there exists a maximal solution $[u^*, \ell^*]$ of the stationary problem, and by comparison principle, if $u_0 \geq u^*$ and $\ell_0 \geq \ell^*$, then for the solution $\{u, \ell\}$, $u(t)$ and $\ell(t)$ converge to u^* and ℓ^*, respectively, as time goes to infinity.

In our problem, since g may not be monotone increasing and data, $b_i^t (i = 0, 1)$ and $f_i (i = 0, 1)$ depend on time variable t, we can not prove the convergence of the solution. So, in order to consider the large time behavior of solutions we discuss a global attractor for the problem SP. Our main results of the present paper are stated as follows:

(i) (Global existence) SP has a solution $\{u, \ell\}$ on $[0, \infty)$ satisfying for $t \geq 0$

$$0 < \inf_{t \geq 0} \ell(t) \leq \sup_{t \geq 0} \ell(t) < 1 \text{ and } |u(t)|_{L^2(0,1)} \leq C(|u_0|_{L^2(0,1)} \exp(-\mu t) + 1),$$

where C and μ are positive constants.

(ii) (Global attractor for the autonomous problem) We put $SP^* = SP(\rho; a; b_0, b_1; \beta, g, f_0^*, f_1^*, u_0, \ell_0)$ where $f_i^* \in L^2(0, 1)$ and b_i is a proper l.s.c. convex function on R for $i = 0, 1$. Then, there is a global attractor $\mathcal{A}$ for the problem SP^*.

(iii) (Asymptotic behavior of solutions to SP)
We suppose that

$$b_i^t \to b_i \quad \text{and} \quad f_i(t) \to f_i^* \quad \text{in some sense as } t \to \infty \text{ for } i = 0, 1,$$

and $\{u, \ell\}$ is a solution of problem SP. Then, we have

$$dist([u(t), \ell(t)], \mathcal{A}) \to 0 \quad \text{as } t \to \infty,$$

where $dist(z, B)$ is an usual distance in $L^2(0, 1) \times R$ between a point $z \in L^2(0, 1) \times R$ and a set $B \subset L^2(0, 1) \times R$.

There are many interesting results dealing with a global attractor of autonomous nonlinear partial differential equations (ex. [16, 9] and etc.). The question concerned with relationship between global attractors of autonomous and non-autonomous problems was earlier discussed by Smiley [14, 15]. Recently, by Ito, Kenmochi and Yamazaki [5] similar results to (iii) were obtained, in which the following non-autonomous problem,

$$\frac{d}{dt} u(t) + \partial \varphi^t(u(t)) + N(u(t)) \ni f(t), \quad t > 0, \quad \text{in } H,$$

was considered, where H is a Hilbert space, φ^t is a proper l.s.c. convex function on H for $t > 0$, $\partial \varphi^t$ is its subdifferential, $N : H \to H$ is Lipschitz continuous and f is a given function. They gave a more general answer for that question. But, our system SP can not be described a single evolution equation of the above form, so that their result is not directly applied to our problem.

The outline of the present paper is as follows. In section 2 we present assumptions and main results. In section 3 we recall known results about problem SP, which are concerned with uniqueness, local existence results in time and energy inequalities. Some uniform estimates for solutions to SP are obtained in section 4, and then used in section 5 to prove global existence for problem SP and existence of a global attractor of the semigroup associated to problem SP^* by applying the theory on dynamical systems in Temam [16]. The asymptotic behavior of solutions to SP is proved in the final section.

Throughout this paper for simplicity we put

$$H := L^2(0,1), X := W^{1,p}(0,1), 2 \le p < \infty;$$

$$(\cdot,\cdot)_H : \text{the standard inner product in } H;$$

$$X_0 := \{z \in X; z(x_0) = 0 \text{ for some } x_0 \in (0,1)\};$$

$$V := \{[z,r] \in H \times (0,1); z \ge 0 \text{ a.e. on } (0,r), z \le 0 \text{ a.e. on } (r,1)\};$$

$$\left.\begin{array}{l} dist([u,\ell],[v,m]) := |u-v|_H + |\ell - m| \\ dist(z,A) := \inf\{dist(z,z'); z' \in A\} \\ dist(A,B) := \sup\{dist(x,B); x \in A\} \end{array}\right\} \quad \begin{array}{l} \text{for } [u,\ell],[v,m], z \in V \\ \text{and } A, B \subset V; \end{array}$$

$$\mathcal{B}(M,\delta) := \{(z,r) \in V; |z|_H \le M, \delta \le r \le 1-\delta\}$$

$$\text{for } M > 0 \text{ and } \delta \in (0,1/2).$$

For a proper l.s.c. convex function ψ on R, $D(\psi) := \{r \in R; \psi(r) < \infty\}$. We refer to Brézis [3] for definitions and basic properties concerned with convex analysis.

2. Main results

Let $p \ge 2$ and $1/p' + 1/p = 1$, and let us begin with the precise assumptions (H1) $\sim$ (H6) on ρ, a, β, g, $b_i^t (i = 0,1)$ and $f_i (i = 0,1)$ under which SP is discussed.

(H1) $\rho : R \to R$ is bi-Lipschitz continuous and increasing function with $\rho(0) = 0$; denote by C_ρ a common Lipschitz constant of ρ and ρ^{-1}.

(H2) $a : R \to R$ is a continuous function such that

$$a_0|r|^p \le a(r)r \le a_1|r|^p \quad \text{for any } r \in R,$$

$$a_0(r-r')^{p-1} \le a(r) - a(r') \quad \text{for any } r, r' \in R \text{ with } r \le r',$$

where a_0 and a_1 are positive constants.

(H3) β is a maximal monotone graph in $R \times R$ such that $\beta(0) \ni 0$ and

$$|r'| \le C_\beta \quad \text{for } r' \in \beta(r) \text{ and } r \in R.$$

(H4) $g : R \to R$ is Lipschitz continuous with $g(0) = 0$ satisfying the condition that there is a positive constant C_g such that

$$g(r)r \geq -C_g \quad \text{and} \quad |g(r) - g(r')| \leq C_g|r - r'| \quad \text{for any } r, r' \in R.$$

(H5) For $i = 0, 1$ and each $t \geq 0$, b_i^t is a proper l.s.c. convex function on R and there is a positive constant d_0 such that

$$D(b_0^t) \subset [d_0, \infty) \quad \text{and} \quad D(b_1^t) \subset (-\infty, -d_0],$$

and there are absolutely continuous functions α_0, α_1 on $[0, \infty)$ such that

$$\alpha_0' \in L^1(0, \infty) \cap L^2(0, \infty) \text{ and } \alpha_1' \in L^1(0, \infty),$$

and for each $0 \leq s \leq t < \infty$ and each $r \in D(b_i^s)$ there exists $r' \in D(b_i^t)$ satisfying

$$|r' - r| \leq |\alpha_0(t) - \alpha_0(s)|(1 + |r| + |b_i^s(r)|^{1/p}),$$

$$b_i^t(r') - b_i^s(r) \leq |\alpha_1(t) - \alpha_1(s)|(1 + |r|^p + |b_i^s(r)|).$$

Also, we suppose that for $i = 0, 1$ there is a function $k_i \in W^{1,\infty}(0, \infty)$ such that $b_i^{(\cdot)}(k_i(\cdot)) \in L^\infty(0, \infty)$.

Furthermore, we assume that for $i = 0, 1$, b_i^t converges to a proper l.s.c. convex function b_i on R as $t \to \infty$ in the sense of Mosco [13], that is, the following conditions (b1) and (b2) hold:

(b1) If $w : [0, \infty) \to R$ and $w(t) \to z$ in R as $t \to \infty$, then

$$\liminf_{t \to \infty} b_i^t(w(t)) \geq b_i(z);$$

(b2) for each $z \in D(b_i)$ there is a function $w : [0, \infty) \to R$ such that $w(t) \to z$ and $b_i^t(w(t)) \to b_i(z)$ as $t \to \infty$.

Remark 2.1. (cf. [1, Section 5]) In the case of Dirichlet or Signorini boundary condition, the following conditions imply the above (H5).
(1) (Dirichlet type).

$$u(t, i) = q_i(t), \quad t \geq 0 \text{ and } i = 0, 1;$$

this is written in the form (1.4) and (1.5) if $b_i^t(\cdot)$ is defined by

$$b_i^t(r) = \begin{cases} 0 & \text{if } r = q_i(t), \\ \infty & \text{if } r \neq q_i(t). \end{cases}$$

We suppose that for $i = 0, 1$

$$\begin{cases} (-1)^i q_i(t) \geq d_0 > 0 \text{ for } t \geq 0, \\ q_i \in C([0, \infty)) \text{ and } q_i' \in L^1(0, \infty) \cap L^2(0, \infty) \cap L^\infty(0, \infty). \end{cases} \tag{2.1}$$

Then, condition (H5) holds.

(2) (Signorini type).

$$\begin{cases} u(\cdot,0) \ge q_0(\cdot) & \text{on } [0,\infty), \\ u_x(\cdot,0+) = 0 & \text{on } \{u(\cdot,0) > q_0(\cdot)\}, \\ u_x(\cdot,0+) \le 0 & \text{on } \{u(\cdot,0) = q_0(\cdot)\}, \end{cases}$$

these conditions are represented in the form (1.4) for b_0^t given by

$$b_0^t(r) = \begin{cases} 0 & \text{if } r \ge q_0(t), \\ \infty & \text{otherwise.} \end{cases}$$

If $q_0(t)$ satisfies condition (2.1), then condition (H5) holds.

Furthermore we suppose:

(H6) For $i = 0, 1$, $f_i : [0,\infty) \to C([0,1])$, $(-1)^i f_i \ge 0$ on $[0,\infty) \times [0,1]$ and

$$f_{it} \in L^1(0,\infty; H).$$

Now, we give the definition of a solution to SP.

Definition 2.1. We say that a pair $\{u, \ell\}$ is a solution of SP on $[0,T]$, $0 < T < \infty$, if the following properties (S1) $\sim$ (S3) are fulfilled:

(S1) $u \in W^{1,2}_{loc}((0,T]; H) \cap L^\infty(0,T;H) \cap L^\infty_{loc}((0,T];X) \cap L^p(0,T;X)$, $\ell \in C([0,T]) \cap W^{1,2}_{loc}((0,T])$ with $0 < \ell < 1$ on $[0,T]$.

(S2) For each $i = 0, 1$ (1.1) holds in the sense of $\mathcal{D}'(Q^{(i)}_\ell(T))$ for some $\xi \in L^2(Q(T))$ with $\xi \in \beta(u(t,x))$ a.e. on $Q(T)$, and (1.2), (1.3), (1.6) and (1.7) are satisfied.

(S3) For $i = 0, 1$, $b_i^{(\cdot)}(u(\cdot,i)) \in L^1(0,T) \cap L^\infty_{loc}((0,T])$, $u(t,i) \in D(\partial b_i^t)$ for a.e. $t \in [0,T]$, and (1.4) and (1.5) hold.

Also, for $0 < T' \le \infty$, a pair $\{u,\ell\}$ is a solution of SP on $[0,T')$ if it is a solution of SP on $[0,T]$ for every $0 < T < T'$ in the above sense. Furthermore, $[0,T^*)$, $0 < T^* \le \infty$ is called the maximal interval of existence of the solution, if the problem has a solution on $[0,T^*)$ and the solution can not be extended in time beyond T^*.

The first main result is concerned with the global existence of a solution to SP.

Theorem 2.1. *Assume that conditions (H1) $\sim$ (H6) hold and* $[u_0, \ell_0] \in V$. *Then, there is one and only one solution* $\{u,\ell\}$ *of* $SP(\rho; a; b_0^t, b_1^t; \beta, g, f_0, f_1, u_0, \ell_0)$ *on* $[0,\infty)$.

Next, we show uniform estimates for solutions of SP.

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, let M be any positive number and $\delta \in (0, 1/2)$ and put*

$$U(M,\delta) := \left\{ [u,\ell]; \begin{array}{l} \{u,\ell\} \textit{ is a solution to } SP(\rho; a; b_0^t, b_1^t; \beta, g, f_0, \\ f_1, u_0, \ell_0) \textit{ on } [0,\infty) \textit{ for } [u_0,\ell_0] \in \mathcal{B}(M,\delta) \end{array} \right\}.$$

Then, there are positive constants M_0, μ_0 and δ_0 such that for any $[u, \ell] \in U(M, \delta)$

$$|u(t)|_H \leq M_0(\exp(-\mu_0 t) + 1) \quad \textit{for any } t \geq 0, \tag{2.2}$$

$$\delta_0 \leq \ell(t) \leq 1 - \delta_0 \quad \textit{for any } t \geq 0. \tag{2.3}$$

Moreover, for any $t_0 > 0$ there exists a positive number $K(t_0)$ satisfying

$$\left. \begin{array}{l} |u(t)|_X \leq K(t_0) \\ |b_0^t(u(t,0))| \leq K(t_0) \\ |b_1^t(u(t,1))| \leq K(t_0) \end{array} \right\} \textit{for any } t \geq t_0 \textit{ and } [u, \ell] \in U(M, \delta). \tag{2.4}$$

Next, we consider a global attractor for the autonomous problem SP^*. For this purpose we give some assumptions and notations.

(H5*) For $i = 0, 1$, b_i is a proper l.s.c. convex function on R and there is a positive constant d_0 such that

$$D(b_0) \subset [d_0, \infty) \quad \text{and} \quad D(b_1) \subset (-\infty, -d_0].$$

(H6*) For $i = 0, 1$, $f_i^* \in C([0,1])$, $(-1)^i f_i^* \geq 0$ on $[0, 1]$.

Corollary 2.1. *Under the assumptions (H1) $\sim$ (H4), (H5*) and (H6*), the problem $SP^*(u_0, \ell_0) := SP(\rho; a; b_0, b_1; \beta, g, f_0^*, f_1^*, u_0, \ell_0)$ has a unique solution $\{u, \ell\}$ on $[0, \infty)$.*

Obviously, (H5*) and (H6*) imply (H5) and (H6), respectively, and hence Corollary 2.1 is a direct consequence of Theorem 2.1. Here, we define a family of operators $S(t)$, $t \geq 0$, by

$$S(t) : V \to V,\ S(t)[u_0, \ell_0] = [u(t), \ell(t)] \quad \text{for } t \geq 0 \text{ and all } [u_0, \ell_0] \in V,$$

where $\{u, \ell\}$ is the solution of $SP^*(u_0, \ell_0)$ on $[0, \infty)$. By Corollary 2.1, $\{S(t); t \geq 0\}$ has the usual semigroup property:

$$S(t + s) = S(t) \cdot S(s) \text{ in } V \text{ for any } s, t \geq 0 \text{ and } S(0) = \text{ Identity in } V.$$

Theorem 2.3. *Suppose that the same assumptions as in Corollary 2.1 hold. Then, for each $t \geq 0$, $S(t)(\cdot)$ is continuous in V, and for any $t_0 > 0$, $M > 0$ and $0 < \delta < 1/2$, $\bigcup_{t \geq t_0} S(t)\mathcal{B}(M, \delta)$ is relatively compact in V.*

Moreover, there is a global attractor $\mathcal{A} \subset V$ for the semigroup $\{S(t); t \geq 0\}$, that is, $\mathcal{A}$ is compact in V,

$$S(t)\mathcal{A} = \mathcal{A} \quad \textit{for any } t \geq 0,$$

and for each $M > 0$ and $\delta \in (0, \frac{1}{2})$ we have

$$dist(S(t)\mathcal{B}(M, \delta), \mathcal{A}) \to 0 \quad \textit{as } t \to \infty.$$

Finally, as to the asymptotic stability of the solution to SP, we prove the following theorem.

Theorem 2.4. *Assume that (H1) $\sim$ (H6), (H5*) and (H6*) hold, and $f_i - f_i^* \in L^2(0,\infty;H)(i=0,1)$. Then, for any solution $\{u,\ell\}$ of SP,*

$$dist([u(t),\ell(t)],\mathcal{A}) \to 0 \quad \text{as } t \to \infty,$$

where $\mathcal{A}$ is the global attractor defined by Theorem 2.3.

Moreover, let $M>0$ and $\delta \in (0,\frac{1}{2})$. Then, for any positive number $\varepsilon > 0$ there is a positive constant $t_0 := t_0(M,\delta,\varepsilon) > 0$ such that

$$dist([u(t),\ell(t)],\mathcal{A}) \le \varepsilon \quad \text{for } t \ge t_0 \text{ and } [u,\ell] \in U(M,\delta),$$

where $\{u,\ell\}$ is a solution of $SP(\rho;a;b_0^t,b_1^t;\beta,g,f_0,f_1,u_0,\ell_0)$ on $[0,\infty)$.

Throughout this paper, we suppose that conditions (H1) $\sim$ (H4) always hold and for simplicity the following notations are used:

$$\tilde{\rho}(r) := \int_0^r \rho^{-1}(\tau)d\tau, \hat{\rho}(r) := \tilde{\rho}(\rho(r)) \text{ and } \hat{a}(r) := \int_0^r a(s)ds.$$

Since ρ and a satisfy (H1) and (H2), respectively, there are positive constants γ_0, γ_1 and a_2 depending only on C_ρ, a_0, a_1 and p such that

$$\gamma_0|r|^2 \le \hat{\rho}(r) \le \gamma_1|r|^2 \quad \text{for all } r \in R,$$

$$(2.5) \quad |r|^p \le a_2\hat{a}(r), \quad \hat{a}(r) \le a_2|r|^p \quad \text{and} \quad |a(r)|^{p'} \le a_2\hat{a}(r) \quad \text{for } r \in R.$$

Putting $\hat{\beta}(r) = \int_0^r \beta(s)ds$ and $\hat{g}(r) = \int_0^r g(s)ds$, we see that

$$0 \le \hat{\beta}(r) \le \frac{C_\beta}{2}r^2 \quad \text{and} \quad |\hat{g}(r)| \le \frac{C_g}{2}r^2 \quad \text{for any } r \in R.$$

Here, we list some useful inequalities:

$$(2.6) \qquad |v|_{L^\infty(0,x_0)} \le |v|_{L^r(0,x_0)} + |v_x|_{L^2(0,x_0)} \quad \text{for } v \in W^{1,2}(0,x_0),$$

$$(2.7) \qquad |v|_{L^\infty(0,x_0)} \le |v_x|_{L^2(0,x_0)} \quad \text{for } v \in W,$$

$$(2.8) \qquad |v|_{L^\infty(0,x_0)} \le \sqrt{2}|v_x|_{L^2(0,x_0)}^{1/2}|v|_{L^2(0,x_0)}^{1/2} \quad \text{for } v \in W,$$

where $x_0 \in (0,1)$ and $r \ge 1$ and $W = \{v \in W^{1,2}(0,x_0); v(x_0) = 0\}$.

3. Preliminaries and known results

First, in this section we recall the results in Aiki-Kenmochi [7, 1, 2] on the local existence, uniqueness and estimates for solutions of SP which are given as follows.

Theorem 3.1. *(cf.* [2, Theorem 1.1] *and* [7, Theorem]*) Under the same assumptions as in Theorem 2.1, for some positive number T, SP has* *a solution* $\{u, \ell\}$ *on* $[0, T]$ *such that*

$$\begin{cases} t^{1/2}u_t \in L^2(0,T;H), t^{1/p}u \in L^\infty(0,T;X), t^{2/(p'+2)}\ell' \in L^{p'+2}(0,T), \\ tb_i^t(u(t,i)) \in L^\infty(0,T), i = 0, 1. \end{cases}$$

Lemma 3.1. *(cf.* [2, Theorem 1.4]*) Suppose that all the assumptions of Theorem 2.1 hold. Let* $\{u, \ell\}$ *be a solution of SP on* $[0, T]$*. Further, assume that for some positive number* δ, $\delta \le \ell \le 1 - \delta$ *on* $[0, T]$*. Then, there is a positive constant* m_1 *depending only on* δ, ρ, a *and* p *such that for* $0 < s \le t \le T$

$$\begin{aligned} &\int_s^t |\ell'(\tau)|^{p'+2} d\tau \\ \text{(3.1)} \quad \le\ & m_1(t-s)|u|^{3p-2}_{L^\infty(s,t;X)} \\ &+ m_1|u|^p_{L^\infty(s,t;X)} \int_s^t (|u_\tau|_H^2 + |\xi|_H^2 + |g(u)|_H^2 + |f_0|_H^2 + |f_1|_H^2) d\tau, \end{aligned}$$

and, moreover,

$$\begin{aligned} &\int_s^t (\tau - s)^2 |\ell'(\tau)|^{p'+2} d\tau \\ \text{(3.2)} \quad \le\ & m_1|(\tau - s)^{1/p}u|^p_{L^\infty(s,t;X)} \times \\ &\times \int_s^t (\tau - s)(|u_\tau|_H^2 + |\xi|_H^2 + |g(u)|_H^2 + |f_0|_H^2 + |f_1|_H^2) d\tau \\ &+ m_1(t-s)^{2/p}|(\tau - s)^{1/p}u|^{3p-2}_{L^\infty(s,t;X)} \quad \textit{for } 0 \le s \le t \le T. \end{aligned}$$

The next lemma is concerned with the boundary conditions.

Lemma 3.2. *(cf.* [4, Lemma 1] *and* [6, Section 1.5]*) For* $i = 0, 1$, *assume that* b_i^t *satisfies (H5). Then, there is a positive number* B_1 *such that*

$$\left.\begin{aligned} b_i^t(r) + B_1|r| + B_1 \ge 0 \\ b_i^t(r) + B_1|r|^p + B_1 \ge 0 \end{aligned}\right\} \quad \textit{for all } r \in R, t \ge 0 \textit{ and } i = 0, 1.$$

For simplicity of notations we put

$$\begin{aligned} E(t,z) &= \int_0^1 \hat{a}(z_x)dx + b_0^t(z(0)) + b_1^t(z(1)), \\ F(t,z) &= B_0\{b_0^t(z(0)) + b_1^t(z(1)) + B_1(|z(0)|^p + |z(1)|^p + 2)\} \\ &\qquad\qquad \text{for } t \ge 0 \text{ and } z \in X, \end{aligned}$$

where B_0 and B_1 are positive constants defined in Lemma 3.2.

According to Lemma 3.2, (2.5), (2.7) and (2.8) it is easy to get the following inequalities:

Lemma 3.3. *For $i = 0, 1$, assume that b_i^t satisfies (H5). Then, there are positive constants μ, B_2 and B_3 depending only on B_1 and a_2 such that*

$$\left.\begin{array}{l} \displaystyle\int_0^1 \hat{a}(z_x)dx \le 2E(t,z) + B_2, \\ \mu|z|_H^2 \le E(t,z) + B_2, \\ |z|_{L^\infty(0,1)} \le B_3(E(t,z) + B_2), \\ 0 \le F(t,z) \le B_0(E(t,z) + B_2 + |z|_H^p), \\ |b_i^t(z)| \le B_3(E(t,z) + B_2), \ (i = 0, 1) \end{array}\right\} \text{for } z \in X_0 \text{ and } t \ge 0. \tag{3.3}$$

Now, we show the useful energy inequality:

Proposition 3.1. *(cf.* [2, Section 3]*) Suppose that the same assumptions as in Theorem 2.1 hold. Let $\{u, \ell\}$ be a solution of SP on $[0, T]$, $0 < T < \infty$. Then, the function $t \to E(t, u(t))$ is of bounded variation on $[s, T]$ for any $s \in (0, T]$ and*

$$E(t, u(t)) - E(s, u(s)) \le \int_s^t \frac{d}{d\tau} E(\tau, u(\tau))d\tau \quad \text{for any } 0 < s \le t \le T,$$

and

$$\begin{aligned} & \frac{d}{dt}E(t, u(t)) + \frac{1}{C_\rho}|u_t(t)|_H^2 \\ \le\ & |\alpha_0'(t)|(|a(u_x)(t, 0+)| + |a(u_x)(t, 1-)|)F(t, u(t))^{1/p} \\ & + |\alpha_1'(t)|F(t, u(t)) - (\xi(t), u_t(t))_H - (g(u)(t), u_t(t))_H \\ & + (f_0(t), u_t(t))_H + (f_1(t), u_t(t))_H \quad \text{for a.e. } t \in [0, T]. \end{aligned}$$

We can prove this proposition in ways similar to those of [2, section 3] with the help of Lemma 3.2, so we omit its proof.

4. Uniform estimates

We use the same notation as in the previous section and prove the following propositions in similar ways to those of [2, Section 3]. In this section we assume that all the assumptions of Theorem 2.1 hold and $\{u, \ell\}$ is a solution of SP on $[0, T]$, $0 < T < \infty$.

Proposition 4.1. *There are positive constants C_1 and μ_1 depending only on ρ, a, β, g, f_0, f_1, b_0^t and b_1^t (independent of T, u_0 and ℓ_0) such that*

$$|u(t)|_H \le C_1\{|u_0|_H \exp(-\mu_1 t) + 1\} \quad \text{for any } t \in [0, T], \tag{4.1}$$

and, for any $0 \le s \le t \le T$,

$$\begin{aligned} & \int_0^1 \{\hat{\rho}(u)(t) - k(t)\rho(u)(t)\}dx + \mu_1 \int_s^t E(\tau, u(\tau))d\tau \\ & \le C_1\{(t - s) + 1\} + \int_0^1 (\hat{\rho}(u)(s) - k(s)\rho(u)(s))dx, \end{aligned} \tag{4.2}$$

where $k(t,x) = (1-x)k_0(t) + xk_1(t)$ *for* $(t,x) \in [0,\infty) \times [0,1]$.

Proof. First, we observe that, for a.e. $t \in [0,T]$,

$$\begin{aligned} &(\rho(u)_t(t), u(t) - k(t))_H \\ &= \frac{d}{dt}\int_0^1 \hat{\rho}(u)(t)dx + (\rho(u)(t), k_t(t))_H - \frac{d}{dt}(\rho(u)(t), k(t))_H. \end{aligned} \tag{4.3}$$

On the other hand, by integration by parts we obtain that, for a.e. $t \in [0,T]$,

$$\begin{aligned} (\rho(u)_t, u-k)_H &= \int_0^{\ell(t)} (a(u_x)_x - \xi - g(u) + f_0)(u-k)\,dx \\ &\quad + \int_{\ell(t)}^1 (a(u_x)_x - \xi - g(u) + f_1)(u-k)\,dx \\ &\leq -E(t,u(t)) + E(t,k(t)) + \ell'(t)k(t,\ell(t)) \\ &\quad + \int_0^1 (\hat{\beta}(k) - \hat{\beta}(u))dx - \int_0^1 g(u)(u-k)dx \\ &\quad + \int_0^{\ell(t)} f_0(u-k)dx + \int_{\ell(t)}^1 f_1(u-k)dx. \end{aligned} \tag{4.4}$$

Here, we note that, for a.e. $t \in [0,T]$,

$$\begin{aligned} \ell'(t)k(t,\ell(t)) &= \ell'(t)k_0(t) - \ell'(t)\ell(t)k_0(t) + \ell'(t)\ell(t)k_1(t) \\ &= \frac{d}{dt}L(t) - \ell(t)k_0'(t) - \frac{1}{2}\ell^2(t)(k_1'(t) - k_0'(t)) \\ &\leq \frac{d}{dt}L(t) + 2(|k_0'(t)| + |k_1'(t)|), \end{aligned} \tag{4.5}$$

where $L(t) = \ell(t)k_0(t) + \frac{1}{2}\ell^2(t)(k_1(t) - k_0(t))$;

$$\begin{aligned} &E(t,u(t)) + \int_0^1 \hat{\beta}(u)dx + (g(u),u)_H \\ &\geq \frac{1}{2}E(t,u(t)) + \frac{\mu}{2}|u(t)|_H^2 - \frac{B_2}{2} - C_g; \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\int_0^1 \hat{\beta}(k)dx + (g(u),k)_H \\ &\leq \frac{C_\beta}{2}|k(t)|^2_{L^\infty(0,1)} + \frac{\mu}{8}|u(t)|_H^2 + \frac{2C_g^2}{\mu}|k(t)|^2_{L^\infty(0,1)}; \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\int_0^{\ell(t)} f_0(u-k)dx + \int_{\ell(t)}^1 f_1(u-k)dx \\ &\leq \frac{\mu}{8}|u(t)|_H^2 + (\frac{2}{\mu} + \frac{1}{2})(|f_0(t)|_H^2 + |f_1(t)|_H^2) + |k(t)|^2_{L^\infty(0,1)}. \end{aligned} \tag{4.8}$$

From (4.3) $\sim$ (4.8) it follows that

$$\begin{aligned}(4.9)\qquad &\frac{d}{dt}\int_0^1 \hat{\rho}(u)(t)dx + \frac{\mu}{8\gamma_1}\int_0^1 \hat{\rho}(u)(t)dx + \frac{1}{2}E(t,u(t))\\ &\le C_2 + \frac{d}{dt}L(t) + \frac{d}{dt}(\rho(u)(t),k(t))_H \quad \text{for a.e. } t\in[0,T],\end{aligned}$$

where C_2 is a positive constant depending only on given data.

Moreover, with $\mu_2 = \dfrac{\mu}{16\gamma_1}$ we see that for a.e. $t\in[0,T]$

$$\begin{aligned}&\frac{d}{dt}(\rho(u)(t),k(t))_H\\ \le\ &\frac{d}{dt}(\rho(u)(t),k(t))_H + \mu_2(\rho(u)(t),k(t))_H + \mu_2|\rho(u)(t)|_H|k(t)|_H\\ \le\ &\frac{d}{dt}(\rho(u)(t),k(t))_H + \mu_2(\rho(u)(t),k(t))_H\\ &+\mu_2\int_0^1 \hat{\rho}(u)(t)dx + \frac{C_\rho^2}{4\gamma_0^2\mu_2}|k(t)|_H^2;\end{aligned}$$

$$\frac{d}{dt}L(t) \le \frac{d}{dt}L(t) + \mu_2 L(t) + \mu_2|L(t)|.$$

Therefore, we infer together with (3.3) that for a.e. $t\in[0,T]$

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \hat{\rho}(u)(t)dx + \mu_2\int_0^1 \hat{\rho}(u)(t)dx\\ \le\ & C_3 + \frac{d}{dt}L(t) + \mu_2 L(t) + \frac{d}{dt}(\rho(u)(t),k(t))_H + \mu_2(\rho(u)(t),k(t))_H\end{aligned}$$

where C_3 is a positive constant independent of T and $|u_0|_H$.

Hence, multiplying the above inequality by $\exp(\mu_2 t)$, we conclude that for a.e. $t\in[0,T]$

$$\begin{aligned}&\frac{d}{dt}\left\{\exp(\mu_2 t)\int_0^1 \hat{\rho}(u)(t)dx\right\}\\ \le\ & C_3\exp(\mu_2 t) + \frac{d}{dt}\{\exp(\mu_2 t)L(t)\} + \frac{d}{dt}\{\exp(\mu_2 t)(\rho(u)(t),k(t))_H\}\end{aligned}$$

so that

$$\begin{aligned}&\int_0^1 \hat{\rho}(u)(t)dx\\ \le\ &\frac{C_3}{\mu_2} + \left\{\int_0^1 \hat{\rho}(u_0)dx - (\rho(u_0),k(0))_H - L(0)\right\}\exp(-\mu_2 t)\\ &+ (\rho(u)(t),k(t))_H + L(t) \quad \text{for any } t\in[0,T].\end{aligned}$$

Thus, we get the assertion (4.1).

Integrating (4.9) over $[s,t_1]$ for $0 < s \le t_1 \le T$, (4.2) is obtained, since $\hat{\rho}$ is nonnegative. ∎

Proposition 4.2. *There is a positive constant $\overline{C}_1$, independent of T, u_0 and ℓ_0, such that the following inequality holds: For a.e. $t \in [0,T]$,*

$$\begin{aligned}\frac{1}{2C_\rho}|u_t(t)|_H^2 + \frac{d}{dt}E(t,u(t)) + \frac{d}{dt}G_1(t) \\ \le \alpha(t)\left(1+|u(t)|_H\right)\left(E(t,u(t)) + |u(t)|_H^p + B_2 + 1\right) + G_2(t)|u(t)|_H,\end{aligned} \tag{4.10}$$

where

$$\alpha(t) = \overline{C}_1(|\alpha_0'(t)| + |\alpha_0'(t)|^2 + |\alpha_1'(t)|),$$

$$G_1(t) = \int_0^1 (\hat\beta(u)(t) + \hat g(u)(t))dx - (\int_0^{\ell(t)} f_0(t)u(t)dx + \int_{\ell(t)}^1 f_1(t)u(t)dx)$$

and

$$G_2(t) = |f_{0t}(t)|_H + |f_{1t}(t)|_H.$$

Proof. It follows from Proposition 3.1 that

$$\begin{aligned}&\frac{d}{dt}E(t,u(t)) + \frac{1}{C_\rho}|u_t(t)|_H^2 \\ \le\ & |\alpha_0'(t)|(|a(u_x)(t,0+)| + |a(u_x)(t,1-)|)F(t,u(t))^{1/p} \\ &+ |\alpha_1'(t)|F(t,u(t)) - (\xi(t),u_t(t))_H - (g(u)(t),u_t(t))_H \\ &+ (f_0(t),u_t(t))_H + (f_1(t),u_t(t))_H \quad \text{for a.e. } t\in[0,T].\end{aligned} \tag{4.11}$$

We see that, for a.e. $t \in [0,T]$,

$$\begin{aligned}&(\xi(t),u_t(t))_H + (g(u)(t),u_t(t))_H \\ &= \frac{d}{dt}\left(\int_0^1 \hat\beta(u)(t)dx + \int_0^1 \hat g(u)(t)dx\right);\end{aligned} \tag{4.12}$$

$$\begin{aligned}&(f_0(t),u_t(t))_H + (f_1(t),u_t(t))_H \\ &\le \frac{d}{dt}(\int_0^{\ell(t)} f_0(t)u(t)dx + \int_{\ell(t)}^1 f_1(t)u(t)dx) + G_2(t)|u(t)|_H.\end{aligned} \tag{4.13}$$

Also, by (2.6), (2.5) and Lemma 3.3 we have

$$\begin{aligned}&|\alpha_0'(t)|\left(|a(u_x)(t,0+)| + |a(u_x)(t,1-)|\right)F(t,u(t))^{1/p} \\ &\le 2|\alpha_0'(t)|F(t,u(t))^{1/p}\Big(|a(u_x)(t)|_{L^{p'}(0,1)} \\ &\quad + |a(u_x)_x(t)|_{L^2(0,\ell(t))} + |a(u_x)_x(t)|_{L^2(\ell(t),1)}\Big) \\ &\le 4a_2^{1/p'}|\alpha_0'(t)|F(t,u(t))^{1/p}(2E(t,u(t)) + B_2)^{1/p'} \\ &\quad + 4|\alpha_0'(t)|F(t,u(t))^{1/p}(|\xi(t)|_H + C_g|u(t)|_H \\ &\quad + |f_0(t)|_H + |f_1(t)|_H) \\ &\quad + \frac{1}{2C_\rho}|u_t(t)|_H^2 + 8C_\rho^3|\alpha_0'(t)|^2F(t,u(t))^{2/p'}, \qquad \text{a.e. on } [0,T].\end{aligned} \tag{4.14}$$

Therefore, we infer from (4.11) $\sim$ (4.14) together with Lemma 3.3, again, that for a.e. $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2C_\rho}|u_t(t)|_H^2 + \frac{d}{dt}E(t,u(t)) + \frac{d}{dt}\int_0^1 (\hat\beta(u)(t) + \hat g(u)(t))dx \\
\le\ & \alpha(t)\left(E(t,u(t)) + |u(t)|_H^p + B_2\right) \\
&+ \alpha(t)\left(E(t,u(t)) + |u(t)|_H^p + B_2 + 1\right)\left(|u(t)|_H + 1\right) \\
&+ \frac{d}{dt}\int_0^{\ell(t)} f_0(t)u(t)dx + \frac{d}{dt}\int_{\ell(t)}^1 f_1(t)u(t)dx + G_2(t)|u(t)|_H,
\end{aligned}$$

where $\overline{C}_1$ is some suitable positive constant.

Thus the proposition has been proved. ∎

5. Global existence and global attractor

The aim of this section is to prove Theorems 2.1, 2.2 and 2.3. In the rest of this paper we shall use same notation as in the previous sections, too.

Proof of Theorem 2.1. Let $[0,T^*)$ be the maximal interval of existence of a solution $\{u,\ell\}$ of SP. Suppose that $T^* < \infty$. Then, by (4.1) and Proposition 4.2 there is a positive constant M_1 such that $|u(t)|_H \le M_1$ for $t \in [0,T^*)$ and for a.e. $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2C_\rho}|u_t(t)|_H^2 + \frac{d}{dt}E(t,u(t)) + \frac{d}{dt}G_1(t) \\
&\le \alpha(t)(1+M_1)(E(t,u(t)) + M_1^p + B_2) + M_1 G_2(t).
\end{aligned} \tag{5.1}$$

For simplicity, putting

$$\tilde E(t) := E(t,u(t)) + M_1^p + B_2 + 1 \quad \text{and} \quad \tilde\alpha(t) = (1+M_1)\alpha(t),$$

and applying the Gronwall's inequality to (5.1) with the aid of integration by parts, we conclude that for $0 < s_0 < t < T^*$

$$\begin{aligned}
\tilde E(t)\ \le\ & \exp\{\int_{s_0}^t \tilde\alpha(\tau)d\tau\} \times \\
&\times \left\{\tilde E(s_0) + \int_{s_0}^t \{M_1 G_2(\tau) - G_{1\tau}(\tau)\exp(\int_{s_0}^\tau \tilde\alpha(s)ds)\}d\tau\right\} \\
\le\ & \exp(\int_{s_0}^\infty \tilde\alpha(\tau)d\tau)\left\{\tilde E(s_0) + \int_{s_0}^t G_1(\tau)\tilde\alpha(\tau)\exp(\int_{s_0}^\tau \tilde\alpha(s)ds)d\tau\right\} \\
&+ \exp\{\int_{s_0}^\infty \tilde\alpha(\tau)d\tau\} \times \\
&\times \left\{-G_1(t)\exp\{-\int_{s_0}^t \tilde\alpha(\tau)d\tau\} + G_1(s_0) + M_1\int_{s_0}^t G_2(\tau)d\tau\right\}.
\end{aligned}$$

Therefore, from the above inequality together with (3.3) there exists a positive constant $M_2 := M_2(s_0)$ independent of T^* such that

$$\left.\begin{array}{l} |u(t)|_X \le M_2 \\ |u(t)|_{L^\infty(0,1)} \le M_2 \\ |b_i^t(u(t,i))| \le M_2 (i=0,1) \end{array}\right\} \quad \text{for } t \in [s_0, T^*).$$

Furthermore, by assumption (H5) we have

$$\begin{aligned} u(t,x) &= \int_0^x u_y(t,y)dy + u(t,0) \\ &\ge -x^{1/p'}|u_x(t)|_{L^p(0,1)} + d_0 \\ &\ge -x^{1/p'}M_2 + d_0 \quad \text{for } (t,x) \in [s_0, T^*) \times [0,1]. \end{aligned} \tag{5.2}$$

This implies that $u(t,x) > 0$ for $(t,x) \in [s_0, T^*) \times [0, (d_0/M_2)^{p'})$, that is, $\ell(t) > (d_0/M_2)^{p'}$ for $t \in [s_0, T^*)$. Similarly, we have $\ell(t) < 1 - (d_0/M_2)^{p'}$ for $t \in [s_0, T^*)$. Hence, by Theorem 3.1 the solution $\{u, \ell\}$ can be extended beyond time T^*. This is a contradiction. Thus, $T^* = \infty$ is obtained, namely, $\{u, \ell\}$ is a solution of SP on $[0, \infty)$. ∎

Proof of Theorem 2.2. First, (2.2) is a direct consequence of (4.1), that is, for $t \ge 0$ and $[u, \ell] \in U(M, \delta)$

$$|u(t)|_H \le C_1(M\exp(-\mu_1 t) + 1) \le C_1(M_1 + 1) := M_3,$$

where C_1 and μ_1 are positive constants defined in Proposition 4.1.

Let t_0 be any positive number. By (4.2) with $s = 0$ it holds that

$$\int_0^{t_0} E(\tau, u(\tau))d\tau \le M_4 \quad \text{for all } [u, \ell] \in U(M, \delta), \tag{5.3}$$

where M_4 is a positive constant depending only on C_1, M and t_0.

On account of Proposition 4.2 we see that, for a.e. $t \in [0, T]$,

$$\begin{aligned} &\frac{1}{2C_\rho}|u_\tau(\tau)|_H^2 + \frac{d}{d\tau}E(\tau, u(\tau)) + \frac{d}{d\tau}G_1(\tau) \\ &\le \alpha(\tau)\left(1 + |u(\tau)|_H\right)\left(E(\tau, u(\tau))\right. \\ &\quad \left. + |u(\tau)|_H^p + B_2 + 1\right) + G_2(\tau)|u(\tau)|_H. \end{aligned} \tag{5.4}$$

Multiplying (5.4) by τ and integrating it over $[0, t]$, $0 < t \le t_0$, we obtain, for all $t \in [0, t_0]$,

$$\begin{aligned} &\frac{1}{2C_\rho}\int_0^t \tau|u_\tau(\tau)|_H^2 d\tau + tE(t, u(t)) + tG_1(t) \\ \le \; &\int_0^t \tau\alpha(\tau)\left(1 + M_3\right)\left(E(\tau, u(\tau)) + M_3^p + B_2 + 1\right)d\tau \\ &+ M_3\int_0^t G_2(\tau)d\tau + \int_0^t E(\tau, u(\tau))d\tau + \int_0^t G_1(\tau)d\tau. \end{aligned} \tag{5.5}$$

Applying Gronwall's inequality to (5.5), for any $t_0 > 0$ there is a positive constant $M_5(t_0)$ such that

$$\left.\begin{array}{l} E(t_0, u(t_0)) \le M_5(t_0) \\ \int_0^{t_0} \tau |u_\tau(\tau)|_H^2 d\tau \le M_5(t_0) \end{array}\right\} \quad \text{for } [u,\ell] \in U(M,\delta). \tag{5.6}$$

From Lemma 3.3 it follows that (2.4) is valid.

Finally, we shall show (2.3). From a similar calculation to (5.2) it holds that for any $t_0 > 0$

$$\frac{1}{2}\left(\frac{d_0}{M_5(t_0)}\right)^{p'} \le \ell(t) \le 1 - \frac{1}{2}\left(\frac{d_0}{M_5(t_0)}\right)^{p'}$$
$$\text{for } t \ge t_0 \text{ and } [u,\ell] \in U(M,\delta),$$

so that in order to prove (2.3) it is sufficient to get the uniform estimate for free boundary near $t = 0$. Let $t_0 > 0$ and $[u,\ell] \in U(M,\delta)$. Then, there is a positive number $t_1 \le t_0$ (which may depend on $[u_0,\ell_0]$) such that $\frac{\delta}{2} \le \ell \le 1 - \frac{\delta}{2}$ on $[0,t_1]$. Therefore, it follows from Lemma 3.1 that there is a positive constant $M_6 = M_6(\delta)$ such that for $0 \le t \le t_1$

$$\begin{aligned} &\int_0^t \tau^2 |\ell'(\tau)|^{p'+2} d\tau \\ \le\ & M_6(t^{2/p}|\tau^{1/p}u|^{3p-2}_{L^\infty(0,t;X)} \\ &+ |\tau^{1/p}u|^p_{L^\infty(0,t;X)} \int_0^t \tau(|u_\tau|_H^2 + |\xi|_H^2 + |g(u)|_H^2 \\ &+ |f_0|_H^2 + |f_1|_H^2) d\tau). \end{aligned} \tag{5.7}$$

Accordingly,

$$\begin{aligned} |\ell(t) - \ell_0| &\le \int_0^t |\ell'(\tau)| d\tau \\ &= \int_0^t \tau^{-2/(p'+2)} \tau^{2/(p'+2)} |\ell'(\tau)| d\tau \\ &\le \left(\frac{p'+1}{p'-1} t\right)^{\frac{p'+1}{p'+2}} (\int_0^t \tau^2 |\ell'(\tau)|^{p'+2} d\tau)^{1/(p'+2)}. \end{aligned} \tag{5.8}$$

By (5.6) $\sim$ (5.8) there exists a positive number $t_2 \in (0, t_0]$ such that

$$\frac{\delta}{2} \le \ell(t) \le 1 - \frac{\delta}{2} \quad \text{for } t \in [0,t_2] \text{ and } [u,\ell] \in U(M,\delta).$$

Thus, Theorem 2.2 has been proved. ∎

Proof of Theorem 2.3. First, [1, Theorem 5.1] implies the continuity of the operator $S(t)$, $t \ge 0$, and Theorem 2.2 shows that for any $t_0 > 0$, $M > 0$ and $\delta \in (0, 1/2)$ the set $\cup_{t \ge t_0} S(t)\mathcal{B}(M,\delta)$ is relatively compact in V. So, in order to accomplish the proof of Theorem 2.3 it is sufficient to show the existence of an absorbing set because of the general theory in [16, Chapter 1, Theorem 1.1]. Namely, we shall show that there are positive constants M^* and $\delta^* \in (0, 1/2)$ such that for any positive numbers $M > 0$ and $\delta \in (0, 1/2)$ there exists a positive number $T_1 = T_1(M, \delta)$ such that

$$S(t)\mathcal{B}(M,\delta) \subset \mathcal{B}(M^*,\delta^*) \quad \text{for } t \geq T_1. \tag{5.9}$$

Let $[u_0, \ell_0] \in \mathcal{B}(M, \delta)$ and $[u(t), \ell(t)] = S(t)[u_0, \ell_0]$. By Proposition 4.1 we have

$$|u(t)|_H \leq C_1(M\exp(-\mu_1 t) + 1) \quad \text{for } t \geq 0,$$

where C_1 and μ_1 are positive constants independent of M and δ.

Then, there is a positive number T_2 depending only on M and μ_1 such that

$$|u(t)|_H \leq 2C_1 =: M_7 \quad \text{for } t \geq T_2,$$

and hence it follows from (4.2) that

$$\int_t^{t+1} E(\tau, u(\tau))d\tau \leq M_8,$$

where $M_8 = M_8(C_1, \mu_1, \rho, k_0, k_1)$ is a positive constant independent of M and δ. It is clear from Proposition 4.2 that for $t \geq T_2$ and a.e. $\tau \in [t, \infty)$

$$\begin{aligned} &(\tau - t)\frac{d}{d\tau}\hat{E}(\tau) + (\tau - t)\frac{d}{d\tau}G_1(\tau) \\ &\leq (1 + M_7)(\tau - t)\alpha(\tau)\hat{E}(\tau) + M_7 G_2(\tau), \end{aligned} \tag{5.10}$$

where $\hat{E}(\tau) = E(\tau, u(\tau)) + M_7^p + B_2$. Integrating (5.10) over $[t, t+1]$, $t \geq T_2$, we see that

$$\begin{aligned} &\hat{E}(t+1) + G_1(t+1) \\ \leq\ &(1 + M_7)\int_t^{t+1}(\tau - t)\alpha(\tau)\hat{E}(\tau)d\tau + M_7\int_t^{t+1} G_2(\tau)d\tau \\ &+ \int_t^{t+1}\hat{E}(\tau)d\tau + \int_t^{t+1} G_1(\tau)d\tau. \end{aligned}$$

Applying Gronwall's inequality to the above inequality, there is a positive constant M_8 independent of M and δ such that $|u(t)|_X \leq M_8$ for $t \geq T_2 + 1$. By virtue of a similar argument to the proof of Theorem 2.1 we infer that

$$(\frac{d_0}{M_8})^{p'} \leq \ell(t) \leq 1 - (\frac{d_0}{M_8})^{p'} \quad \text{for } t \geq T_2 + 1.$$

Thus, putting $T_1 = T_2 + 1$, $M^* = M_7$ and $\delta^* = (\frac{d_0}{M_8})^{p'}$, we get (5.9). ∎

6. Asymptotic behavior as $t \to \infty$

In order to prove Theorem 2.4 we give the following proposition which is concerned with the asymptotic convergence of solutions of SP.

Lemma 6.1. *Suppose that all the assumptions of Theorem 2.1 hold. Let $M > 0$, $\delta \in (0, 1/2)$, $T_0 > 0$ and $\varepsilon > 0$. Then, there is a positive number s^* satisfying the following condition (*):*

(*) For any $[u_0, \ell_0] \in \mathcal{B}(M, \delta)$ denote by $\{u(\cdot; u_0, \ell_0), \ell(\cdot; u_0, \ell_0)\}$ a solution of $SP(\rho; a; b_0^t, b_1^t; \beta, g, f_0, f_1, u_0, \ell_0)$ on $[0, \infty)$. Then,

$$\sup_{t\in[0,T_0]} dist([u(t+s; u_0, \ell_0), \ell(t+s; u_0, \ell_0)], S(t)[u(s; u_0, \ell_0), \ell(s; u_0, \ell_0)]) < \varepsilon$$
$$\text{for } s \geq s^* \text{ and } [u_0, \ell_0] \in \mathcal{B}(M, \delta).$$

Proof. We suppose that condition (*) does not hold. Then, there are a positive number $\varepsilon_0 > 0$, $s_n \geq n (n = 1, 2, \cdots)$ and $[u_{0n}, \ell_{0n}] \in \mathcal{B}(M, \delta)$ $(n = 1, 2, \cdots)$ such that

$$(6.1) \sup_{t\in[0,T_0]} dist([u(t+s_n; u_{0n}, \ell_{0n}), \ell(t+s_n; u_{0n}, \ell_{0n})], S(t)[u(s_n; u_{0n}, \ell_{0n}), \ell(s_n; u_{0n}, \ell_{0n})]) \geq \varepsilon_0 \quad \text{for each } n = 1, 2, \cdots.$$

Here, in order to avoid surplus confusion for notation put $u_n = u(\cdot + s_n; u_{0n}, \ell_{0n})$, $\ell_n = u(\cdot + s_n; u_{0n}, \ell_{0n})$, $b_{in}^t = b_i^{t+s_n}$, $i = 0, 1$, $f_{in}(t) = f_i(t + s_n)$, $i = 0, 1$, $\hat{u}_{0n} = u(s_n; u_{0n}, \ell_{0n})$ and $\hat{\ell}_{0n} = u(s_n; u_{0n}, \ell_{0n})$. Clearly, $\{u_n, \ell_n\}$ is a solution of
$SP(\rho; a; b_{0n}^t, b_{1n}^t; \beta, g, f_{0n}, f_{1n}, \hat{u}_{0n}, \hat{\ell}_{0n})$ on $[0, T_0]$, $b_{in}^t \to b_i$ on R as $n \to \infty$ in the sense of Mosco for each $t \in [0, T_0]$ and $i = 0, 1$, and $f_{in} \to f_i^*$ in $L^2(0, T_0; H)$ as $n \to \infty$ for $i = 0, 1$.

By Theorem 2.2 and Lemma 3.1 there are positive constants K_1 and δ_1 such that for each n

$$|u_n|_{W^{1,2}(0,T_0;H)} \leq K_1,$$
$$|u_n|_{L^\infty(0,T_0;X)} \leq K_1,$$
$$|b_{in}^{(\cdot)}(u_n(\cdot, i))|_{L^\infty(0,T_0)} \leq K_1 \quad \text{for } i = 0, 1,$$
$$|\ell_n|_{L^{p'+2}(0,T_0)} \leq K_1,$$
$$\delta_1 \leq \ell_n \leq 1 - \delta_1 \quad \text{on } [0, T_0].$$

In particular, we have

$$|\hat{u}_{0n}|_X \leq K_1 \quad \text{and} \quad \delta_1 \leq \hat{\ell}_{0n} \leq 1 - \delta_1 \quad \text{for each } n.$$

Then, without loss of generality we may assume that $\hat{u}_{0n} \to \hat{u}_0$ in H and weakly in X, and $\hat{\ell}_{0n} \to \hat{\ell}_0$ in R for some $\hat{u}_0 \in X$ and $\hat{\ell}_0 \in (0, 1)$. Hence, we can obtain the following convergence in a similar way to [8, Theorem 2.4]:

(6.2) $u_n \to \hat{u}$ in $C([0,T_0];H)$ and $\ell_n \to \hat{\ell}$ in $C([0,T_0])$ as $n \to \infty$,

where $\{\hat{u}, \hat{\ell}\}$ is a solution of $SP(\rho; a; b_0, b_1; \beta, g, f_0^*, f_1^*, \hat{u}_0, \hat{\ell}_0)$ on $[0,T_0]$.

Similarly, by putting $[\tilde{u}_n, \tilde{\ell}_n] = S(\cdot)[u(s_n; u_{0n}, \ell_{0n}), \ell(s_n; u_{0n}, \ell_{0n})]$, we have

$$\tilde{u}_n \to \hat{u} \text{ in } C([0,T_0];H) \text{ and } \tilde{\ell}_n \to \hat{\ell} \text{ in } C([0,T_0]) \text{ as } n \to \infty.$$

Obviously, (6.2) and (6.3) contradict (6.1). Thus, this lemma has been proved. ∎

Proof of Theorem 2.4. Let $M > 0$, $\delta \in (0, 1/2)$ and $\varepsilon > 0$. We denote by $\{u(\cdot; u_0, \ell_0), \ell(\cdot; u_0, \ell_0)\}$ a solution of $SP(\rho; a; b_0^t, b_1^t; \beta, g, f_0, f_1, u_0, \ell_0)$ on $[0, \infty)$. By Propositions 4.1 and 4.2 the set $U_1 := \{[u(t; u_0, \ell_0), \ell(t; u_0, \ell_0)] \in V;\ [u_0, \ell_0] \in \mathcal{B}(M, \delta),\ t \geq 0\}$ is a subset of $\mathcal{B}(M_*, \ell_*)$ for some $M_* > 0$ and $\delta_* \in (0, 1/2)$. Since $\mathcal{A}$ is a global attractor of the semigroup $\{S(t); t \geq 0\}$, there is a positive number T_0 such that

$$dist(S(t)U_1, \mathcal{A}) < \frac{\varepsilon}{2} \quad \text{for } t \geq T_0.$$

This implies that for $t \geq T_0$, $s \geq 0$ and $[u_0, \ell_0] \in \mathcal{B}(M, \delta)$

$$dist(S(t)[u(s; u_0, \ell_0), \ell(s; u_0, \ell_0)], \mathcal{A}) < \frac{\varepsilon}{2}. \tag{6.3}$$

It follows from Lemma 6.1 that there exists a positive constant s^* such that for $s \geq s^*$ and $[u_0, \ell_0] \in \mathcal{B}(M, \delta)$

$$dist([u(s + T_0; u_0, \ell_0), \ell(s + T_0; u_0, \ell_0)], S(T_0)[u(s; u_0, \ell_0), \ell(s; u_0, \ell_0)]) < \frac{\varepsilon}{2}.$$

Hence, on account of (6.3) and the above inequality, we have

$$dist([u(t; u_0, \ell_0), \ell(t; u_0, \ell_0)], \mathcal{A}) < \varepsilon l \text{ for } t \geq T_0 + s^* \text{ and } [u_0, \ell_0] \in \mathcal{B}(M, \delta).$$

This is the conclusion of Theorem 2.4. ∎

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