LOCAL SOLVABILITY OF A CONSTRAINED GRADIENT SYSTEM OF TOTAL VARIATION

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A 1-harmonic map flow equation, a gradient system of total variation where values of unknowns are constrained in a compact manifold in $\mathbb{R}^N$, is formulated by the use of subdifferentials of a singular energy—the total variation. An abstract convergence result is established to show that solutions of approximate problem converge to a solution of the limit problem. As an application of our convergence result, a local-in-time solution of 1-harmonic map flow equation is constructed as a limit of the solutions of p-harmonic ($p > 1$) map flow equation, when the initial data is smooth with small total variation under periodic boundary condition.

1. Introduction

We consider a gradient system of total variation of mappings with constraint of their values. We are interested in the solvability of its initial value problem.

To see the difficulty, we write the equation at least formally. For a mapping $u : \Omega \to \mathbb{R}^N$, let $E_p(u)$ denote its energy:

$$E_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$ and $p \geq 1$. The energy E_1 is the total variation of u. Let M be a smoothly embedded compact submanifold (without boundary) of $\mathbb{R}^N$. Then the gradient system for $u : \Omega \times (0,T) \to \mathbb{R}^N$ of E_p with constraint of values in M is of the form

$$u_t(x,t) = -\pi_{u(x,t)}\big(- \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)(x,t)\big); \tag{1.2}$$

here, π_v denotes the orthogonal projection of $\mathbb{R}^N$ to the tangent space T_vM of M at $v \in M$ and $u_t = \partial u/\partial t$. This equation is called the p-harmonic map flow equation since the case $p = 2$ is called the harmonic map flow equation. Because of π, the values of a solution of (1.2) are constrained in M if they are in M initially. If M is a unit sphere S^{N-1}, then the

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explicit form of (1.2) is of the form

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |\nabla u|^p u \tag{1.3}$$

since $\pi_v(w) = w - \langle w, v\rangle v$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^N$. An explicit form for (1.2) is given, for example, in [23]. Our constrained gradient system of total variation of mapping is the 1-harmonic flow of the form (1.2) for $p = 1$, that is,

$$u_t = -\pi_u\left(-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right). \tag{1.4}$$

This equation has a strong singularity at $\nabla u = 0$ so that the evolution speed is expected to be determined by a nonlocal quantity. Even if one considers the corresponding unconstrained problem

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right), \tag{1.5}$$

the speed where u is constant is determined by a nonlocal quantity (like the length of spatial interval where u is a constant when $n = 1$) [13, 14, 19]. The equation (1.5) is a nonlocal diffusion equation, so even the notion of a solution is a priori not clear. Fortunately, for (1.5), a general nonlinear semigroup theory (initiated by Kōmura [21]) applies under appropriate boundary conditions since the energy is convex. The theory yields the unique global solvability of the initial value problem for (1.5) under Dirichlet boundary condition (see, e.g., [6, 8] and also [13, 17, 19]), for a recent L^1-theory, see [1, 2, 3, 7]. However, for (1.4), such a theory does not apply since it cannot be viewed as a gradient system of a convex functional. For a scalar function, a more general form of (1.4) without gradient structure is studied when $n = 1$ by extending the notion of viscosity solution [11, 12]. However, such a theory does not apply since (1.4) has no pointwise order-preserving structure. For other examples of singular diffusion equations with nonlocal effects, the reader is referred to a recent review article [14].

Our goal is to give a suitable notion of a solution of (1.4) and to solve its initial value problem under suitable boundary condition. We formulate (1.4) with Dirichlet boundary condition and periodic boundary condition by using the subdifferential of energy, which is an extended notion of differentials for nonsmooth functional like E_1. A similar formulation is given in a recent work in [15]. In fact, they constructed a global solution for any piecewise constant initial data, when $n = 1$, $N = 2$, and $M = S^1$, under Dirichlet boundary condition. They also studied its behavior and provided a numerical simulation. However, their analysis is limited to one-dimensional piecewise constant mappings although their formulation of the problem is general. Our formulation is close to theirs, but is slightly different since we use the subdifferential of space-time functional $\int_0^T E_1(u)dt$ instead of E_1 itself.

To solve (1.4), we prepare an abstract convergence result. Roughly speaking, it asserts that if a sequence of approximate energy converges to our energy in the sense of Mosco, the corresponding sequence of the solutions of the approximate problem converges to our original problem. (For this purpose, the interpretation of $-\operatorname{div}(\nabla u/|\nabla u|)$ by a subdifferential of $\int_0^T E_1(u)dt$ is convenient.) We use this abstract result by approximating E_1 by

E_p $(1 < p < 2)$. Compared with the harmonic map flow equation, less is known for (1.2) for $p \in (1,2)$. Misawa [24] proved the global existence of weak solution of the initial value problem with a Dirichlet boundary condition when $M = S^{N-1}$. However, his existence result is not enough to apply our abstract theory since it is not clear that $\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p)$ is in $L^2(\Omega\times(0,T))$ for his solution u_p of (1.2). Our formulation unfortunately requires such a structure. Moreover, we need the condition that $\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p)$ is bounded in $L^2(\Omega\times(0,T))$ as $p\downarrow 1$ to apply our existence theorem. Recently, Fardoun and Regbaoui [9] constructed a unique global weak solution for a general target manifold when Ω is a compact manifold without boundary for smooth initial data of small E_p energy. Since we need to establish a bound of $\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) \in L^2(\Omega\times(0,T))$, we estimate the Lipschitz norm. Fortunately, we establish a uniform spatially Lipschitz bound for u_p in a small time interval, and we are able to prove the local solvability of (1.4) under a periodic boundary condition when initial data is smooth with small total variation. The constructed solution is spatially Lipschitz-continuous. Of course, since the results in [9] are for a general source manifold, our results easily extend to such a general manifold by interpreting the gradient in an appropriate way. If u has a jump, the dynamics given by (1.4) depends not only on the metric of M but also on the metric of ambient space $\mathbb{R}^N$ outside M. This is a serious difference between 1-harmonic flow equation and (1.2) for $p > 1$. Fortunately, our solution does not depend on that quantity since it has no jumps. We note that the notion of BV for mapping in M is not clear as pointed out in [10].

Problem (1.4) for the case $n = 2$ and $M = S^{N-1}$ is proposed in [27] in image processing. If we let $I(x,y,0):\Omega\to\mathbb{R}^N$ represent the color data whose components stand for the brightness of each color pixel of the image at $(x,y)\in\Omega$, then its pixel's chromaticity $u(x,y,0):\Omega\to S^{N-1}$ is expressed by the normalized vector $u(x,y,0) := I(x,y,0)/|I(x,y,0)|$. System (1.4) for the scaled chromaticity $u(x,y,t)$ describes the process to remove the noise from original $u(x,y,0)$ maintaining the unit norm constraint and preserving chroma discontinuities. See [25] for background of our problem (1.4) and other PDEs from image processing. This type of constrained problems also naturally arises in the modeling of multigrain boundaries [20] where u represents a direction of grains embedded in a larger crystal of fixed orientation in the two-dimensional frame.

We will formulate (1.4) by using the notion of subdifferential in Section 2. In Section 3, we will state three main theorems, which are as follows: an abstract theorem providing the framework of our convergence results, convergence theorem obtained by applying abstract theorem, and local existence theorem following from convergence theorem by applying the result of [9]. From Section 4 to Section 6, we will prove these main theorems. In addition, we will prove some properties of general convex functionals, which are used to show convergence theorem in the appendix.

2. Formulation of the problems

In this Section, we formulate the initial value problem with periodic boundary condition:

$$\begin{aligned} u_t &= -\pi_u\left(-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) \quad \text{in } \mathbb{T}^n\times(0,T], \\ u &= u_0 \quad \text{on } \mathbb{T}^n\times\{0\}, \end{aligned} \tag{2.1}$$

where $\mathbb{T}^n := \prod_{i=1}^n (\mathbb{R}/\omega_i\mathbb{Z})$ for given $\omega_i > 0$ ($i = 1,2,\ldots,n$) and the given initial data u_0 is a map from $\mathbb{T}^n$ to M. We also formulate the initial boundary value problem

$$\begin{aligned} u_t &= -\pi_u\left(-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) \quad \text{in } \Omega \times (0,T], \\ u &= u_0 \quad \text{on } \partial\Omega \times [0,T] \cup \Omega \times \{0\}, \end{aligned} \tag{2.2}$$

where Ω denotes a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ and the initial data $u_0 : \bar{\Omega} \to M$ is Lipschitz-continuous.

We formulate (2.1) and (2.2) as evolution equations on L^2-space. Since some notations are different for each case, we state the formulation of each problem individually. Let M denote a smoothly embedded compact manifold in $\mathbb{R}^N$ and let π_v denote the orthogonal projection from $\mathbb{R}^N$ to the tangent space T_vM of M at $v \in M$. Note that the inner product of $L^2(\Omega,\mathbb{R}^N)$ is defined by $\langle f,g\rangle_{L^2(\Omega,\mathbb{R}^N)} := \int_\Omega \langle f,g\rangle dx$, where $\langle\cdot,\cdot\rangle$ represents the standard inner product of $\mathbb{R}^N$. The inner product of $L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ is also defined by $\langle f,g\rangle_{L^2(0,T;L^2(\Omega,\mathbb{R}^N))} := \int_0^T \langle f,g\rangle_{L^2(\Omega,\mathbb{R}^N)} dt$.

2.1. Subdifferential formulation of the problem with a periodic boundary condition. We formulate the initial value problem of constrained total variation flow equation with a periodic boundary condition (2.1). First, we define the energy functional ϕ_{pe} of total variation of each function $u \in L^2(\mathbb{T}^n,\mathbb{R}^N)$ by

$$\phi_{\text{pe}}(u) := \begin{cases} \displaystyle\int_{\mathbb{T}^n} |\nabla u(x)|\, dx & \text{if } u \in \mathrm{BV}(\mathbb{T}^n,\mathbb{R}^N) \cap L^2(\mathbb{T}^n,\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$

where $\mathrm{BV}(\mathbb{T}^n,\mathbb{R}^N)$ denotes the space of functions of bounded variation on $\mathbb{T}^n$ with values in $\mathbb{R}^N$.

It is easy to see that ϕ_{pe} is a proper, convex, and lower semicontinuous functional on $L^2(\mathbb{T}^n,\mathbb{R}^N)$ (see[16]).

We also consider a functional Φ_{pe}^T on $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$ by $\Phi_{\text{pe}}^T(u) := \int_0^T \phi_{\text{pe}}(u(t))dt$.

Proposition 2.1. *The functional Φ_{pe}^T is proper, convex, and lower semicontinuous on $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$.*

Proof. The functional Φ_{pe}^T is obviously proper and convex on $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$. We will show that Φ_{pe}^T is lower semicontinuous.

Assume that $u_m \to u$ strongly in $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$ and $\Phi_{\text{pe}}^T(u_m) \le \lambda$ for any $m \in \mathbb{N}$. Since $\mathrm{BV}(\mathbb{T}^n,\mathbb{R}^N)$ is compactly embedded in $L^1(\mathbb{T}^n,\mathbb{R}^N)$ (see [16]), by taking some subsequence of $\{u_m\}_{m=1}^{+\infty}$, we have that

$$u_m(t) \longrightarrow u(t) \quad \text{strongly in } L^2(\mathbb{T}^n,\mathbb{R}^N) \text{ for a.e. } t \in [0,T]. \tag{2.4}$$

Then, the lower semicontinuity of ϕ_{pe} and Fatou's lemma yield

$$\lambda \geq \liminf_{m\to+\infty} \int_0^T \phi_{\mathrm{pe}}(u_m(t))\,dt \geq \int_0^T \liminf_{m\to+\infty} \phi_{\mathrm{pe}}(u_m(t))\,dt \geq \Phi_{\mathrm{pe}}^T(u). \tag{2.5}$$

This implies that Φ_{pe}^T is lower semicontinuous on $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$. □

Now we formally calculate the variational derivative of this Φ_{pe}^T with respect to the metric of $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$. For any $h \in C_0^\infty(\mathbb{T}^n \times (0,T),\mathbb{R}^N)$, we see that

$$\frac{d\Phi_{\mathrm{pe}}^T(u+\varepsilon h)}{d\varepsilon}\bigg|_{\varepsilon=0} = \left\langle -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right), h\right\rangle_{L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))}. \tag{2.6}$$

Therefore, the variational derivative $\delta\Phi_{\mathrm{pe}}^T(u)/\delta u$ of Φ_{pe}^T in $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$ can be formally written as

$$\frac{\delta\Phi_{\mathrm{pe}}^T}{\delta u}(u) = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \quad \text{in } L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)). \tag{2.7}$$

We need several other notations to complete the formulation of (2.1). Let $L^2(\mathbb{T}^n,M)$ denote the closed subset of $L^2(\mathbb{T}^n,\mathbb{R}^N)$ defined by $L^2(\mathbb{T}^n,M) := \{u \in L^2(\mathbb{T}^n,\mathbb{R}^N) \mid u(x) \in M \text{ a.e. } x \in \mathbb{T}^n\}$.

Let $L^2(0,T;L^2(\mathbb{T}^n,M))$ denote the set of all L^2-mappings from $[0,T]$ to $L^2(\mathbb{T}^n,M)$. For any $g \in L^2(0,T;L^2(\mathbb{T}^n,M))$, we define a map $P_g(\cdot) : L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)) \to L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$ by

$$P_g(f)(x,t) = \pi_{g(x,t)}(f(x,t)) \quad \text{for a.e. } (x,t) \in \mathbb{T}^n \times [0,T], \tag{2.8}$$

for any $f \in L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$.

By these notations of the function space, (2.7), and (2.8), (2.1) is formally of the form

$$\begin{aligned} u_t &= -P_u\left(\frac{\delta\Phi_{\mathrm{pe}}^T}{\delta u}(u)\right) \quad \text{in } L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)),\\ u|_{t=0} &= u_0 \quad \text{in } L^2(\mathbb{T}^n,M). \end{aligned} \tag{2.9}$$

The initial value problem (2.9) does not have a rigorous mathematical meaning since the energy functional Φ_{pe}^T is not always differentiable. We need the notion of subdifferential to handle the problem caused by this singularity of the gradient of our Φ_{pe}^T and to complete the mathematical formulation of (2.9). We recall this definition.

Definition 2.2 (subdifferential). Let ψ be a proper, convex functional on a real Hilbert space H equipped with the inner product $\langle\cdot,\cdot\rangle_H$. Define the subdifferential of ψ denoted by $\partial\psi(u)$ as

$$\partial\psi(u) := \{v \in H \mid \psi(u+h) \geq \psi(u) + \langle v,h\rangle_H \text{ for any } h \in H\}. \tag{2.10}$$

Using the subdifferential $\partial\Phi_{\mathrm{pe}}^T$ of Φ_{pe}^T, we are now able to formulate (2.9) as an evolution equation in $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$ of the form

$$\begin{aligned} u_t \in -P_u\big(\partial\Phi_{\mathrm{pe}}^T(u)\big) \quad &\text{in } L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)), \\ u|_{t=0} = u_0 \quad &\text{in } L^2(\mathbb{T}^n,M), \end{aligned} \tag{2.11}$$

where $u_0 \in L^2(\mathbb{T}^n,M)$ is a given initial data. The initial value problem (2.11) can be regarded as a mathematical formulation of (2.1).

Our goal is to show the existence of a solution of (2.1), the definition of a solution is given below.

Definition 2.3. Call a function $u : \mathbb{T}^n \times [0,T] \to \mathbb{R}^N$ a solution of (2.1) if u belongs to $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)) \cap C([0,T],L^2(\mathbb{T}^n,\mathbb{R}^N))$ and satisfies (2.11).

2.2. Subdifferential formulation of the problem with a Dirichlet boundary condition. In this section, we formulate the initial value problem of constrained total variation flow equation with a Dirichlet boundary condition (2.2). Let $L^2(\Omega,M)$ be the closed subset of $L^2(\Omega,\mathbb{R}^N)$ of the form

$$L^2(\Omega,M) := \{v \in L^2(\Omega,\mathbb{R}^N) \mid v(x) \in M \text{ a.e. } x \in \Omega\}. \tag{2.12}$$

We always choose an initial data v_0 which is a Lipschitz continuous map from $\overline{\Omega}$ to M.

Let $\tilde{v}_0$ denote a Lipschitz extension of v_0 to $\mathbb{R}^n$. We define the energy functional ϕ_D with a Dirichlet boundary condition on $L^2(\Omega,\mathbb{R}^N)$ as follows:

$$\phi_D(v) := \begin{cases} \displaystyle\int_{\overline{\Omega}} |\nabla\tilde{v}(x)|\,dx & \text{if } \tilde{v} \in \mathrm{BV}(\Omega,\mathbb{R}^N) \cap L^2(\Omega,\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.13}$$

where $\tilde{v}$ denotes an extension of $v \in L^2(\Omega,\mathbb{R}^N)$ to $\mathbb{R}^n$ such that $\tilde{v}(x) = \tilde{v}_0(x)$ for $x \in \mathbb{R}^n \setminus \Omega$. The definition is independent of the way of extension.

It is easy to check that ϕ_D is a proper, convex, and lower semicontinuous functional on $L^2(\Omega,\mathbb{R}^N)$ (see [16]). Note that the energy ϕ_D also measures the discrepancy of v from v_0 on the boundary $\partial\Omega$.

If we define a functional Φ_D^T on $L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ by $\Phi_D^T(v) = \int_0^T \phi_D(v)\,dt$, then like Φ_{pe}^T, we obtain the following proposition.

Proposition 2.4. *The functional Φ_D^T is proper, convex, and lower semicontinuous on $L^2(0,T;L^2(\Omega,\mathbb{R}^N))$.*

Since the proof parallels that of Proposition 2.1, we do not give it.

For $g \in L^2(0,T;L^2(\Omega,M))$, we define a map $P_g(\cdot) : L^2(0,T;L^2(\Omega,\mathbb{R}^N)) \to L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ by

$$P_g(f)(x,t) := \pi_{g(x,t)}(f(x,t)) \quad \text{for } f \in L^2(0,T;L^2(\Omega,\mathbb{R}^N)). \tag{2.14}$$

Since the variational derivative $\delta\Phi_D^T(v)/\delta v$ at $v \in L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ is formally given by

$$\frac{\delta\Phi_D^T(v)}{\delta v} = -\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) \quad \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^N)), \tag{2.15}$$

(2.2) is formally of the form

$$\begin{aligned} v_t &= -P_v\left(\frac{\delta\Phi_D^T}{\delta v}(v)\right) \quad \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^N)), \\ v|_{t=0} &= v_0 \quad \text{in } L^2(\Omega,M). \end{aligned} \tag{2.16}$$

Note that each solution of (2.16) moves, satisfying the Dirichlet boundary condition in order to keep minimizing the energy due to the discrepancy on the boundary. The notion of subdifferential of Φ_D^T allows us to formulate the formal equation (2.16) as an evolution equation in $L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ of the form

$$\begin{aligned} v_t &\in -P_v(\partial\Phi_D^T(v)) \quad \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^N)), \\ v|_{t=0} &= v_0 \quad \text{in } L^2(\Omega,M). \end{aligned} \tag{2.17}$$

Definition 2.5. Call a function $v : \Omega \times [0,T] \to \mathbb{R}^N$ a solution of (2.2) if v belongs to $L^2(0,T;L^2(\Omega,\mathbb{R}^N)) \cap C([0,T],L^2(\Omega,\mathbb{R}^N))$ and solves (2.17).

3. Convergence results

In this section, we state three main theorems. The first theorem shows the validity of our scheme to construct a solution of the equations formulated in the previous section. For applications, we state the theorem in a general setting.

Let H be a real Hilbert space and let G be a nonvoid closed subset of H. Let $L^2(0,T;G)$ denote the closed subset of $L^2(0,T;H)$ of the form $L^2(0,T;G) := \{u \in L^2(0,T;H) \mid u(t) \in G \text{ a.e. } t \in [0,T]\}$. Let B_R denote a closed ball of $L^2(0,T;H)$ defined by $B_R := \{u \in L^2(0,T;H) \mid \|u\|_{L^2(0,T;H)} \le R\}$ for $R > 0$.

Let $P(\cdot)(\cdot) : L^2(0,T;G) \times L^2(0,T;H) \to L^2(0,T;H)$ be an operator satisfying the following properties.

(i) For any $u \in L^2(0,T;G)$, $P(u)(\cdot)$ is a bounded linear operator from $L^2(0,T;H)$ to $L^2(0,T;H)$ (i.e., $P(u)(\cdot) \in \mathcal{L}(L^2(0,T;H),L^2(0,T;H))$).

(ii) There exists a constant $K > 0$ such that $\sup_{u \in L^2(0,T;G)} \|P(u)(\cdot)\|_{\mathcal{L}} \le K$.

(iii) If a sequence $\{u_k\}_{k=1}^{+\infty} \subset L^2(0,T;G)$ strongly converges to some u in $L^2(0,T;H)$, then there exists a subsequence $\{u_{k(l)}\}_{l=1}^{+\infty} \subset \{u_k\}_{k=1}^{+\infty}$ such that $P(u_{k(l)})^*(v)$ strongly converges to $P(u)^*(v)$ in $L^2(0,T;H)$ for any $v \in L^2(0,T;H)$, where $P(u)^*(\cdot)$ denotes the adjoint operator of $P(u)(\cdot)$.

Theorem 3.1 (abstract theorem). *Let Ψ_m $(m=1,2,\dots)$ and Ψ be proper, convex, lower semicontinuous functionals on $L^2(0,T;H)$. Assume that $\partial\Psi_m$ converges to $\partial\Psi$ in the sense of Graph (see Remark 3.2). Assume that $u_m \in L^2(0,T;H)$ $(m=1,2,\dots)$ satisfies the following conditions:*

$$\begin{gathered} u_{m,t} \in -P(u_m)\left(\partial\Psi_m(u_m)\cap B_R\right) \quad \text{in } L^2(0,T;H), \\ u_m \in L^2(0,T;G), \\ u_m|_{t=0} = u_{0,m}, \end{gathered} \tag{3.1}$$

where $u_{0,m} \in G$. In addition, assume that

$$\begin{aligned} u_m &\longrightarrow u \quad \text{in } C([0,T],H), \\ u_{0,m} &\longrightarrow u_0 \quad \text{strongly in } H. \end{aligned} \tag{3.2}$$

Then, u satisfies that

$$\begin{gathered} u_t \in -P(u)\left(\partial\Psi(u)\right) \quad \text{in } L^2(0,T;H), \\ u \in L^2(0,T;G), \\ u|_{t=0} = u_0, \end{gathered} \tag{3.3}$$

where $u_0 \in G$.

Remark 3.2. For (multivalued) operators A_m $(m=1,2,\dots)$ and A on a real Hilbert space H, we say that A_m converges to A in the sense of graph as $m\to+\infty$ if for any $(u,v)\in A$, there exists $(u_m,v_m)\in A_m$ such that $u_m\to u$ and $v_m\to v$ strongly in H as $m\to+\infty$.

Applying Theorem 3.1 to our cases, we obtain more explicit statements. Before we give the second theorem, we define approximate energies $\Phi^T_{\mathrm{pe},m}$ and $\Phi^T_{D,m}$ $(m=1,2,\dots)$ for our original energies Φ^T_{pe} and Φ^T_D, respectively:

$$\begin{aligned} &\phi_{\mathrm{pe},m}(u) \\ &\quad := \begin{cases} \dfrac{1}{1+1/m}\displaystyle\int_{\mathbb{T}^n} \left|\nabla u(x)\right|^{1+1/m} dx & \text{if } u \in W^{1,1+1/m}(\mathbb{T}^n,\mathbb{R}^N)\cap L^2(\mathbb{T}^n,\mathbb{R}^N), \\ +\infty & \text{otherwise}, \end{cases} \\ &\phi_{D,m}(v) \\ &\quad := \begin{cases} \dfrac{1}{1+1/m}\displaystyle\int_{\overline{\Omega}} \left|\nabla \tilde{v}(x)\right|^{1+1/m} dx & \text{if } \tilde{v} \in W^{1,1+1/m}(\Omega,\mathbb{R}^N)\cap L^2(\Omega,\mathbb{R}^N), \\ +\infty & \text{otherwise}, \end{cases} \end{aligned} \tag{3.4}$$

where $\tilde{v}$ denotes the extension of $v\in L^2(\Omega,\mathbb{R}^N)$ to $\mathbb{R}^n$ such that $\tilde{v}(x)=\widetilde{v_{0,m}}(x)$ for $x\in\mathbb{R}^n\setminus\Omega$, for a Lipschitz map $v_{0,m}:\overline{\Omega}\to M$.

Note that these energy functionals are equivalent to p-energy in p-harmonic map flow equation for $p=1+1/m$.

We again associate Φ^T's with ϕ's:

$$\begin{aligned}\Phi^T_{\mathrm{pe},m}(u) &:= \int_0^T \phi_{\mathrm{pe},m}(u)dt \quad \text{for } u \in L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)),\\ \Phi^T_{D,m}(v) &:= \int_0^T \phi_{D,m}(v)dt \quad \text{for } v \in L^2(0,T;L^2(\Omega,\mathbb{R}^N)).\end{aligned} \tag{3.5}$$

It is not difficult to see that these functionals $\Phi^T_{\mathrm{pe},m}$ and $\Phi^T_{D,m}$ are proper, convex, and lower semicontinuous.

We are now in position to state the second theorem.

Theorem 3.3 (convergence theorem). *The following statements hold.*

(1) *(The case with a periodic boundary condition.) Assume that* $u_m \in L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$ $(m = 1,2,\dots)$ *satisfies*

$$\begin{aligned}u_{m,t} &\in -P_{u_m}\big(\partial\Phi^T_{\mathrm{pe},m}(u_m)\cap B_R\big) \quad \textit{in } L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)),\\ u_m|_{t=0} &= u_{0,m} \quad \textit{in } L^2(\mathbb{T}^n,M),\end{aligned} \tag{3.6}$$

with $R>0$ *independent of* m, *where* $u_{0,m} \in L^2(\mathbb{T}^n,M)$. *Moreover, assume that*

$$\begin{aligned}&u_{0,m} \longrightarrow u_0 \quad \textit{strongly in } L^2(\mathbb{T}^n,\mathbb{R}^N), \textit{ as } m \longrightarrow +\infty,\\ &\limsup_{m\to+\infty} \phi_{\mathrm{pe},m}(u_{0,m}) \le \phi_{\mathrm{pe}}(u_0).\end{aligned} \tag{3.7}$$

Then, there exists a function $u \in C([0,T],L^2(\mathbb{T}^n,\mathbb{R}^N))$ *such that*

$$\begin{aligned}u_t &\in -P_u\big(\partial\Phi^T_{\mathrm{pe}}(u)\big) \quad \textit{in } L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)),\\ u|_{t=0} &= u_0 \quad \textit{in } L^2(\mathbb{T}^n,M),\end{aligned} \tag{3.8}$$

and u *satisfies the energy equality*

$$\int_0^t\int_{\mathbb{T}^n} |u_t(x,\tau)|^2 dx\,d\tau + \phi_{\mathrm{pe}}(u(t)) = \phi_{\mathrm{pe}}(u_0) \quad \textit{for any } t \in [0,T]. \tag{3.9}$$

This means that u *is a solution of (2.1) in the sense of Definition 2.3.*

(2) *(The case with a Dirichlet boundary condition.) Assume that* $v_m \in L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ $(m = 1,2,\dots)$ *satisfies*

$$\begin{aligned}v_{m,t} &\in -P_{v_m}\big(\partial\Phi^T_{D,m}(v_m)\cap B_R\big) \quad \textit{in } L^2(0,T;L^2(\Omega,\mathbb{R}^N)),\\ v_m|_{t=0} &= v_{0,m} \quad \textit{in } L^2(\Omega,M),\end{aligned} \tag{3.10}$$

with $R>0$ *independent of* m, *where the function* $v_{0,m}$ *is a Lipschitz continuous map from* $\overline{\Omega}$ *to* M. *Moreover, assume that*

$$\begin{aligned}&v_{0,m} \longrightarrow v_0 \quad \textit{strongly in } L^2(\Omega,\mathbb{R}^N), \textit{ as } m \longrightarrow +\infty,\\ &\limsup_{m\to+\infty} \phi_{D,m}(v_{0,m}) \le \phi_D(v_0),\end{aligned} \tag{3.11}$$

where v_0 is a Lipschitz continuous map from $\overline{\Omega}$ to M. Then there exists a function $v \in C([0,T], L^2(\Omega, \mathbb{R}^N))$ such that

$$\begin{aligned} v_t \in -P_v\left(\partial\Phi_D^T(v)\right) \quad & \text{in } L^2\left(0,T; L^2(\Omega,\mathbb{R}^N)\right), \\ v|_{t=0} = v_0 \quad & \text{in } L^2(\Omega, M), \end{aligned} \tag{3.12}$$

and v satisfies the energy equality

$$\int_0^t \int_\Omega \left| v_t(x,\tau) \right|^2 dx\, d\tau + \phi_D\left(v(t)\right) = \phi_D\left(v_0\right) \quad \text{for any } t \in [0,T]. \tag{3.13}$$

This means that v is a solution of (2.2) in the sense of Definition 2.5.

In some situations, our Theorem 3.3 actually yields a solution of our limit problem. Indeed, the solvability result of p-harmonic map flow equation in [9] ($1 < p < 2$) with Theorem 3.3 and a priori estimate yield local existence of a solution of (2.1) in the sense of Definition 2.3.

Theorem 3.4 (local existence theorem). *For any $K > 0$, there exists $\varepsilon_0 > 0$ depending only on $\mathbb{T}^n$, M, and K such that if the initial data $u_0 : \mathbb{T}^n \to M$ satisfies the conditions:*

(i) $u_0 \in C^{2+\alpha}(\mathbb{T}^n, \mathbb{R}^N)$ $(0 < \alpha < 1)$,
(ii) $\|\nabla u_0\|_{L^\infty(\mathbb{T}^n)} \le K$,
(iii) *there exists $m_0 \in \mathbb{N}$, $m_0 \ge 3$, such that*

$$\phi_{\mathrm{pe},m_0}\left(u_0\right) + \frac{1}{m_0+1} \prod_{i=1}^n \omega_i \le \varepsilon_0, \tag{3.14}$$

then, for any $T \in (0, 2/C\sqrt{\max\{1,K^2\}})$, where C is a positive constant depending only on M, there exists a function $u \in C([0,T], L^2(\mathbb{T}^n, M))$ solving (2.11) for this T and satisfying the energy equality

$$\int_0^t \int_{\mathbb{T}^n} \left| u_t(x,\tau) \right|^2 dx\, d\tau + \phi_{\mathrm{pe}}\left(u(t)\right) = \phi_{\mathrm{pe}}\left(u_0\right) \quad \text{for any } t \in [0,T]. \tag{3.15}$$

Remark 3.5. It was proved in [24] that the global weak solution which solves the initial value problem of p-harmonic map flow equation ($1 < p < 2$) with a Dirichlet boundary condition for the case that the target manifold is S^{N-1} is an element of $L^\infty((0,\infty); W^{1,p}(\Omega, S^{N-1})) \cap W^{1,2}((0,\infty); L^2(\Omega, \mathbb{R}^N))$. This regularity of the solution is not sufficient to be a solution of our approximate problem $v_t \in -P_v(\partial\Phi_{D,m}^T(v))$, since we are considering this evolution equation in $L^2(0,T; L^2(\Omega,\mathbb{R}^N))$. All the terms of the equation $v_t = \operatorname{div}(|\nabla v|^{1/m-1}\nabla v) + |\nabla v|^{1/m+1} v$ must belong to $L^2(0,T; L^2(\Omega,\mathbb{R}^N))$ to be a solution of our approximate problem. Therefore, we are unable to apply our convergence theorem (Theorem 3.3) in this setting. So even local existence is unknown for the Dirichlet problem (2.17).

4. Proof of abstract theorem

We need a notion of convergence of sets in a Hilbert space to carry out the proof. We give the definition of the convergence first.

Definition 4.1. Let H be a real Hilbert space and let $\{S_m\}_{m=1}^{+\infty}$ be a sequence of subsets of H. Define *sequentially weak upper limit* of $\{S_m\}_{m=1}^{+\infty}$ denoted by *sqw* $-\operatorname{Lim\,sup}_{m\to+\infty} S_m$ as

$$\begin{aligned}\underset{m\to+\infty}{\text{sqw-Limsup}}\ S_m := \Big\{x\in H \mid\ & \text{there exist } \{m_k\}_{k=1}^{+\infty}\subset\mathbb{N},\\ & \text{and } x_k\in S_{m_k}\ (k=1,2,\dots)\\ & \text{such that } x_k\rightharpoonup x \text{ weakly in } H \text{ as}\\ & k\longrightarrow+\infty\Big\}.\end{aligned}\tag{4.1}$$

Remark 4.2. If H is separable, then for any bounded set $B\subset H$, we can introduce a topology τ by a suitable countable family of seminorms on H into B so that (B,τ) is a first countable topological space and the weak topology is equivalent to τ. In this case, if $\{S_m\}_{m=1}^{+\infty}$ is bounded, our definition of $\text{sqw-Limsup}_{m\to+\infty} S_m$ agrees with the usual notion of τ-upper limit of $\{S_m\}_{m=1}^{+\infty}$ (see, e.g., [5]).

We prepare two important propositions to prove the theorem.

Proposition 4.3. *Let $\{A_m\}_{m=1}^{+\infty}$ be a sequence of monotone operators and let A be a maximal monotone operator from a real Hilbert space H to 2^H. Assume that A_m converges to A in the sense of Graph as $m\to+\infty$. Take a sequence $\{u_m\}_{m=1}^{+\infty}\subset H$ with*

$$u_m\longrightarrow u\quad\textit{strongly in } H,\qquad A_m(u_m)\neq\emptyset\quad\textit{for any } m\in\mathbb{N}.\tag{4.2}$$

Then $\text{sqw-Limsup}_{m\to+\infty} A_m(u_m)\subset A(u)$.

Proof. By definition, for any $v\in\text{sqw-Limsup}_{m\to+\infty} A_m(u_m)$, there exist $\{m_k\}_{k=1}^{+\infty}\subset\mathbb{N}$ and $v_k\in A_{m_k}(u_{m_k})$ $(k=1,2,\dots)$ such that

$$v_k\rightharpoonup v\quad\text{weakly in } H,\ \text{as } k\longrightarrow+\infty.\tag{4.3}$$

We take any $(f,g)\in A$ and fix it. Since A_{m_k} converges to A as Graph, we see that there exists a sequence $(f_k,g_k)\in A_{m_k}$ $(k=1,2,\dots)$ such that

$$f_k\longrightarrow f,\quad g_k\longrightarrow g\quad\text{strongly in } H,\ \text{as } k\longrightarrow+\infty.\tag{4.4}$$

By the convergences (4.3), (4.4) and the fact that any weakly convergent sequence is bounded in H, we see that

$$\begin{aligned}&|\langle v-g,u-f\rangle_H-\langle v_k-g_k,u_{m_k}-f_k\rangle_H|\\ &\quad\le|\langle v,u-f\rangle_H-\langle v_k,u-f\rangle_H|+|\langle v_k,u-f\rangle_H-\langle v_k,u_{m_k}-f_k\rangle_H|\\ &\qquad+|\langle -g,u-f\rangle_H-\langle -g_k,u-f\rangle_H|\\ &\qquad+|\langle -g_k,u-f\rangle_H-\langle -g_k,u_{m_k}-f_k\rangle_H|\\ &\quad\le|\langle v-v_k,u-f\rangle_H|+\|v_k\|_H\|(u-f)-(u_{m_k}-f_k)\|_H\\ &\qquad+\|-g+g_k\|_H\|u-f\|_H+\|g_k\|_H\|(u-f)-(u_{m_k}-f_k)\|_H\\ &\quad\longrightarrow 0\quad(k\longrightarrow+\infty).\end{aligned}\tag{4.5}$$

Thus, we obtain

$$\langle v-g,u-f\rangle_H=\lim_{k\to+\infty}\langle v_k-g_k,u_{m_k}-f_k\rangle_H\geq 0 \tag{4.6}$$

since A_{m_k} $(k=1,2,\dots)$ are monotone operators.

Therefore, if we define an operator $\tilde{A}:H\to 2^H$ by $\tilde{A}:=(u,v)\cup A$, then by (4.6), we see that $\tilde{A}$ is a monotone operator which includes A. The maximality of A yields that $\tilde{A}=A$, thus $v\in A(u)$. □

Corollary 4.4. *Let Ψ_m $(m=1,2,\dots)$ and Ψ be proper, convex, and lower semicontinuous functionals on a real Hilbert space H. Assume that $\partial\Psi_m$ converges to $\partial\Psi$ in the sense of Graph. Let $\{u_m\}_{m=1}^{+\infty}$ be a sequence of H satisfying that $u_m\to u$ strongly in H as $m\to+\infty$ with $\partial\Psi_m(u_m)\neq\varnothing$ $(m=1,2,\dots)$.*

Then

$$\operatorname*{sqw-Limsup}_{m\to+\infty}\partial\Psi_m(u_m)\subset\partial\Psi(u). \tag{4.7}$$

Proof. Since $\partial\Psi_m$ and $\partial\Psi$ are maximal monotone operators in H, the proof is a direct consequence of the previous proposition. □

Proposition 4.5. *Under the notations of Theorem 3.1, let $\{u_m\}_{m=1}^{+\infty}\subset L^2(0,T;G)$ be a sequence such that $u_m\to u$ strongly in $L^2(0,T;H)$ as $m\to+\infty$ and that $\partial\Psi_m(u_m)\cap B_R\neq\varnothing$ $(m=1,2,\dots)$. Then*

$$\operatorname*{sqw-Limsup}_{m\to+\infty}P(u_m)\left(\partial\Psi_m(u_m)\cap B_R\right)\subset P(u)\left(\partial\Psi(u)\right). \tag{4.8}$$

Proof. By definition, for $f\in\text{sqw-Limsup}_{m\to+\infty}P(u_m)(\partial\Psi_m(u_m)\cap B_R)$, there exist $\{m_k\}_{k=1}^{+\infty}\subset\mathbb{N}$ and $f_k\in P(u_{m_k})(\partial\Psi_{m_k}(u_{m_k})\cap B_R)$ such that

$$f_k\rightharpoonup f\quad\text{weakly in } L^2(0,T;H),\text{ as } k\longrightarrow+\infty. \tag{4.9}$$

Moreover, for any $k\in\mathbb{N}$, there exists $v_k\in\partial\Psi_{m_k}(u_{m_k})\cap B_R$ such that $f_k=P(u_{m_k})(v_k)$. Since $\{v_k\}_{k=1}^{+\infty}$ is bounded, by choosing some subsequence if necessary, we see that there exists $v\in L^2(0,T;H)$ such that

$$v_k\rightharpoonup v\quad\text{weakly in } L^2(0,T;H),\text{ as } k\longrightarrow+\infty. \tag{4.10}$$

Then, by the definition of sequentially weak upper limit and Corollary 4.4, we obtain that

$$v\in\operatorname*{sqw-Limsup}_{k\to+\infty}\left(\partial\Psi_{m_k}(u_{m_k})\cap B_R\right)\subset\partial\Psi(u). \tag{4.11}$$

We will show that

$$P(u_{m_k})(v_k)\rightharpoonup P(u)(v)\quad\text{weakly in } L^2(0,T;H),\text{ as } k\longrightarrow+\infty, \tag{4.12}$$

by taking a suitable subsequence of $\{P(u_{m_k})(v_k)\}_{k=1}^{+\infty}$ (still denoted by $\{P(u_{m_k})(v_k)\}_{k=1}^{+\infty}$). Indeed, if we choose some subsequence of $\{u_{m_k}\}_{k=1}^{+\infty}$ so that condition (iii) for $P(\cdot)(\cdot)$ holds, then we see that for any $h \in L^2(0,T;H)$,

$$\begin{aligned}
&\left|\langle P(u_{m_k})(v_k) - P(u)(v), h\rangle_{L^2(0,T;H)}\right| \\
&\quad \leq \left|\langle v_k, P(u_{m_k})^*(h) - P(u)^*(h)\rangle_{L^2(0,T;H)}\right| + \left|\langle v_k - v, P(u)^*(h)\rangle_{L^2(0,T;H)}\right| \\
&\quad \leq R\left\|P(u_{m_k})^*(h) - P(u)^*(h)\right\|_{L^2(0,T;H)} + \left|\langle v_k - v, P(u)^*(h)\rangle_{L^2(0,T;H)}\right| \\
&\quad \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty.
\end{aligned} \tag{4.13}$$

Here, we have used the convergences that $P(u_{m_k})^*(h) \to P(u)^*(h)$ strongly in $L^2(0,T;H)$ by condition (iii) and (4.10).

Therefore, by sending $k \to +\infty$ in both sides of $f_k = P(u_{m_k})(v_k)$, we have $f = P(u)(v)$ by (4.9) and (4.12). Moreover, the inclusion (4.11) yields that $f \in P(u)(\partial\Psi(u))$ holds.

Then the desired inclusion has been proved. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By condition (ii), we see that $\{u_{m,t}\}_{m=1}^{+\infty}$ is bounded. Thus, one can choose a subsequence $\{u_{m_k,t}\}_{k=1}^{+\infty} \subset \{u_{m,t}\}_{m=1}^{+\infty}$ so that $u_{m_k,t}$ converges weakly to some $\tilde{u}$ in $L^2(0,T;H)$. Moreover, the convergence $u_{m_k} \to u$ in $L^2(0,T;H)$ yields that $\tilde{u} = u_t$ and $u_{m_k,t} \to u_t$ weakly in $L^2(0,T;H)$.

Since $u_{m_k,t} \in -P(u_{m_k})(\partial\Psi_{m_k}(u_{m_k}) \cap B_R)$, the definition of sequentially weak upper limit and Proposition 4.5 assure that

$$\begin{aligned}
u_t &\in \operatorname*{sqw-Limsup}_{k\to+\infty}\left(-P(u_{m_k})\left(\partial\Psi_{m_k}(u_{m_k}) \cap B_R\right)\right) \\
&\subset -P(u)\left(\partial\Psi(u) \cap B_R\right).
\end{aligned} \tag{4.14}$$

The properties $u \in L^2(0,T;G)$ and $u|_{t=0} = u_0$ obviously follow from the assumptions. The proof is now complete. □

5. Proof of convergence theorem

In this section, we prove Theorem 3.3 as an application of Theorem 3.1. We will check that the situation of Theorem 3.3 satisfies the assumptions of Theorem 3.1. Some convergence results of the convex functionals assure that Theorem 3.1 is available for our problem. Especially, we show that the functionals $\phi_{\mathrm{pe},m}, \phi_{D,m}, \Phi^T_{\mathrm{pe},m}$, and $\Phi^T_{D,m}$ defined in Section 3 converge to our original energy functionals in the sense of Mosco. The following lemma proved in [26] is the first step. We give its proof for the completeness, only under Dirichlet boundary condition, since the proof under periodic boundary condition is easier.

Proposition 5.1 (see [26]). *The functional $\phi_{D,m}$ ($\phi_{\mathrm{pe},m}$) converges to ϕ_D ($\phi_{\mathrm{pe},m}$) in the sense of Mosco as $m \to +\infty$.*

Remark 5.2. For proper, convex, and lower semicontinuous functionals Ψ_m $(m = 1,2,\dots)$ and Ψ on a real Hilbert space H, we say that Ψ_m converges to Ψ in the sense of Mosco as $m \to +\infty$, if the following statements hold.

(i) If $u_m \rightharpoonup u$ weakly in H, then $\Psi(u) \le \liminf_{m\to+\infty} \Psi_m(u_m)$.

(ii) For any $u \in D(\Psi)$, there exists $\{u_m\}_{m=1}^{+\infty} \subset H$ such that $u_m \to u$ strongly and $\Psi_m(u_m) \to \Psi(u)$ as $m \to +\infty$.

Proof. We first show condition (i) of Mosco convergence. Assume that $u_m \rightharpoonup u$ weakly in $L^2(\Omega,\mathbb{R}^N)$. It is sufficient to show the case that $u_m \in D(\phi_{D,m})$. Thus, we may assume that $\widetilde{u_m} \in W^{1,1+1/m}(\Omega,\mathbb{R}^N)$. By Hölder's inequality, we see that $\widetilde{u_m} \in \mathrm{BV}(\Omega,\mathbb{R}^N)$ and

$$\begin{aligned}\phi_D(u_m) &= \int_{\overline{\Omega}} |\nabla \widetilde{u_m}|\,dx \\ &\le \left(\int_{\overline{\Omega}} |\nabla \widetilde{u_m}|^{1+1/m} dx\right)^{1/(1+1/m)} \cdot |\overline{\Omega}|^{1-1/(1+1/m)} \\ &\le \phi_{D,m}(u_m) + \frac{1}{m+1}|\overline{\Omega}|.\end{aligned} \tag{5.1}$$

Thus, by the lower semicontinuity of ϕ_D, we obtain

$$\phi_D(u) \le \liminf_{m\to+\infty} \phi_D(u_m) \le \liminf_{m\to+\infty} \phi_{D,m}(u_m). \tag{5.2}$$

This implies that (i) holds.

Next we show that condition (ii) of Mosco convergence is satisfied. Take any $u \in D(\phi_D)$ and fix it. Since $\tilde{u} \in \mathrm{BV}(\Omega,\mathbb{R}^N)$, by [16, Remark 2.12], we see that there exists $\{u_j\}_{j=1}^{+\infty} \subset C^\infty(\Omega,\mathbb{R}^N)$ such that

$$\begin{gathered} u_j \longrightarrow u \quad \text{strongly in } L^2(\Omega,\mathbb{R}^N), \\ \int_\Omega |\nabla u_j|\,dx \longrightarrow \int_\Omega |\nabla u|dx \quad \text{as } j \longrightarrow +\infty, \\ \text{and the trace of } u_j \text{ on } \partial\Omega \text{ is equivalent to the trace of u.}\end{gathered} \tag{5.3}$$

The properties (5.3) yield that $u_j \in D(\phi_D)$ and

$$\phi_D(u_j) \longrightarrow \phi_D(u) \quad \text{as } j \longrightarrow +\infty. \tag{5.4}$$

Moreover, we observe that $u_j \in D(\phi_{D,m})$, for any $m \in \mathbb{N}$, and

$$\begin{aligned}\phi_{D,m}(u_j) &= \frac{1}{1+1/m}\int_{\overline{\Omega}} |\nabla \widetilde{u_j}|^{1+1/m} dx \\ &\longrightarrow \int_{\overline{\Omega}} |\nabla \widetilde{u_j}|\,dx = \phi_D(u_j) \quad \text{as } m \longrightarrow +\infty.\end{aligned} \tag{5.5}$$

Thus, we can choose a subsequence $\{i_j^*\}_{j=1}^{+\infty} \subset \mathbb{N}$ so that

$$\begin{gathered} i_j^* \ge j, \qquad i_{j+1}^* \ge i_j^*, \\ |\phi_{D,i}(u_j) - \phi_D(u_j)| \le \frac{1}{j} \quad \text{for any } i \in \{i_j^*,\dots,i_{j+1}^*\} \text{ and any } j \in \mathbb{N}.\end{gathered} \tag{5.6}$$

We take

$$\varepsilon_i := \frac{1}{j}, \quad \hat{u}_i := u_j \quad \text{for any } i \in \{i_j^*, \dots, i_{j+1}^*\} \text{ and any } j \in \mathbb{N}, \qquad \hat{u}_i := u_1 \quad \text{for } i \in \{1, \dots, i_1^* - 1\}, \tag{5.7}$$

and observe that

$$\hat{u}_i \in D(\phi_{D,i}) \quad \text{for any } i \in \mathbb{N}, \qquad \hat{u}_i \longrightarrow u \quad \text{in } L^2(\Omega, \mathbb{R}^N), \text{ as } i \longrightarrow +\infty. \tag{5.8}$$

Moreover, for $i \geq i_1^*$,

$$\begin{aligned} |\phi_{D,i}(\hat{u}_i) - \phi_D(u)| &\leq |\phi_{D,i}(\hat{u}_i) - \phi_D(\hat{u}_i)| + |\phi_D(\hat{u}_i) - \phi_D(u)| \\ &\leq \varepsilon_i + |\phi_D(\hat{u}_i) - \phi_D(u)| \\ &\longrightarrow 0 \quad (i \longrightarrow +\infty). \end{aligned} \tag{5.9}$$

This implies that condition (ii) of Mosco convergence holds. □

Proposition 5.3. *The operator $\Phi_{D,m}^T$ (resp., $\Phi_{\mathrm{pe},m}^T$) converges to Φ_D^T (resp., Φ_{pe}^T) in the sense of Mosco. Moreover, $\partial\Phi_{D,m}^T$ (resp., $\partial\Phi_{\mathrm{pe},m}^T$) converges to $\partial\Phi_D^T$ (resp.,$\partial\Phi_{\mathrm{pe}}^T$) in the sense of Graph as $m \to +\infty$.*

It needs some technical arguments to prove this proposition. We will give the proof in a general setting in the appendix. The consequence follows from Propositions 5.1, .7, and .9 which will be proved in the appendix (see also [4, 5]).

We can derive energy equalities which are necessary to prove Theorem 3.3 by applying Proposition .6 also shown later in the appendix.

Proposition 5.4. *Assume the same hypotheses of Theorem 3.3.*

(1) *(The case with a periodic boundary condition.) $u_m \in L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$ $(m = 1,2, \dots)$ satisfies*

$$\int_0^t \int_{\mathbb{T}^n} |u_{m,t}(x,\tau)|^2 dx\, d\tau + \phi_{\mathrm{pe},m}(u_m(t)) = \phi_{\mathrm{pe},m}(u_{0,m}) \quad \textit{for any } t \in [0,T]. \tag{5.10}$$

(2) *(The case with a Dirichlet boundary condition.) $v_m \in L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ $(m = 1,2, \dots)$ satisfies*

$$\int_0^t \int_{\Omega} |v_{m,t}(x,\tau)|^2 dx\, d\tau + \phi_{D,m}(v_m(t)) = \phi_{D,m}(v_{0,m}) \quad \textit{for any } t \in [0,T]. \tag{5.11}$$

Proof. We only prove (5.11). We can show (5.10) by the same argument as below.

There exists $w_m \in \partial\Phi_{D,m}^T(v_m)$ such that $v_{m,t}(x,t) = -\pi_{v_m(x,t)}(w_m(x,t))$. Noting that $v_{m,t}(x,t) \in T_{v_m(x,t)}M$ for a.e. $(x,t) \in \Omega \times [0,T]$, we see that

$$\begin{aligned} \int_\Omega |v_{m,t}(x,t)|^2 dx &= \int_\Omega \langle v_{m,t}(x,t), -\pi_{v_m(x,t)}(w_m(x,t))\rangle dx \\ &= -\langle v_{m,t}(t), w_m(t)\rangle_{L^2(\Omega,\mathbb{R}^N)} \quad \text{for a.e. } t \in [0,T]. \end{aligned} \tag{5.12}$$

Since the inclusion $w_m \in \partial\Phi^T_{D,m}(v_m)$ yields that $w_m(t) \in \partial\phi_{D,m}(v_m(t))$ for a.e. $t \in [0,T]$, Proposition .6 which will be proved in the appendix assures that

$$\langle v_{m,t}(t), w_m(t)\rangle_{L^2(\Omega,\mathbb{R}^N)} = \frac{d}{dt}\phi_{D,m}(v_m(t)) \quad \text{for a.e. } t \in [0,T]. \tag{5.13}$$

Combining (5.13) with (5.12) and integrating both sides in $(0,T)$, we obtain equality (5.11). □

Now we show Theorem 3.3.

Proof of Theorem 3.3. We present the proof only under Dirichlet boundary condition, since the proof is similar for periodic boundary value problem.

First we note that Proposition 5.3 actually gives the assumption for the Graph convergence of the subdifferential of energy functionals in Theorem 3.1.

We will check that our projection $P_\cdot(\cdot)$ satisfies the conditions of Theorem 3.1. Since it is easy to check that conditions (i), (ii) hold, we only show that condition (iii) holds.

Assume that $u_k \to u$ strongly in $L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ and $u_k \in L^2(0,T;L^2(\Omega,M))$ ($k = 1,2,\dots$). Then one can choose some subsequence $\{u_{k(l)}\}_{l=1}^{+\infty} \subset \{u_k\}_{k=1}^{+\infty}$ such that

$$u_{k(l)}(x,t) \longrightarrow u(x,t) \quad \text{as } l \longrightarrow +\infty, \text{ for a.e. } (x,t) \in \Omega \times [0,T]. \tag{5.14}$$

For any $v \in L^2(0,T;L^2(\Omega,\mathbb{R}^N))$, we observe that

$$\begin{aligned}|P_{u_{k(l)}}(v)(x,t) - P_u(v)(x,t)|^2 &\le 4\Big(\sup_{w\in M}\sup_{y\in\mathbb{R}^N,|y|\le 1}|\pi_w(y)|\Big)^2|v(x,t)|^2\\ &\in L^1(\Omega\times[0,T],\mathbb{R}^N).\end{aligned} \tag{5.15}$$

By (5.14) and (5.15), one is able to apply Lebesgue's theorem to get that

$$P_{u_{k(l)}}(v) \longrightarrow P_u(v) \quad \text{strongly in } L^2(0,T;L^2(\Omega,\mathbb{R}^N)), \text{ as } l \longrightarrow +\infty. \tag{5.16}$$

In addition, since $\pi_u(\cdot)$ is a symmetric matrix for any $u \in M$, we easily see that the bounded linear operator P_w is selfadjoint, that is, $P_w^* = P_w$ for any $w \in L^2(0,T;L^2(\Omega,M))$. Thus, the convergence (5.16) assures that condition (iii) holds.

We next show that there exists a subsequence of $\{v_m\}_{m=1}^{+\infty}$ such that it converges in $C([0,T],L^2(\Omega,\mathbb{R}^N))$. By the assumption that $\limsup_{m\to+\infty}\phi_{D,m}(v_{0,m}) \le \phi_D(v_0)$ and equality (5.11), there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}&\int_0^t\int_\Omega |v_{m,t}(x,\tau)|^2\,dx\,d\tau + \phi_{D,m}(v_m(t))\\ &\qquad \le \phi_D(v_0) + 1 \quad \text{for any } m \ge k \text{ and any } t \in [0,T].\end{aligned} \tag{5.17}$$

Moreover, we observe that

$$|v_m(t) - v_m(s)| \le \int_s^t |v_{m,t}(\tau)|\,d\tau \le \Big(\int_s^t |v_{m,t}(\tau)|^2 d\tau\Big)^{1/2}|t-s|^{1/2}. \tag{5.18}$$

This inequality, together with (5.17), yields

$$\begin{aligned}&\|v_m(t)-v_m(s)\|_{L^2(\Omega,\mathbb{R}^N)}\\&\qquad\le\left(\phi_D(v_0)+1\right)^{1/2}|t-s|^{1/2}\text{for any } s,t\in[0,T] \text{ with } s\le t \text{ and any } m\ge k.\end{aligned}\tag{5.19}$$

This implies that $\{v_m(t)\}_{m=k}^{+\infty}\subset C([0,T],L^2(\Omega,\mathbb{R}^N))$ is equicontinuous.

In addition, since each v_m takes its values in M, it is obvious that $\{v_m(t)\}_{m=k}^{+\infty}\subset C([0,T],L^2(\Omega,\mathbb{R}^N))$ is uniformly bounded.

By using inequality (5.17) again, we can calculate as follows:

$$\begin{aligned}\int_\Omega|\nabla v_m(t)|\,dx&\le\left(\int_\Omega|\nabla v_m(t)|^{1+1/m}dx\right)^{m/(m+1)}|\Omega|^{1/(m+1)}\\&\le\left(\phi_D(v_0)+1\right)\left(|\Omega|+1\right)\quad\text{for any } m\ge k,\ t\in[0,T].\end{aligned}\tag{5.20}$$

Thus, by compactness [16, Theorem 1.19], this BV bound implies that the sequence $\{v_m(t)\}_{m\ge k}$ is relatively compact in $L^1(\Omega,\mathbb{R}^N)$ for any $t\in[0,T]$. Since $\{v_m(t)\}_{m\ge k}$ is bounded in $L^\infty(\Omega,\mathbb{R}^N)$, it is easy to see that $\{v_m(t)\}_{m\ge k}$ is also relatively compact in $L^2(\Omega,\mathbb{R}^N)$ for any $t\in[0,T]$.

We are now able to use Ascoli-Arzela's theorem (for $C([0,T],L^2(\Omega,\mathbb{R}^N))$) and conclude that there exist a subsequence $\{v_{m(l)}\}_{l=1}^{+\infty}\subset\{v_m\}_{m=1}^{+\infty}$ and $v\in C([0,T],\mathbb{R}^N)$ such that $v_{m(l)}$ converges to v in $C([0,T],\mathbb{R}^N)$.

We now observe that all the assumptions of Theorem 3.1 are fulfilled. Thus, Theorem 3.1 yields the desired result. □

6. Proof of local existence theorem

Since we have already established convergence theorem, it is sufficient to find approximate solutions of p-harmonic map flow equation which satisfies the assumptions of convergence theorem.

First of all, we calculate $\partial\Phi^T_{\mathrm{pe},m}$ to see that solutions of p-harmonic map flow equation solve our approximate problem in our notation with $\partial\Phi^T_{\mathrm{pe},m}$.

Lemma 6.1. *The subdifferential $\partial\Phi^T_{\mathrm{pe},m}$ is of the form*

$$\partial\Phi^T_{\mathrm{pe},m}(u)=\left\{-\operatorname{div}\left(|\nabla u|^{1/m-1}\nabla u\right)\right\}\quad\textit{for } u\in D(\partial\Phi^T_{\mathrm{pe},m}).\tag{6.1}$$

Proof. Let $v\in\partial\Phi^T_{\mathrm{pe},m}(u)$. Then, by the definition of subdifferential, for any $f\in C_0^\infty(\mathbb{T}^n\times[0,T],\mathbb{R}^N)$ and $\varepsilon>0$,

$$\begin{aligned}&\frac{1}{1+1/m}\int_0^T\int_{\mathbb{T}^n}|\nabla u+\varepsilon\nabla f|^{1+1/m}dx\,dt\\&\qquad\ge\frac{1}{1+1/m}\int_0^T\int_{\mathbb{T}^n}|\nabla u|^{1+1/m}dx\,dt+\int_0^T\int_{\mathbb{T}^n}\langle\varepsilon f,v\rangle dx\,dt.\end{aligned}\tag{6.2}$$

Moreover,

$$\begin{aligned}
&\frac{1}{1+1/m}\int_0^T\int_{\mathbb{T}^n}|\nabla u+\varepsilon\nabla f|^{1+1/m}dx\,dt\\
&\quad=\frac{1}{1+1/m}\int_0^T\int_{\mathbb{T}^n}|\nabla u|^{1+1/m}dx\,dt+\varepsilon\int_0^T\int_{\mathbb{T}^n}|\nabla u|^{1/m-1}\langle\nabla u,\nabla f\rangle dx\,dt+o(\varepsilon).
\end{aligned}\tag{6.3}$$

Thus, we have

$$\varepsilon\int_0^T\int_{\mathbb{T}^n}|\nabla u|^{1/m-1}\langle\nabla u,\nabla f\rangle dx\,dt+o(\varepsilon)\geq\varepsilon\int_0^T\int_{\mathbb{T}^n}\langle f,v\rangle dx\,dt.\tag{6.4}$$

By dividing both sides by ε, sending $\varepsilon\downarrow 0$, and integrating by parts, we obtain that

$$\int_0^T\int_{\mathbb{T}^n}\langle v,f\rangle dx\,dt\leq\int_0^T\int_{\mathbb{T}^n}\langle -\operatorname{div}(|\nabla u|^{1/m-1}\nabla u),f\rangle dx\,dt.\tag{6.5}$$

By taking negative $\varepsilon<0$ and sending $\varepsilon\uparrow 0$ in the same way, we also obtain

$$\int_0^T\int_{\mathbb{T}^n}\langle -\operatorname{div}(|\nabla u|^{1/m-1}\nabla u),f\rangle dx\,dt\leq\int_0^T\int_{\mathbb{T}^n}\langle v,f\rangle dx\,dt.\tag{6.6}$$

Combining (6.5) with (6.6), we have

$$\int_0^T\int_{\mathbb{T}^n}\langle v+\operatorname{div}(|\nabla u|^{1/m-1}\nabla u),f\rangle dx\,dt=0\quad\text{for any } f\in C_0^\infty(\Omega\times[0,T],\mathbb{R}^N).\tag{6.7}$$

This implies that $v=-\operatorname{div}(|\nabla u|^{1/m-1}\nabla u)$. The proof is now complete. □

We need to know the solvability result of p-harmonic map flow equation as an approximate solution for our problem. By Lemma 6.1, we safely transfer the result of [9] into our setting.

Proposition 6.2 (global solvability of p-harmonic map flow equation [9]). *For $m\in\mathbb{N}$ and $K>0$, there exists $\varepsilon_0>0$ depending only on $K,M,\mathbb{T}^n$, and m such that for the initial data $u_{0,m}:\mathbb{T}^n\to M$ satisfying the conditions*

(i) $u_{0,m}\in C^{2+\alpha}(\mathbb{T}^n,\mathbb{R}^N)(0<\alpha<1)$,
(ii) $\phi_m(u_{0,m})\leq\varepsilon_0$,
(iii) $\|\nabla u_{0,m}\|_{L^\infty(M)}\leq K$,

then, there exists uniquely a function $u_m:\mathbb{T}^n\times[0,\infty)\to M$ satisfying

$$\begin{aligned}
u_{m,t}\in -P_{u_m}(\partial\Phi^T_{\mathrm{pe},m}(u_m))\quad &\textit{in } L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N)),\\
u_m\big|_{t=0}=u_{0,m}\quad &\textit{in } L^2(\mathbb{T}^n,M),
\end{aligned}\tag{6.8}$$

and the energy inequality

$$\int_0^T\int_{\mathbb{T}^n}|u_{m,t}(x,\tau)|^2dx\,d\tau+\phi_{\mathrm{pe},m}(u_m(T))\leq\phi_{\mathrm{pe},m}(u_{0,m})\tag{6.9}$$

for any $T > 0$. In addition,

$$\begin{aligned} u_{m,t} &\in L^2(\mathbb{T}^n \times [0,\infty), \mathbb{R}^N), \\ u_m &\in C^\beta(\mathbb{T}^n \times [0,\infty), \mathbb{R}^N), \\ \nabla u_m &\in C^\beta(\mathbb{T}^n \times [0,\infty), \mathbb{R}^{nN}), \quad \textit{where } 0 < \beta < 1. \end{aligned} \tag{6.10}$$

Remark 6.3. In [9], this theorem was proved not only for our manifold $\mathbb{T}^n$, but also for a general compact Riemannian manifold without boundary. The dependence of ε_0 with respect to m is not explicitly stated in [9]. However, if one examines the proof, one concludes that ε_0 can be chosen independently of $m \geq 3$ as stated below.

Corollary 6.4. *For any $K > 0$, there exists $\varepsilon_0 > 0$ which depends only on $K, \mathbb{T}^n$, and M such that for any $m \geq 3$, if the initial data $u_{0,m}$ satisfies conditions (i), (ii), (iii) of Proposition 6.2, then there exists uniquely a function $u_m : \mathbb{T}^n \times [0,\infty) \to M$ satisfying all the consequences of Proposition 6.2.*

Proof. We follow the arguments in [9] briefly. In [9], the global solution was obtained as a limit of a function $u_{\delta,m} : \mathbb{T}^n \times [0, T_\delta) \to M$, which is a solution of the following regularized problem as $\delta \downarrow 0$:

$$\begin{aligned} u_{\delta,m,t} = -\pi_{u_{\delta,m}}\Big(-\operatorname{div}\Big(\big(|\nabla u|^2 + \delta \big)^{1/m-1/2} \nabla u \Big) \Big) \quad &\text{in } \mathbb{T}^n \times (0, T_\delta), \\ u_{\delta,m}\big|_{t=0} = u_0 \quad &\text{in } \mathbb{T}^n. \end{aligned} \tag{6.11}$$

Set $f_{\delta,m} := |du_{\delta,m}|^2 + \delta$, where $|du|^2$ is written in local coordinate $u = (u^1, u^2, \ldots, u^l)$ and by the metric h of M as $|du|^2 = \sum_{i,j,\alpha,\beta} h_{\alpha\beta}(u) \partial u^\alpha / \partial x_i \partial u^\beta / \partial x_j$. The following regularity property was proved in [9, Lemma 2]:

> "Let K be any positive constant such that $\|\nabla u_0\|_{L^\infty(\mathbb{T}^n)} \leq K$. There exists a positive constant ε_1 depending on $K, \mathbb{T}^n, M$, and m such that if $\sup_{0 \leq t < T_\delta} \|f_{\delta,m}(t,\cdot)\|_{L^{n/2}(\mathbb{T}^n)} \leq \varepsilon_1$, then $\|f_{\delta,m}\|_{L^\infty(\mathbb{T}^n \times [0,T_\delta))} \leq C$, where C is a constant depending on $K, \mathbb{T}^n, M$ and m."

By using these constants ε_1 and C, the constant $\varepsilon_0 > 0$ of Proposition 6.2 can be taken as

$$\varepsilon_0 := \frac{C^{(1+1/m-n)/2} \varepsilon_1^{n/2}}{(1+1/m) \sup\{1, 2^{(1/m-1)/2}\} 2^{1+n/2}}. \tag{6.12}$$

Now, by calculation, we can check that $\varepsilon_1' := \inf_{m \geq 3} \varepsilon_1$ is still positive and there exists $C' > 0$ independent of $m \geq 3$ such that for any $m \geq 3$, if $\sup_{0 \leq t < T_\delta} \|f_{\delta,m}(t,\cdot)\|_{L^{n/2}(\mathbb{T}^n)} \leq \varepsilon_0'$, then

$$\|f_{\delta,m}\|_{L^\infty(\mathbb{T}^n \times [0,T_\delta))} \leq C'. \tag{6.13}$$

Using these $\varepsilon_1' > 0$ and $C' > 0$, we define $\varepsilon_0' > 0$ by

$$\varepsilon_0' := \inf_{m \geq 3} \frac{C'^{(1+1/m-n)/2} \varepsilon_1'^{n/2}}{(1+1/m) \sup\{1, 2^{(1/m-1)/2}\} 2^{1+n/2}}. \tag{6.14}$$

Then, by the proof of [9, Theorem 1], one is able to prove that $u_0 \in C^{2+\alpha}(\mathbb{T}^n,\mathbb{R}^N)$, $\phi_m(u_0) \le \varepsilon_0'$, and $\|\nabla u_0\|_{L^\infty(M)} \le K$ yield the consequences of Proposition 6.2. □

Corollary 6.5. *For any $K > 0$, there exists $\varepsilon_0 > 0$ depending only on $\mathbb{T}^n, M$, and K such that if the initial data $u_0 : \mathbb{T}^n \to M$ satisfies the conditions:*

(i) $u_0 \in C^{2+\alpha}(\mathbb{T}^n,\mathbb{R}^N)$ $(0 < \alpha < 1)$,
(ii) $\|\nabla u_0\|_{L^\infty(\mathbb{T}^n)} \le K$,
(iii) *there exists $m_0 \in \mathbb{N}$, $m_0 \ge 3$, such that (3.14) holds,*

then, for any $m \ge m_0$, there exists uniquely a function $u_m : \mathbb{T}^n \times [0,\infty) \to M$ which satisfies all the consequences of Proposition 6.2 for the initial data u_0.

Proof. For $K > 0$, let $\varepsilon_0 > 0$ be the positive constant defined in Corollary 6.4.

Suppose that $u_0 : \mathbb{T}^n \to M$ satisfies conditions (i), (ii), (iii). For any $m \ge m_0$, we see that

$$\begin{aligned}
\phi_{\mathrm{pe},m}(u_0) &\le \frac{1}{1+1/m}\int_{\mathbb{T}^n}\left(\frac{1+1/m}{1+1/m_0}\,|\nabla u_0(x)|^{1+1/m_0} + \frac{1/m_0 - 1/m}{1+1/m_0}\right)dx \\
&\le \phi_{\mathrm{pe},m_0}(u_0) + \frac{1}{m_0+1}\prod_{i=1}^{n}\omega_i \\
&\le \varepsilon_0.
\end{aligned} \tag{6.15}$$

Thus, Corollary 6.4 assures the existence of $u_m : \mathbb{T}^n \times [0,\infty) \to M$ with the desired properties. □

We are now in position to prove the local existence theorem.

Proof of Theorem 3.4. It is sufficient to show that there exist $R > 0$ and $T > 0$ such that for approximate solutions u_m whose existence is assured by Corollary 6.5, the inclusion

$$\partial\Phi^T_{\mathrm{pe},m}(u_m) \subset B_R \tag{6.16}$$

holds for any $m \ge m_0$. Then, all the assumptions of Theorem 3.3 are satisfied and Theorem 3.3 yields the existence of a solution of (2.11) for this $T > 0$.

We see that the approximate equation $u_{m,t} \in -P_{u_m}(\partial\Phi^T_{\mathrm{pe},m}(u_m))$ is equivalent to the following equation:

$$u_{m,t} = \operatorname{div}\left(|\nabla u_m|^{1/m-1}\nabla u_m\right) + |\nabla u_m|^{1/m-1}A(u_m)(\nabla u_m,\nabla u_m), \tag{6.17}$$

where $A(u)$ denotes the second fundamental form of M at $u \in M$. Since the coefficients of $A(u)(\nabla u,\nabla u)$ smoothly depend on the value u on M, one can estimate that

$$\begin{aligned}
&|\nabla u_m(x,t)|^{1/m-1}\,|A(u_m(x,t))(\nabla u_m(x,t),\nabla u_m(x,t))| \\
&\quad \le C\,|\nabla u_m(x,t)|^{1/m+1} \quad \text{for any } (x,t) \in \mathbb{T}^n \times (0,+\infty),
\end{aligned} \tag{6.18}$$

where C is a positive constant depending only on M. By inequality (6.9) and assumption (iii) of Theorem 3.3, we know that there exists $R > 0$ such that $u_{m,t} \in B_R$ for any $m \ge m_0$.

Thus, if we prove that there exist $K' > 0$ and $T > 0$ such that

$$||\nabla u_m||_{L^\infty(\mathbb{T}^n\times[0,T])} \leq K' \quad \forall m \geq m_0, \tag{6.19}$$

then, by (6.17) and (6.18), we have that

$$\operatorname{div}\left(|\nabla u_m|^{1/m-1}\nabla u_m\right) \in B_{R'} \quad \forall m \geq m_0, \tag{6.20}$$

for some $R' > 0$ independent of m. This inclusion implies that (6.16) holds.

We will show inequality (6.19).

Fix $m \geq m_0$. We set $U := \{(x,t) \in \mathbb{T}^n \times [0,\infty) \mid \nabla u_m(x,t) \neq 0\}$. Since $\nabla u_m \in C^\beta(\mathbb{T}^n \times [0,\infty), \mathbb{R}^{nN})$, by a standard argument for a system of uniform parabolic equations (see [22]), we conclude that $u_m \in C^\infty(U)$.

We put $w_m(x,t) := |\nabla u_m(x,t)|^2$ and differentiate both sides in time. Noting equality (6.17), we see that

$$\begin{aligned}
w_{m,t} &= 2\langle \nabla u_m, \nabla u_{m,t}\rangle \\
&= 2\left\langle \nabla u_m, \nabla \operatorname{div}\left(|\nabla u_m|^{1/m-1}\nabla u_m\right)\right\rangle \\
&\quad + 2\sum_{l=1}^{N}\left\langle \nabla u_m^l, \nabla\left(|\nabla u_m|^{1/m-1}\right)\right\rangle A_l(u_m)(\nabla u_m, \nabla u_m) \\
&\quad + 2\langle \nabla u_m, \nabla A(u_m)(\nabla u_m, \nabla u_m)\rangle |\nabla u_m|^{1/m-1}.
\end{aligned} \tag{6.21}$$

Moreover, by calculation, we obtain

$$\begin{aligned}
&2\left\langle \nabla u_m, \nabla \operatorname{div}\left(|\nabla u_m|^{1/m-1}\nabla u_m\right)\right\rangle \\
&\quad = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} w_m + \sum_{i=1}^{n} b_i^1(x,t)\frac{\partial}{\partial x_i} w_m \\
&\qquad - 2|\nabla u_m|^{1/m-1}\sum_{l=1}^{N}\sum_{i,j=1}^{n}\left(\frac{\partial^2}{\partial x_i \partial x_j}u_m^l\right)^2, \\
&2\sum_{l=1}^{N}\left\langle \nabla u_m^l, \nabla\left(|\nabla u_m|^{1/m-1}\right)\right\rangle A_l(u_m)(\nabla u_m, \nabla u_m) = \sum_{i=1}^{n} b_i^2(x,t)\frac{\partial}{\partial x_i} w_m, \\
&2\langle \nabla u_m, \nabla A(u_m)(\nabla u_m, \nabla u_m)\rangle |\nabla u_m|^{1/m-1} \leq \sum_{i=1}^{n} b_i^3(x,t)\frac{\partial}{\partial x_i} w_m + C' {w_m}^{(3+1/m)/2},
\end{aligned} \tag{6.22}$$

where a_{ij}, b_i^1, b_i^2, b_i^3 are continuous functions in U, and C' is a positive constant depending only on M. More precisely, we see that a_{ij} is written as

$$a_{ij} = \left(\frac{1}{m} - 1\right)|\nabla u_m|^{1/m-3}\sum_{l=1}^{N}\frac{\partial}{\partial x_i}u_m^l \frac{\partial}{\partial x_j}u_m^l + |\nabla u_m|^{1/m-1}\delta_{ij}, \tag{6.23}$$

where δ_{ij} is Kronecker's delta. We can check that $(a_{ij}) > 0$ in U. Indeed, by Schwarz's inequality,

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j &= \left(\frac{1}{m}-1\right)\left|\nabla u_m\right|^{1/m-3}\sum_{l=1}^{N}\left(\langle\nabla u_m^l,\xi\rangle\right)^2 + \left|\nabla u_m\right|^{1/m-1}|\xi|^2 \\ &\geq \left(\frac{1}{m}-1\right)\left|\nabla u_m\right|^{1/m-3}\left|\nabla u_m\right|^2|\xi|^2 + \left|\nabla u_m\right|^{1/m-1}|\xi|^2 \\ &= \frac{1}{m}\left|\nabla u_m\right|^{1/m-1}|\xi|^2 > 0 \quad \text{for any } \xi \in \mathbb{R}^N \setminus \{0\}.\end{aligned} \tag{6.24}$$

Substituting the (in)equalities (6.22) into (6.21), we obtain the inequality

$$\begin{aligned}w_{m,t} \leq &\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j}w_m \\ &+\sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i}w_m + C' w_m{}^{(3+1/m)/2} \quad \text{for any } (x,t) \in U.\end{aligned} \tag{6.25}$$

Here, we have set $b_i := b_i^1 + b_i^2 + b_i^3$.

Let $f_m(t)$ be a solution of the following initial value problem:

$$\begin{aligned}f_{m,t} &= C' f_m^{(3+1/m)/2}, \\ f_m\big|_{t=0} &= \max\{1,K^2\}.\end{aligned} \tag{6.26}$$

Then f_m is of the form

$$f_m(t) = \left(\left(\max\{1,K^2\}\right)^{-(1+1/m)/2} - \frac{C'}{2}\left(1+\frac{1}{m}\right)t\right)^{-2/(1+1/m)}. \tag{6.27}$$

Evidently, f_m is strictly increasing and blows up when $t = t_m$, where t_m is given by

$$t_m := \frac{2}{C'(1+1/m)\left(\max\{1,K^2\}\right)^{(1+1/m)/2}}. \tag{6.28}$$

Set $v_m := w_m - f_m$. Since $w_m \in C(\mathbb{T}^n \times [0,+\infty))$, there exists $\delta > 0$ such that

$$v_m \leq 0 \quad \text{in } \mathbb{T}^n \times [t_m - \delta, t_m). \tag{6.29}$$

Plug v_m into (6.21); we obtain that

$$\begin{aligned}v_{m,t} \leq &\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j}v_m \\ &+\sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i}v_m + d(x,t)v_m \quad \text{in } U \cap (\mathbb{T}^n \times [0,t_m-\delta)),\end{aligned} \tag{6.30}$$

where $d(x,t)$ is a continuous function in $U \cap (\mathbb{T}^n \times [0,t_m-\delta))$.

For a positive constant $\lambda > 0$, we set $v_{m,\lambda} := e^{-\lambda t} v_m$ and differentiate both sides in time. Then, by (6.30), we observe that

$$
\begin{aligned}
v_{m,\lambda,t} \le \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} v_{m,\lambda} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial}{\partial x_i} v_{m,\lambda} \\
+ (d(x,t) - \lambda) v_{m,\lambda} \quad \text{in } U \cap (\mathbb{T}^n \times [0, t_m - \delta)).
\end{aligned}
\tag{6.31}
$$

By taking λ sufficiently large, we may assume that $d(x,t) - \lambda < 0$ in $U \cap (\mathbb{T}^n \times [0, t_m - \delta))$. Thus, the standard maximum principle for parabolic equations assures that there exists a boundary point $(\hat{x}, \hat{t}) \in \partial(U \cap (\mathbb{T}^n \times [0, t_m - \delta)))$ such that

$$
v_{m,\lambda}(\hat{x}, \hat{t}) = \sup_{(x,t) \in U \cap (\mathbb{T}^n \times [0, t_m - \delta))} v_{m,\lambda}(x,t). \tag{6.32}
$$

We obviously observe that at least one of the following properties holds:

(1) $\hat{t} = 0$,
(2) $\hat{t} = t_m - \delta$,
(3) $(\hat{x}, \hat{t}) \notin U$.

For each case, it is easy to check that $v_m(\hat{x}, \hat{t}) \le 0$. In conclusion, the inequality $w_m \le f_m$ holds in $U \cap (\mathbb{T}^n \times [0, t_m - \delta))$. Moreover, the definition of U and (6.29) yield that

$$
w_m \le f_m \quad \text{in } \mathbb{T}^n \times [0, t_m), \ \forall m \ge m_0. \tag{6.33}
$$

By (6.27) and (6.28), we obtain that if $m_1 \le m_2$, then

$$
t_{m_1} \le t_{m_2}, \quad f_{m_1}(t) \ge f_{m_2}(t) \quad \text{in } [0, t_{m_1}]. \tag{6.34}
$$

Let $f(t)$ be a solution of

$$
\begin{aligned}
f_t &= C' f^{1+1/2}, \\
f|_{t=0} &= \max\{1, K^2\}.
\end{aligned}
\tag{6.35}
$$

Then f blows up when $t_0 = 2/(C' \sqrt{\max\{1, K^2\}})$ and we observe that $t_m < t_0$ for any $m \in \mathbb{N}$ and $t_m \nearrow t_0$ as $m \to +\infty$.

Now take any $T \in (0, t_0)$ and fix it. Then there exists a natural number $m_T \ge m_0$ such that the blowup time t_{m_T} of f_{m_T} is larger than T. Noting (6.33) and (6.34), we see that for any $m \ge m_T$,

$$
w_m \le f_m \le f_{m_T} \quad \text{in } \mathbb{T}^n \times [0, T]. \tag{6.36}
$$

In other words,

$$
|\nabla u_m(x,t)| \le \sqrt{f_{m_T}(T)} \quad \text{for any } (x,t) \in \mathbb{T}^n \times [0, T] \text{ and any } m \ge m_T. \tag{6.37}
$$

If we set

$$K' := \max_{m_0 \leq m \leq m_T - 1} \left\{ \|\nabla u_m\|_{L^\infty(\mathbb{T}^n \times [0,T])}, \sqrt{f_{m_T}(T)} \right\}, \tag{6.38}$$

then we finally obtain (6.19).

Thus, Theorem 3.3 yields the existence of a solution of (2.11) in $L^2(0,T;L^2(\mathbb{T}^n,\mathbb{R}^N))$. The energy equality (3.15) follows by the same argument as Proposition 5.4. □

Appendix

Here, we state several propositions which are used to prove Propositions 5.3 and 5.4 and need some technical arguments to be shown for convex functionals in a general setting. First we give one proposition which is necessary to show Proposition 5.4. The result was proved in [8]. But we give the proof for completeness.

PROPOSITION .6 (see [8, Lemma 3.3]). *Let ϕ be a proper, lower semicontinuous, convex functional on H and $v \in W^{1,2}(0,T;H)$ with $v(t) \in D(\partial\phi)$ a.e. $t \in (0,T)$. Then, the function $t \mapsto \phi(v(t))$ is absolutely continuous on $[0,T]$. Moreover*

$$\frac{d}{dt}\phi(v(t)) = \left\langle h, \frac{dv}{dt}(t) \right\rangle_H, \quad \forall h \in \partial\phi(v(t)),\ a.e.\ t \in (0,T). \tag{.1}$$

Proof. For each $\lambda > 0$, we put $g_\lambda(t) = \partial\phi^\lambda(v(t))$, where $\phi^\lambda(v(t)) = 1/2\lambda\|x - J^\lambda v(t)\|_H^2 + \phi(J^\lambda v(t))$ and $J^\lambda v(t) := (I + \lambda\partial\phi)^{-1}v(t)$. Here, we note that by using the canonical extension of $\partial\phi$ to $L^2(0,T;H)$, we can take $l \in L^2(0,T;H)$ such that $l(t) \in \partial\phi(v(t))$ a.e. $t \in (0,T)$. Then, we easily see that

$$\begin{gathered} \|g_\lambda(t)\|_H \leq \|\partial^0\phi(v(t))\|_H \leq \|l(t)\|_H, \quad \forall t \in (0,T), \\ g_\lambda(t) \longrightarrow \partial^0\phi(v(t)) \quad \text{a.e. } t \in (0,T), \text{ as } \lambda \longrightarrow 0, \end{gathered} \tag{.2}$$

where $\partial^0\phi(v(t))$ denotes the minimal section of $\partial\phi(v(t))$. It follows from (.2) that

$$g_\lambda \longrightarrow \partial^0\phi(v) \quad \text{in } L^2(0,T;H), \text{ as } \lambda \longrightarrow 0. \tag{.3}$$

Since $d\phi^\lambda(v(t))/dt = \langle \partial\phi^\lambda(v(t)), dv(t)/dt\rangle_H$ a.e. $t \in (0,T)$, we see that

$$\phi^\lambda(v(t_2)) - \phi^\lambda(v(t_1)) = \int_{t_1}^{t_2} \left\langle \partial\phi^\lambda(v(t)), \frac{dv}{dt}(t) \right\rangle_H dt, \quad \forall t_1, t_2 \in [0,T]. \tag{.4}$$

Passing in (.4) to the limit with $\lambda \to 0$, we get

$$\phi(v(t_2)) - \phi(v(t_1)) = \int_{t_1}^{t_2} \left\langle \partial^0\phi(v(t)), \frac{dv}{dt}(t) \right\rangle_H dt, \tag{.5}$$

which implies that the function $t \mapsto \phi(v(t))$ is absolutely continuous on $[0,T]$.

Now we define the set

$$E := \{t \in (0,T) \mid v(t) \text{ and } \phi(v(t)) \text{ are differentiable at } t,\ v(t) \in D(\partial\phi)\}. \tag{.6}$$

For any $t \in E$ and $h \in \partial\phi(v(t))$, we have

$$\phi(z) - \phi(v(t)) \geq \langle h, z - v(t)\rangle_H, \quad \forall z \in H. \tag{.7}$$

By taking in (.7) $z = v(t+\varepsilon)$ with $\varepsilon > 0$, dividing by ε, and passing to the limit with $\varepsilon \to 0$, we get

$$\frac{d}{dt}\phi(v(t)) \geq \left\langle h, \frac{dv}{dt}(t)\right\rangle_H. \tag{.8}$$

Similarly, by taking in (.7) $z = v(t-\varepsilon)$ with $\varepsilon > 0$, we get

$$\frac{d}{dt}\phi(v(t)) \leq \left\langle h, \frac{dv}{dt}(t)\right\rangle_H. \tag{.9}$$

Therefore, it follows from (.8) and (.9) that

$$\frac{d}{dt}\phi(v(t)) = \left\langle h, \frac{dv}{dt}(t)\right\rangle_H, \quad \forall h \in \partial\phi(v(t)),\ \forall t \in E. \tag{.10}$$

□

Next we show one proposition for Mosco convergence of convex functional, which assures the statement of Proposition 5.3. We follow the arguments in [4]. We set some notations used below in advance. Let H denote a real Hilbert space and let ϕ_m $(m = 1,2,\ldots)$ and ϕ be proper, convex, and lower semicontinuous functionals on H. Define functionals Φ_m $(m = 1,2,\ldots)$ and Φ on $L^2(0,T;H)$ by $\Phi_m(u) := \int_0^T \phi_m(u)dt$ and $\Phi(u) := \int_0^T \phi(u)dt$ for $u \in L^2(0,T;H)$.

The proposition we are going to prove can be stated as follows.

Proposition .7. *If ϕ_m converges to ϕ on H in the sense of Mosco as $m \to +\infty$, then Φ_m also converges to Φ on $L^2(0,T;H)$ in the sense of Mosco.*

Remark .8. This is generalized to time-dependent ϕ_m^t, ϕ^t by Kenmochi [18] under suitable assumptions.

We recall a property for Mosco-converging energy functional.

Proposition .9 (see [4] or [5]). *The following properties are equivalent:*

(a) $\phi_m \to \phi$ *in the sense of Mosco,*

(b) $\partial\phi_m \to \partial\phi$ *in the sense of resolvent, that is,*

$$(I + \lambda\partial\phi_m)^{-1}x \longrightarrow (I + \lambda\partial\phi)^{-1}x \quad \text{in } H \text{ for any } \lambda > 0 \text{ and any } x \in H, \tag{.11}$$

and there exist $(u,v) \in \partial\phi$ and $(u_m, v_m) \in \partial\phi_m$ such that $u_m \to u, v_m \to v$ strongly, and $\phi_m(u_m) \to \phi(u)$.

Remark .10. Note that the convergence $\partial\phi_m \to \partial\phi$ in the sense of resolvent is equivalent to the convergence $\partial\phi_m \to \partial\phi$ in the sense of Graph (see [4] or [5]).

The previous proposition means that to show property (b) for Φ_m and Φ is sufficient to attain our purpose. We prepare some lemmas to show property (b).

LEMMA .11. *Assume that ϕ_m converges to ϕ on H in the sense of Mosco. Then the following properties hold.*

(i) *There exist constants $c_1, c_2 > 0$ such that*

$$\phi_m(x) + c_1\|x\|_H + c_2 \geq 0 \quad \text{for any } x \in H \text{ and any } m \in \mathbb{N}. \tag{.12}$$

(ii) *For any $\lambda > 0$ and $x \in H$,*

$$\phi_m^\lambda(x) \longrightarrow \phi^\lambda(x) \quad \text{as } m \longrightarrow +\infty, \tag{.13}$$

where

$$\begin{aligned} \phi_m^\lambda(x) &:= \frac{1}{2\lambda}\|x - J_m^\lambda x\|_H^2 + \phi_m(J_m^\lambda x), \\ \phi^\lambda(x) &:= \frac{1}{2\lambda}\|x - J^\lambda x\|_H^2 + \phi(J^\lambda x), \\ J_m^\lambda x &:= (I + \lambda\partial\phi_m)^{-1}x, \qquad J^\lambda x := (I + \lambda\partial\phi)^{-1}x. \end{aligned} \tag{.14}$$

Proof. (i) Suppose that the conclusions were false. Then there would exist a subsequence $\{\phi_{m_k}\}_{k=1}^{+\infty} \subset \{\phi_m\}_{m=1}^{+\infty}$ and a sequence $\{y_k\}_{k=1}^{+\infty} \subset H$ such that

$$\phi_{m_k}(y_k) + k^2\|y_k\| + k^2 < 0 \quad \text{for any } k \in \mathbb{N}. \tag{.15}$$

Fix $x_0 \in D(\phi)$. The definition of Mosco convergence yields that there exists $\{x_m\}_{m=1}^{+\infty} \subset H$ such that

$$x_m \longrightarrow x_0 \quad \text{strongly in } H, \qquad \phi_m(x_m) \longrightarrow \phi(x_0) \quad \text{as } m \longrightarrow +\infty. \tag{.16}$$

For each $k \in \mathbb{N}$, set

$$z_k := \varepsilon_k y_k + (1 - \varepsilon_k)x_{m_k}, \qquad \varepsilon_k := \frac{1}{k(1 + \|y_k\|_H)}. \tag{.17}$$

By (.16) and (.17), we obviously see that

$$\begin{aligned} \varepsilon_k &\longrightarrow 0 \quad \text{as } k \longrightarrow +\infty, \\ 0 < \varepsilon_k &< 1 \quad \text{for any } k \in \mathbb{N}, \\ z_k &\longrightarrow x_0 \quad \text{strongly in } H, \text{ as } k \longrightarrow +\infty. \end{aligned} \tag{.18}$$

Moreover, the convexity of ϕ_{m_k}, (.15), and (.17) yield that

$$\begin{aligned} \phi_{m_k}(z_k) &\leq \varepsilon_k\phi_{m_k}(y_k) + (1 - \varepsilon_k)\phi_{m_k}(x_{m_k}) \\ &< -k^2\varepsilon_k(\|y_k\|_H + 1) + (1 - \varepsilon_k)\phi_{m_k}(x_{m_k}) \\ &= -k + (1 - \varepsilon_k)\phi_{m_k}(x_{m_k}). \end{aligned} \tag{.19}$$

Sending $k \to +\infty$ in (.19), by (.16) and (.18), we observe that

$$\limsup_{k\to+\infty} \phi_{m_k}(z_k) = -\infty. \tag{.20}$$

On the other hand, the second convergence of (.18) and the definition of Mosco convergence assure that

$$\liminf_{k\to+\infty} \phi_{m_k}(z_k) \geq \phi(x_0). \tag{.21}$$

We have by (.20) and (.21) that $\phi(x_0) = -\infty$. This is a contradiction since we took $x_0 \in D(\phi)$.

(ii) Since $\phi_m \to \phi$ on H in the sense of Mosco, Proposition .9 implies that

$$(I + \lambda\partial\phi_m)^{-1}x \longrightarrow (I + \lambda\partial\phi)^{-1}x \quad \text{in } H, \tag{.22}$$

and there are $(\xi_m, \eta_m) \in \partial\phi_m$ and $(\xi, \eta) \in \partial\phi$ such that

$$\xi_m \longrightarrow \xi, \quad \eta_m \longrightarrow \eta, \quad \phi_m(\xi_m) \longrightarrow \phi(\xi) \quad \text{as } m \longrightarrow +\infty. \tag{.23}$$

Now, for a fixed $\lambda > 0$, we put $z_m = \xi_m + \lambda\eta_m$ and $z = \xi + \lambda\eta$. Then we easily see that

$$z_m \longrightarrow z \quad \text{in } H, \text{ as } m \longrightarrow +\infty. \tag{.24}$$

In addition, since $z_m \in (I + \lambda\phi_m)(\xi_m)$, we have $\xi_m = J_m^\lambda z_m$. Similarly, we can get $\xi = J^\lambda z$. Thus we see that

$$\phi_m(J_m^\lambda z_m) \longrightarrow \phi(J^\lambda z) \quad \text{as } m \longrightarrow +\infty, \tag{.25}$$

which implies that $\phi_m^\lambda(z_m) \to \phi^\lambda(z)$ as $m \to +\infty$. Therefore, we observe that

$$\begin{aligned} &\phi_m^\lambda(x) = \phi_m^\lambda(z_m) + \int_0^1 \langle \partial\phi_m^\lambda(z_m + \tau(x - z_m)), x - z_m\rangle_H d\tau \\ &\Longrightarrow \phi^\lambda(x) = \phi^\lambda(z) + \int_0^1 \langle \partial\, \phi^\lambda(z + \tau(x - z)), x - z\rangle_H d\tau. \end{aligned} \tag{.26}$$

□

Lemma .12. *Assume that ϕ_m converges to ϕ in the sense of Mosco. Then there exists a sequence $\{b_m\}_{m=1}^{+\infty} \subset W^{1,2}(0,T;H)$ such that*

$$\begin{aligned} &\|b_m(t)\|_H \leq M, \quad \phi_m(b_m(t)) \leq M, \\ &\|b_m'\|_{L^2(0,T;H)} \leq M \quad \textit{for any } m \in \mathbb{N} \textit{ and a.e. } t \in [0,T], \end{aligned} \tag{.27}$$

where M is a positive constant independent of m and t.

Proof. Fix any $b_0 \in D(\phi)$. Then, by the definition of Mosco convergence, we obtain $\{b_{0,m}\}_{m=1}^{+\infty} \subset H$ such that $b_{0,m} \to b_0$ strongly and $\phi_m(b_{0,m}) \to \phi(b_0)$ as $m \to +\infty$.

Let $b_m(t) \in C([0,T],H)$ be a solution of

$$\begin{aligned} b_{m,t} &\in -\partial\phi_m(b_m(t)) \quad \text{in } H, \\ b_m(0) &= b_{0,m}. \end{aligned} \tag{.28}$$

Then, $\{b_m\}_{m=1}^{+\infty}$ is the desired sequence. □

Lemma .13. *Assume $\phi_m \to \phi$ in the sense of Mosco. Then, $\partial\Phi_m \to \partial\Phi$ in the sense of resolvent.*

Proof. Take any $f \in L^2(0,T;H)$ and set $u_m := (I+\lambda\partial\Phi_m)^{-1}f$ and $u := (I+\lambda\partial\Phi)^{-1}f$.

We show that $u_m \to u$ strongly in $L^2(0,T;H)$ as $m \to +\infty$.

Note that

$$u_m(t) = \left(I+\lambda\partial\phi_m\right)^{-1}f(t), \quad u(t) = (I+\lambda\partial\phi)^{-1}f(t) \quad \text{a.e. } t \in [0,T]. \tag{.29}$$

Since ϕ_m converges to ϕ in the sense of Mosco, Proposition .9 yields that $\partial\phi_m$ converges to $\partial\phi$ in the sense of resolvent. Thus, we obtain that

$$u_m(t) \longrightarrow u(t) \quad \text{strongly in } H, \text{ for a.e. } t \in [0,T]. \tag{.30}$$

Since $(f(t)-u_m(t))/\lambda \in \partial\phi_m(u_m(t))$ a.e. $t \in [0,T]$, we obtain by the definition of subdifferential that

$$\phi_m(b_m(t)) \geq \phi_m(u_m(t)) + \left\langle \frac{f(t)-u_m(t)}{\lambda}, b_m(t)-u_m(t)\right\rangle_H \quad \text{for a.e. } t \in [0,T], \tag{.31}$$

where $\{b_m\}_{m=1}^{+\infty}$ is the sequence given in Lemma .12.

Combining the inequality of Lemma .11 with (.31), we see that

$$\begin{aligned} \phi_m(b_m(t)) &\geq -c_1\|u_m(t)\|_H - c_2 + \frac{1}{\lambda}\langle f(t),b_m(t)\rangle_H - \frac{1}{\lambda}\langle f(t),u_m(t)\rangle_H \\ &\quad - \frac{1}{\lambda}\langle u_m(t),b_m(t)\rangle_H + \frac{1}{\lambda}\|u_m(t)\|_H^2 \\ &\geq -\varepsilon\|u_m(t)\|_H^2 - \frac{1}{4\varepsilon}c_1^2 - c_2 - \frac{1}{\lambda}\|f(t)\|_H\|b_m(t)\|_H \\ &\quad - \varepsilon\|u_m(t)\|_H^2 - \frac{1}{4\varepsilon\lambda^2}\|f(t)\|_H^2 \\ &\quad - \varepsilon\|u_m(t)\|_H^2 - \frac{1}{4\varepsilon\lambda^2}\|b_m(t)\|_H^2 + \frac{1}{\lambda}\|u_m(t)\|_H^2 \\ &= \left(\frac{1}{\lambda} - 3\varepsilon\right)\|u_m(t)\|_H^2 - \frac{1}{4\varepsilon}c_1^2 - c_2 - \frac{1}{\lambda}\|f(t)\|_H\|b_m(t)\|_H \\ &\quad - \frac{1}{4\varepsilon\lambda^2}\|f(t)\|_H^2 - \frac{1}{4\varepsilon\lambda^2}\|b_m(t)\|_H^2. \end{aligned} \tag{.32}$$

Now, putting $\varepsilon = 1/6\lambda$ into (.32) and by Lemma .12, we obtain that

$$\begin{aligned}
\|u_m(t)\|_H^2 &\le 2\lambda\phi_m(b_m(t)) \\
&\quad + 2\lambda\left(\frac{3\lambda}{2}c_1^2 + c_2 + \frac{1}{\lambda}\|f(t)\|_H\|b_m(t)\|_H + \frac{3}{2\lambda}\|f(t)\|_H^2 + \frac{3}{2\lambda}\|b_m(t)\|_H^2\right) \\
&\le 2\lambda M + 3\lambda^2 c_1^2 + 2\lambda c_2 + 2M\|f(t)\|_H + 3\|f(t)\|_H^2 + 3M^2 := g(t) \\
&\in L^1(0,T) \quad \text{for any } m \in \mathbb{N}.
\end{aligned} \tag{.33}$$

Thus, by (.30), Lebesgue's convergence theorem yields that $u_m \to u$ in $L^2(0,T;H)$ as $m \to +\infty$.

We have shown that $\partial\Phi_m \to \partial\Phi$ in the sense of resolvent. □

Lemma .14. *Assume that $\phi_m \to \phi$ in the sense of Mosco. Then there exist $(u,v) \in \partial\Phi$ and $(u_m, v_m) \in \partial\Phi_m$ such that $u_m \to u, v_m \to v$ strongly in $L^2(0,T;H)$, and $\Phi_m(u_m) \to \Phi(u)$ as $m \to +\infty$.*

Proof. Take any $f \in L^2(0,T;H)$ and fix it. If we set $u_m := (I + \lambda\partial\Phi_m)^{-1}f$ and $u := (I + \lambda\partial\Phi)^{-1}f$, then Lemma .13 yields that

$$u_m \longrightarrow u \quad \text{strongly in } L^2(0,T;H), \text{ as } m \longrightarrow +\infty. \tag{.34}$$

Therefore, if we set $v_m := (f - u_m)/\lambda$ and $v := (f - u)/\lambda$, then we easily observe that $v_m \in \partial\Phi_m(u_m), v \in \partial\Phi(u)$, and $v_m \to v$ strongly in $L^2(0,T;H)$ as $m \to +\infty$.

It is sufficient to show that $\Phi_m(u_m) \to \Phi(u)$ in $L^2(0,T;H)$ as $m \to +\infty$ to attain our purpose.

Lemma .11 (ii) assures that for any $\lambda > 0$ and $x \in H$,

$$\phi_m^\lambda(x) \longrightarrow \phi^\lambda(x) \quad \text{as } m \longrightarrow +\infty. \tag{.35}$$

Now, by (.34), we also see that

$$\begin{aligned}
\partial\Phi_m^\lambda(f) &= \frac{f - J_m^\lambda f}{\lambda} = \frac{f - u_m}{\lambda} \\
&\longrightarrow \partial\Phi^\lambda(f) = \frac{f-u}{\lambda} \quad \text{in } L^2(0,T;H), \text{ as } m \longrightarrow +\infty.
\end{aligned} \tag{.36}$$

Therefore (by taking a subsequence of $\{m\}$ if necessary),

$$\partial\phi_m^\lambda(f(t)) \longrightarrow \partial\phi^\lambda(f(t)) \quad \text{strongly in } H, \text{ a.e. } t \in [0,T]. \tag{.37}$$

Note that

$$\begin{aligned}
\phi_m^\lambda(f(t)) &= \frac{1}{2\lambda}\|f(t) - J_m^\lambda f(t)\|_H^2 + \phi(J_m^\lambda f(x)) \\
&= \frac{\lambda}{2}\|\partial\phi_m^\lambda(f(t))\|_H^2 + \phi(u_m(t)).
\end{aligned} \tag{.38}$$

Then, by the convergences (.35), (.37) and equality (.38), we see that

$$\begin{aligned}\phi_m(u_m(t)) &= \phi_m^\lambda(f(t)) - \frac{\lambda}{2}\|\partial\phi_m^\lambda(f(t))\|_H^2 \\ &\longrightarrow \phi^\lambda(f(t)) - \frac{\lambda}{2}\|\partial\phi^\lambda(f(t))\|_H^2 = \phi^\lambda(J^\lambda f(t)) = \phi(u(t))\end{aligned} \tag{.39}$$

as $m \to +\infty$ for a.e. $t \in [0,T]$ and any $\lambda > 0$.

Now inequalities (.31), (.33) and Lemma .12 yield that

$$\begin{aligned}\phi_m(u_m(t)) &\le \phi_m(b_m(t)) + \left\|\frac{f(t)-u_m(t)}{\lambda}\right\|_H \|b_m(t)-u_m(t)\|_H \\ &\le M + \frac{1}{\lambda}\left(\|f(t)\|_H + \sqrt{g(t)}\right)\left(M + \sqrt{g(t)}\right) \\ &\le M + \frac{M}{\lambda}\|f(t)\|_H + \frac{1}{2\lambda}\|f(t)\|_H^2 \\ &\quad + \frac{1}{2\lambda}g(t) + \frac{M^2}{2\lambda} + \frac{1}{2\lambda}g(t) + \frac{1}{\lambda}g(t) := l(t).\end{aligned} \tag{.40}$$

Note that $l(t) \in L^1(0,T)$.

Moreover, Lemma .11, (.33), and (.40) yield that

$$\begin{aligned}|\phi_m(u_m(t))| &\le \phi_m(u_m(t)) + 2(c_1\|u_m(t)\|_H + c_2) \\ &\le l(t) + 2c_1\sqrt{g(t)} + 2c_2 \\ &\le l(t) + c_1^2 + g(t) + 2c_2 \in L^1(0,T).\end{aligned} \tag{.41}$$

By (.39) and (.41), we can apply Lebesgue's convergence theorem and obtain that $\Phi_m(u_m) \to \Phi(u)$ as $m \to +\infty$. □

All the necessary lemmas have been prepared to show Proposition .7.

Proof of Propsition .7. Lemmas .13 and .14 imply that condition (b) of Proposition .9 holds for Φ_m $(m = 1,2,\dots)$ and Φ. Thus, the Mosco convergence $\Phi_m \to \Phi$ follows by Proposition .9. □

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