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Research Article

Continuous-Time Multiobjective Optimization Problems via Invexity

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We introduce some concepts of generalized invexity for the continuous-time multiobjective programming problems, namely, the concepts of Karush-Kuhn-Tucker invexity and Karush-Kuhn-Tucker pseudoinvexity. Using the concept of Karush-Kuhn-Tucker invexity, we study the relationship of the multiobjective problems with some related scalar problems. Further, we show that Karush-Kuhn-Tucker pseudoinvexity is a necessary and suffcient condition for a vector Karush-Kuhn-Tucker solution to be a weakly efficient solution.

1. Introduction

In this work, we regard the continuous-time multiobjective optimization problem,

$$\begin{aligned} &\text{Minimize } \phi(x) = \left(\int_0^T f_1(x(t),t)\,dt,\dots,\int_0^T f_p(x(t),t)\,dt\right) \\ &\text{subject to } g_i(x(t),t) \le 0 \quad \text{a.e. in } [0,T],\ i=1,\dots,m,\ x\in X. \end{aligned} \tag{CMP}$$

Here X is a nonempty open convex subset of the Banach space $L_\infty^n[0,T]$, $\phi : X \to \mathbb{R}^p$, $f_j(x(t),t) = \xi_j(x)(t)$, $j \in J := \{1,\dots,p\}$, $g_i(x(t),t) = \gamma_i(x)(t)$, $i \in I := \{1,\dots,m\}$, where $\xi_j : X \to \Lambda_1^1[0,T]$, $j \in J$, and $\gamma_i : X \to \Lambda_1^1[0,T]$, $i \in I$. $L_\infty^n[0,T]$ denotes the space of all n-dimensional vector-valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0,T] \subset \mathbb{R}$, with norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1\le j\le n} \operatorname{ess\,sup}\{|x_j(t)|,\ 0 \le t \le T\}, \tag{1.1}$$

where for each $t \in [0,T]$, $x_j(t)$ is the jth component of $x(t) \in \mathbb{R}^n$ and $\Lambda_1^m[0,T]$ denotes the space of all m-dimensional vector-valued functions which are essentially bounded and Lebesgue measurable, defined on $[0,T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1\le j\le m} \int_0^T |y_j(t)|\, dt. \tag{1.2}$$

The mono-objective version of this class of problems was introduced by Bellman [1] in connection with production-inventory "bottleneck processes." He considered a type of optimization problems, which is now known as continuous-time linear programming, formulated its dual and provided duality relations. He also suggested some computational procedures.

Since then, various authors have extended his theory to wider classes of continuous-time problems (e.g., [2–12]). In these articles, the authors study the mono-objective case, but in many applications it is necessary to minimize not only one objective. So, the multiobjective problem is more general and more suitable for many applications. The development of the necessary and sufficient optimality conditions for (CMP) was done in [13].

Our aim in this paper is to provide necessary and sufficient conditions for global optimality of a vector Karush-Kuhn-Tucker solution as well for a vector Karush-Kuhn-Tucker solution to solve a related scalar problem. Our results extend the finite dimensional case studied in [14, 15] and the continuous-time mono-objective case studied in [16].

For more literature about these issues, we refer the reader to [14–16] and the bibliography cited therein.

We organized this work into four sections. In Section 2, we give some preliminaries. We introduce KKT pseudoinvexity for (CMP) and state our first mean result in Section 3. The notion of KKT invexity and our second mean result are given in Section 4.

2. Preliminaries

Let V be an open subset of $\mathbb{R}^n$ containing the set $\{x(t) \in \mathbb{R}^n : x \in X,\ t \in [0,T]\}$. We assume that f_j, $j \in J$, and g_i, $i \in I$, are real functions defined on $V \times [0,T]$. The functions $t \mapsto f_j(x(t),t)$, $j \in J$, and $t \mapsto g_i(x(t),t)$, $i \in I$, are assumed to be Lebesgue measurable and integrable for all $x \in X$. We assume also that the functions f_j, $j \in J$, and g_i, $i \in I$, are continuously differentiable with respect to their first arguments.

Let $\mathbb{F}$ be the set of all feasible solutions of problem (CMP) (which we suppose non empty), that is, let

$$\mathbb{F} = \{x \in X : g_i(x(t),t) \le 0 \text{ a.e. in } [0,T],\ i \in I\}. \tag{2.1}$$

Given $x \in \mathbb{F}$, for each $i \in I$ we denote by $A_i(x)$ the subset of $[0,T]$ where the ith constraint is active, that is,

$$A_i(x) = \{t \in [0,T] : g_i(x(t),t) = 0\}. \tag{2.2}$$

In this paper, all vectors are collum vectors. We use a prime to denote transposition. Besides, given $w \in \mathbb{R}^p$, $w \leq 0$ means that $w_i \leq 0$ for $i = 1,2,\dots,p$, and $w < 0$ means that $w_i < 0$ for $i = 1,2,\dots,p$.

In what follows, we state a result which will be useful for the proof of our results. This result can be viewed as a generalized Motzkin theorem of the alternative. It is the continuous-time analogue of the theorem given by Mangasarian [17, page 66] and its proof is almost identical to the one given in the Mangasarian's book.

Theorem 2.1. *Let $Z \subseteq L_\infty^n[0,T]$ be a nonempty convex subset. Let $p : W \times [0,T] \to \mathbb{R}^m$ and $q : W \times [0,T] \to \mathbb{R}^k$ be mappings given by $p(z(t),t) = \pi(z)(t)$ and $q(z(t),t) = B(t)z(t) - b(t)$, respectively, where $W \subseteq \mathbb{R}^n$ is an open subset, π is a mapping from Z into $\Lambda_1^m[0,T]$, $B(t)$ is a $k \times n$ matrix, and $b(t) \in \mathbb{R}^k$. Assume that p is convex with respect to its first argument in W throughout $[0,T]$ and that there does not exist $v \in L_\infty^k[0,T] \setminus \{0\}$, $v(t) \geq 0$ a.e. in $[0,T]$ such that*

$$B'(t)v(t) = 0 \quad \text{a.e. in } [0,T]. \tag{2.3}$$

Then exactly one of the following systems is consistent:

(I) $p(z(t),t) < 0$, $B(t)z(t) \leq b(t)$ *a.e. in* $[0,T]$ *has a solution* $z \in Z$;

(II) $\int_0^T \{u'(t)p(z(t),t) + v'(t)[B(t)z(t) - b(t)]\}dt \geq 0$ *for all* $z \in Z$, *for some* $u \in L_\infty^m[0,T]$, $u(t) \geq 0$, $u(t) \neq 0$ *a.e. in* $[0,T]$ *and for some* $v \in L_\infty^k[0,T]$, $v(t) \geq 0$ *a.e. in* $[0,T]$.

Proof. Quite similar to the proof of Theorem 3.4 in [10, page 137]. □

In the two following definitions, we define as in [13] a weakly efficient solution and a vector Karush-Kuhn-Tucker solution.

Definition 2.2. A feasible solution y is said to be a *weakly efficient solution* of (CMP) if and only if there does not exist another feasible solution x such that $\phi(x) < \phi(y)$.

Definition 2.3. A feasible solution y is said to be a *vector Karush-Kuhn-Tucker solution (or vector KKT solution)* for problem (CMP) if there exist $\mu \in \mathbb{R}^p$ and $\lambda \in L_\infty^m[0,T]$ such that

$$\int_0^T \left[\sum_{j \in J} \mu_j \nabla f_j'(y(t),t) + \sum_{i \in I} \lambda_i(t) \nabla g_i'(y(t),t) \right] h(t)dt = 0 \quad \forall h \in L_\infty^n[0,T], \tag{2.4}$$

$$\lambda_i(t) g_i(y(t),t) = 0 \quad \text{a.e. in } [0,T],\ i \in I, \tag{2.5}$$

$$\lambda_i(t) \geq 0 \quad \text{a.e. in } [0,T],\ i \in I, \tag{2.6}$$

$$\mu_j \geq 0, \quad j \in J,\ \mu \neq 0. \tag{2.7}$$

At last we give a constraint qualification in the continuous-time setting.

Definition 2.4. The constraints g_i, $i \in I$, satisfy *(CQ)* at $y \in \mathbb{F}$ if there do not exist $v_i \in L_\infty[0,T]$, $v_i(t) \geq 0$ a.e. in $[0,T]$, $i \in I$, not all zero, such that

$$\sum_{i\in I}\int_{A_i(y)} v_i(t)\nabla g_i(y(t),t)h(t)dt \geq 0 \quad \forall h \in L_\infty^n[0,T]. \tag{2.8}$$

3. KKT-pseudoinvexity and optimality conditions

In this section, we introduce the notion of Karush-Kuhn-Tucker pseudoinvexity for (CMP). Further, we state and prove a result which provides necessary and sufficient conditions for global optimality of a vector Karush-Kuhn-Tucker solution.

Definition 3.1. The problem (CMP) is said to be *Karush-Kuhn-Tucker pseudoinvex (or KKT-pseudoinvex)* if there exists a function $\eta : V \times V \times [0,T] \to \mathbb{R}^n$ such that $t \mapsto \eta(x(t), y(t),t) \in L_\infty^n[0,T]$, and

$$\phi(x) < \phi(y) \Longrightarrow \int_0^T \nabla f_j'(y(t),t)\eta(x(t),y(t),t)dt < 0, \quad j \in J, \tag{3.1}$$

$$-\nabla g_i'(y(t),t)\eta(x(t),y(t),t) \geq 0 \quad \text{a.e. in } A_i(y),\ i \in I, \tag{3.2}$$

for all $x, y \in \mathbb{F}$.

THEOREM 3.2. *Assume that the constraints g_i, $i \in I$, satisfy (CQ) at each $y \in \mathbb{F}$. Then every vector KKT-solution is a weak efficient solution of (CMP) if and only if (CMP) is KKT-pseudoinvex.*

Proof. Let y be a vector KKT-solution and suppose that (CMP) is KKT-pseudoinvex. Suppose that there exists a feasible solution x such that $\phi(x) < \phi(y)$. As (CMP) is KKT-pseudoinvex, using (3.1), we obtain

$$\int_0^T \nabla f_j'(y(t),t)\eta(x(t),y(t),t)dt < 0, \quad j \in J. \tag{3.3}$$

Since y is a vector KKT-solution, there exist $\mu \in \mathbb{R}^p$ and $\lambda \in L_\infty^m[0,T]$ satisfying (2.4)–(2.7). By (2.7) and (3.3), we have

$$\int_0^T \sum_{j\in J}\mu_j \nabla f_j'(y(t),t)\eta(x(t),y(t),t)dt < 0. \tag{3.4}$$

Using (2.4) with $h(t) = \eta(x(t),y(t),t)$, $t \in [0,T]$, we obtain

$$\int_0^T \sum_{i\in I}\lambda_i(t)\nabla g_i'(y(t),t)\eta(x(t),y(t),t)dt > 0. \tag{3.5}$$

By the other hand, from (2.6) and (3.2), since by (2.5) $\lambda_i(t) = 0$, $t \notin A_i(y)$, $i \in I$, it follows that

$$\int_0^T \sum_{i\in I} \lambda_i(t) \nabla g_i'(y(t),t)\eta(x(t),y(t),t)\,dt \le 0, \tag{3.6}$$

which is a contradiction to (3.5). Therefore y is a weakly efficient solution.

Conversely, suppose that every vector KKT-solution is a weakly efficient solution. Let $x, y \in \mathbb{F}$ be such that $\phi(x) < \phi(y)$. Then y is not a weakly efficient solution, so that, by hypothesis, y is not a vector Karush-Kuhn-Tucker solution. So the system

$$\begin{gathered}\int_0^T \left[\sum_{j\in J} \mu_j \nabla f_j'(y(t),t) + \sum_{i\in I} \lambda_i(t) \nabla g_i'(y(t),t)\right] h(t)\,dt = 0 \quad \forall h \in L_\infty^n[0,T],\\ \lambda_i(t) g_i(y(t),t) = 0 \quad \text{a.e. in } [0,T],\ i \in I,\\ \lambda_i(t) \ge 0 \quad \text{a.e. in } [0,T],\ i \in I,\\ \mu_j \ge 0, \quad j \in J,\ \mu \ne 0,\end{gathered} \tag{3.7}$$

has no solution $(\mu,\lambda) \in \mathbb{R}^p \times L_\infty^m[0,T]$. Equivalently, the system

$$\begin{gathered}\int_0^T \left[\sum_{j\in J} \mu_j \nabla f_j'(y(t),t) + \sum_{i\in I} \lambda_i(t)\chi_i(t) \nabla g_i'(y(t),t)\right] h(t)\,dt = 0 \quad \forall h \in L_\infty^n[0,T],\\ \lambda_i(t) \ge 0 \quad \text{a.e. in } [0,T],\ i \in I,\\ \mu_j \ge 0, \quad j \in J,\ \mu \ne 0,\end{gathered} \tag{3.8}$$

has no solution $(\mu,\lambda) \in \mathbb{R}^p \times L_\infty^m[0,T]$, where $\chi_i : [0,T] \to \mathbb{R}$ is defined, for each $i \in I$, by

$$\chi_i(t) = \begin{cases} 1 & \text{if } t \in A_i(y), \\ 0 & \text{if } t \notin A_i(y). \end{cases} \tag{3.9}$$

As the constraint qualification holds by hypothesis, the condition (2.3) in Theorem 2.1 is verified. Applying that theorem, it follows that there exists $h \in L_\infty^n[0,T]$ such that

$$\begin{gathered}\int_0^T \nabla f_j'(y(t),t) h(t)\,dt < 0, \quad j \in J,\\ \chi_i(t) \nabla g_i'(y(t),t) h(t) \le 0 \quad \text{a.e. in } [0,T],\ i \in I.\end{gathered} \tag{3.10}$$

Define $\eta(x(t),y(t),t) = h(t)$ a.e. in $[0,T]$. Therefore

$$\begin{gathered}\int_0^T \nabla f_j'(y(t),t)\eta(x(t),y(t),t)\,dt < 0, \quad j \in J,\\ -\nabla g_i'(y(t),t)\eta(x(t),y(t),t) \ge 0 \quad \text{a.e. in } A_i(y),\ i \in I.\end{gathered} \tag{3.11}$$

Thus there exists a function $\eta: V \times V \times [0,T] \to \mathbb{R}^n$ such that $t \mapsto \eta(x(t),y(t),t) \in L_\infty^n[0,T]$ and

$$\begin{aligned} \phi(x) < \phi(y) \Longrightarrow \int_0^T \nabla f_j'(y(t),t)\eta(x(t),y(t),t)\,dt < 0, \quad j \in J, \\ -\nabla g_i'(y(t),t)\eta(x(t),y(t),t) \geq 0 \quad \text{a.e. in } A_i(y),\ i \in I, \end{aligned} \tag{3.12}$$

for all $x, y \in \mathbb{F}$, so that (CMP) is KKT-pseudoinvex. □

4. Karush-Kuhn-Tucker invexity and scalar problems

In this section, we generalize to the continuous-time context the notion of Karush-Kuhn-Tucker invexity introduced in [15] for finite-dimensional multiobjective problems. In addition, we give necessary and sufficient conditions for a vector KKT-solution to solve a related weighting scalar problem.

We will regard the following weighting scalar problem related with (CMP):

$$\begin{aligned} &\text{Minimize } \Phi(x) = \int_0^T \sum_{j\in J} \mu_j f_j(x(t),t)\,dt \\ &\text{subject to } g_i(x(t),t) \leq 0 \quad \text{a.e. in } [0,T],\ i \in I,\ x \in X, \end{aligned} \tag{4.1}$$

where $\mu_j \in \mathbb{R}$, $j \in J$. This is one of the most known scalar problems associated with multiobjective optimization problems.

Theorem 4.1. *Every optimal solution of a weighting scalar problem with $\mu_j \geq 0$, $j \in J$, not all zero, is a weakly efficient solution of (CMP).*

Proof. Let y be an optimal solution of a scalar problem with $\mu_j \geq 0$, $j \in J$, not all zero, and let us suppose that there exists $x \in \mathbb{F}$ such that $\phi(x) < \phi(y)$. Then $\phi_j(x) < \phi_j(y)$, $j \in J$, so that

$$\mu_j\phi_j(x) \leq \mu_j\phi_j(y), \quad j \in J, \tag{4.2}$$

since $\mu_j \geq 0$, $j \in J$. Provided μ_j, $j \in J$, are not all zero, there exists at least one $j \in J$ such that $\mu_j > 0$. Therefore, in the inequalities above, at least one holds strictly. So summing over J,

$$\sum_{j\in J}\mu_j\phi_j(x) < \sum_{j\in J}\mu_j\phi_j(y), \tag{4.3}$$

which contradicts the optimality of y. □

In order to establish the reciprocal of Theorem 4.1, we need some qualifications on the constraints and, furthermore, we need some generalized convexity hypothesis.

Definition 4.2. The problem (CMP) is said to be *Karush-Kuhn-Tucker invex (or KKT-invex)* if there exists a function $\eta : V \times V \times [0,T] \to \mathbb{R}^n$ such that $t \mapsto \eta(x(t),y(t),t) \in L_\infty^n[0,T]$ and

$$\begin{gathered}\phi_j(x)-\phi_j(y) \geq \int_0^T \nabla f_j'(y(t),t)\eta(x(t),y(t),t)\,dt, \quad j \in J, \\ -\nabla g_i'(y(t),t)\eta(x(t),y(t),t) \geq 0 \quad \text{a.e. in } A_i(y),\ i \in I,\end{gathered} \tag{4.4}$$

for all $x, y \in \mathbb{F}$.

Theorem 4.3. *Assume that the constraints g_i, $i \in I$, satisfy (CQ) at each $y \in \mathbb{F}$. If (CMP) is KKT-invex, then every weakly efficient solution solves a weighting scalar problem with $\mu_j \geq 0$, $j \in J$, not all zero.*

Proof. Let y be a weakly efficient solution of (CMP). Then, by [13, Theorem 3.3, page 9], there exist $\mu \in \mathbb{R}^p$ and $\lambda \in L_\infty^m[0,T]$ satisfying

$$\begin{gathered}\int_0^T \left[\sum_{j\in J}\mu_j \nabla f_j'(y(t),t) + \sum_{i\in I}\lambda_i(t)\nabla g_i'(y(t),t)\right]h(t)\,dt = 0 \quad \forall h \in L_\infty^n[0,T], \\ \lambda_i(t)g_i(y(t),t) = 0 \quad \text{a.e. in } [0,T],\ i \in I, \\ \mu_j \geq 0, \quad j \in J, \quad \lambda_i(t) \geq 0 \quad \text{a.e. in } [0,T],\ i \in I, \\ (\mu,\lambda(t)) \neq 0 \quad \text{a.e. in } [0,T].\end{gathered} \tag{4.5}$$

If $\mu = 0$, then $\lambda(t) \neq 0$ a.e in $[0,T]$, and

$$\int_0^T \left[\sum_{i\in I}\lambda_i(t)\nabla g_i'(y(t),t)\right]h(t)\,dt = 0 \quad \forall h \in L_\infty^n[0,T]. \tag{4.6}$$

From $\lambda_i(t)g_i(y(t),t) = 0$, a.e. in $[0,T]$, $i \in I$, it follows that $\lambda_i(t) = 0$, $t \notin A_i(y)$. So we have

$$\sum_{i\in I}\int_{A_i(y)} \lambda_i(t)\nabla g_i'(y(t),t)h(t)\,dt = 0 \quad \forall h \in L_\infty^n[0,T], \tag{4.7}$$

with $\lambda_i(t) \geq 0$ a.e. in $[0,T]$, $i \in I$, not all zero, which contradicts the hypothesis that (CQ) holds at y. Thus y is a vector KKT-solution, that is, there exist $\mu \in \mathbb{R}^p$ and $\lambda \in L_\infty^m[0,T]$ satisfying (2.4)–(2.7).

As by (2.7) $\mu_j \geq 0$, $j \in J$, and by (2.6) $\lambda_i(t) \geq 0$ a.e. in $[0,T]$, $i \in I$, using (4.4) we obtain

$$\int_0^T \sum_{j\in J}\mu_j[f_j(x(t),t) - f_j(y(t),t)]\,dt \geq \int_0^T \sum_{j\in J}\mu_j \nabla f_j'(y(t),t)\eta(x(t),y(t),t)\,dt, \tag{4.8}$$

$$-\lambda_i(t)\nabla g_i'(y(t),t)\eta(x(t),y(t),t) \geq 0 \quad \text{a.e. in } A_i(y),\ i \in I, \tag{4.9}$$

for all $x \in \mathbb{F}$. Remembering that $\lambda_i(t) = 0$, $t \notin A_i(y)$, $i \in I$, integrating the inequalities in (4.9) over $[0, T]$ and summing over I, we obtain

$$-\int_0^T \sum_{i \in I} \lambda_i(t) \nabla g_i'(y(t), t) \eta(x(t), y(t), t)\, dt \geq 0. \tag{4.10}$$

From (4.8) and (4.10), it follows that

$$\begin{aligned}
&\int_0^T \sum_{j \in J} \mu_j [f_j(x(t), t) - f_j(y(t), t)]\, dt \\
&\quad \geq \int_0^T \left[\sum_{j \in J} \mu_j \nabla f_j'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g_i'(y(t), t)\right] \eta(x(t), y(t), t)\, dt,
\end{aligned} \tag{4.11}$$

for all $x \in \mathbb{F}$. Setting $h(t) = \eta(x(t), y(t), t)$ a.e. in $[0, T]$, it follows from (2.4) that the integral in the second line above is null. Thus y solves a weighting scalar problem with $\mu_j \geq 0$, $j \in J$, not all zero. □

Theorem 4.4. *If (CMP) is KKT-invex, then every vector KKT-solution solves a weighting scalar problem with $\mu_j \geq 0$, $j \in J$, not all zero.*

Proof. Similar to the proof of Theorem 4.3. □

The theorems above show us that under the assumptions that (CMP) is KKT-invex and the constraints satisfy (CQ) at each $y \in \mathbb{F}$, the sets of vector KKT-solutions, weakly efficient solutions, and optimal solutions of weighting scalar problems are equal.

Under the hypothesis that the constraints satisfy (CQ), we can establish a reciprocal of Theorem 4.4.

Theorem 4.5. *Assume that the constraints g_i, $i \in I$, satisfy (CQ) at each $y \in \mathbb{F}$. Then every vector KKT-solution solves a weighting scalar problem if and only if (CMP) is KKT-invex.*

Proof. The sufficiency part was proved in Theorem 4.4. Let us proceed to the necessity part.

Assume that every vector KKT-solution solves a weighting scalar problem. Then the system

$$\begin{gathered}
\int_0^T \left[\sum_{j \in J} \mu_j \nabla f_j'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g_i(y(t), t)\right] h(t)\, dt = 0 \quad \forall h \in L_\infty^n[0, T], \\
\lambda_i(t) g_i(y(t), t) = 0 \quad \text{a.e. in } [0, T],\ i \in I, \\
\lambda_i(t) \geq 0 \quad \text{a.e. in } [0, T],\ i \in I, \\
\mu_j \geq 0, \quad j \in J,\ \mu \neq 0, \\
\int_0^T \sum_{j \in J} \mu_j [f_j(x(t), t) - f_j(y(t), t)]\, dt < 0,
\end{gathered} \tag{4.12}$$

has no solution $(\mu,\lambda) \in \mathbb{R}^p \times L_\infty^m[0,T]$ for any $x, y \in \mathbb{F}$. Equivalently, the system

$$\int_0^T \left[\sum_{j\in J} \mu_j \nabla f_j'(y(t),t) + \sum_{i\in I} \lambda_i(t)\chi_i(t)\nabla g_i(y(t),t) \right] h(t)dt = 0 \quad \forall h \in L_\infty^n[0,T],$$
$$\lambda_i(t) \geq 0 \quad \text{a.e. in } [0,T],\ i \in I,$$
$$\mu_j \geq 0, \quad j \in J,\ \mu \neq 0, \tag{4.13}$$
$$\int_0^T \sum_{j\in J} \mu_j [f_j(x(t),t) - f_j(y(t),t)]dt < 0,$$

has no solution $(\mu,\lambda) \in \mathbb{R}^p \times L_\infty^m[0,T]$ for any $x, y \in \mathbb{F}$, where χ_i is defined as in the proof of Theorem 3.2. It is easy to see that this implies that the system

$$\int_0^T \begin{bmatrix} \nu & \mu_1 & \cdots & \mu_p \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \nabla f_1'(y(t),t) & f_1(x(t),t) - f_1(y(t),t) \\ \vdots & \vdots \\ \nabla f_p'(y(t),t) & f_p(x(t),t) - f_p(y(t),t) \end{bmatrix} \begin{bmatrix} h(t) \\ \alpha \end{bmatrix} dt$$
$$+ \int_0^T \begin{bmatrix} \lambda_1(t) & \cdots & \lambda_m(t) \end{bmatrix} \begin{bmatrix} \chi_1(t)\nabla g_1'(y(t),t) & 0 \\ \vdots & \vdots \\ \chi_m(t)\nabla g_m'(y(t),t) & 0 \end{bmatrix} \begin{bmatrix} h(t) \\ \alpha \end{bmatrix} dt = 0 \tag{4.14}$$
$$\forall h \in L_\infty^n[0,T], \quad \forall \alpha \in \mathbb{R},$$
$$\nu > 0, \quad \mu_j \geq 0, \quad j \in J, \quad \mu \neq 0, \quad \lambda_i(t) \geq 0 \quad \text{a.e. in } [0,T],\ i \in I,$$

has no solution $(\nu,\mu,\lambda) \in \mathbb{R}^{p+1} \times L_\infty^m[0,T]$ for any $x, y \in \mathbb{F}$. Since (CQ) holds by hypothesis, the condition (2.3) in Theorem 2.1 is satisfied. Therefore, applying that theorem, it follows that there exist $\alpha \in \mathbb{R}$ and $h \in L_\infty^n[0,T]$ such that

$$\int_0^T \begin{bmatrix} 0 & 1 \\ \nabla f_1'(y(t),t) & f_1(x(t),t) - f_1(y(t),t) \\ \vdots & \vdots \\ \nabla f_p'(y(t),t) & f_p(x(t),t) - f_p(y(t),t) \end{bmatrix} \begin{bmatrix} h(t) \\ \alpha \end{bmatrix} dt < 0, \tag{4.15}$$
$$\chi_i(t)\nabla g_i'(y(t),t)h(t) \leq 0 \quad \text{a.e. in } [0,T],\ i \in I,$$

for any $x, y \in \mathbb{F}$. Then $\alpha < 0$ and we can set $\alpha = -1$. Defining

$$\eta(x(t),y(t),t) = h(t) \quad \text{a.e. in } [0,T], \tag{4.16}$$

we have

$$\int_0^T \nabla f_j'(y(t),t)\eta(x(t),y(t),t)\,dt < \int_0^T \left[f_j(x(t),t) - f_j(y(t),t)\right]dt, \quad j \in J,$$
$$\nabla g_i'(y(t),t)\eta(x(t),y(t),t) \le 0 \quad \text{a.e. in } A_i(y),\ i \in I. \tag{4.17}$$

Therefore (CMP) is KKT-invex. □

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