MULTIPLICITY OF SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEM IN EXTERIOR DOMAINS WITH NEUMANN CONDITIONS

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Received 7 January 2003

We study the existence and multiplicity of solutions for a class of quasilinear elliptic problem in exterior domain with Neumann boundary conditions.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of solutions for the following class of quasilinear elliptic problem with Neumann conditions:

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = Q(x)f(u) \quad \text{in } \mathbb{R}^N \setminus \Omega, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $1 < p < N$, and $\Delta_p u$ is the p-Laplacian operator, that is,

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right), \tag{1.2}$$

Q is a continuous function satisfying

$$Q(x) > 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \qquad \lim_{|x|\to\infty} Q(x) = \bar{Q} > 0, \tag{1.3}$$

and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is an odd function of C^1 class satisfying the following hypotheses.

(f_1) There exists $2 \le p < q+1 < \eta+1 < p^* = Np/(N-p)$ verifying

$$\lim_{|s|\to 0} \frac{|f'(s)|}{|s|^{q-1}} = 0, \qquad \limsup_{|s|\to\infty} \frac{|f'(s)|}{|s|^{\eta-1}} < +\infty. \tag{1.4}$$

Abstract and Applied Analysis 2004:3 (2004) 251–268
2000 Mathematics Subject Classification: 35A17, 35H30, 35J65
URL: http://dx.doi.org/10.1155/S1085337504310018

(f_2) There exists $\theta \in (p, \eta+1]$ such that

$$0 < \theta F(s) \leq s f(s) \quad \forall s \neq 0. \tag{1.5}$$

(f_3) The function $s \to f(s)/s^{p-1}$ is increasing in $(0, +\infty)$.

In [5], Benci and Cerami studied the problem (1.1) assuming that $p = 2$, $Q \equiv 1$, and $f(u) = |u|^{\eta-1}u$ with $1 < \eta < (N+2)/(N-2)$. They showed that (1.1), with Dirichlet condition, has not a *ground-state* solution. However, Esteban [8] proved that the same problem with Neumann condition has a *ground-state*. We recall that by a *ground-state* we mean solution of (1.1) with minimum energy.

In [6], Cao also studied the problem (1.1) for $p = 2$, $f(u) = |u|^{\eta-1}u$, and Q satisfying the condition (1.3). The author showed that the problem has at least two solutions, where the first solution is related to the minimization problem

$$I(\Omega) = \inf_{u \in H^1(\mathbb{R}^N \setminus \Omega)} \left\{ \int_{\mathbb{R}^N \setminus \Omega} \left(|\nabla u|^2 + |u|^2\right) : \int_{\mathbb{R}^N \setminus \Omega} Q(x)|u|^{\eta+1} = 1 \right\} \tag{1.6}$$

and the second solution is *nodal*, that is, a solution of (1.1) with change of sign. In that paper, one of the main points is a compactness global result proved in [5].

In this work, motivated by [6], we prove the existence of *ground-state* and *nodal* solutions to (1.1). We used variational methods such as mountain pass theorem without Palais-Smale condition (see [14]) to obtain a positive ground-state solution. In relation to nodal solutions, we apply the implicit function theorem. Here, we adapt to p-Laplacian operator and to a general nonlinearity f some ideas found in [5, 6, 13]. However, the arguments explored in the above articles cannot be carried out straightforwardly in our case because some estimates become more subtle to be established. A main point in this paper is a version of a *compactness global lemma (CGL)* to study the behavior of Palais-Smale sequences, which is a version for p-Laplacian from a result shown by Benci and Cerami in [5].

To state our main results, we need some definitions and notations.

If h is a Lebesgue integrable function and B is a measurable set, we write $\int_B h$ for $\int_B h\,dx$. Moreover, if $h \in W^{1,p}(\mathbb{R}^N \setminus \Omega)$, we denote by $\|h\|$ its usual norm. We denote by $I : W^{1,p}(\mathbb{R}^N \setminus \Omega) \to \mathbb{R}$ the functional related to (1.1) given by

$$I(u) = \frac{1}{p} \int_{\mathbb{R}^N \setminus \Omega} \left(|\nabla u|^p + |u|^p\right) - \int_{\mathbb{R}^N \setminus \Omega} Q(x)F(u), \tag{1.7}$$

where $F(u) = \int_0^u f(t)\,dt$. We have the following problem:

$$-\Delta_p u + |u|^{p-2}u = \bar{Q} f(u) \quad \text{in } \mathbb{R}^N,\ u \in W^{1,p}(\mathbb{R}^N), \tag{1.8}$$

and by $I_\infty : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ the functional related to (1.8) given by

$$I_\infty(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left(|\nabla u|^p + |u|^p\right) - \int_{\mathbb{R}^N} \bar{Q} F(u). \tag{1.9}$$

Concerning the existence of *ground-state*, we have the following result.

Theorem 1.1. *Suppose that f satisfies (f_1), (f_2), and (f_3), $p \geq 2$ and the function Q satisfies (1.3) and*

$$Q(x) \geq \bar{Q} - Ce^{-m|x|}, \quad |x| \longrightarrow \infty, \tag{1.10}$$

where C is a positive constant and $m > p(q+1)/((q+1)-p)$. Then, (1.1) has a positive ground-state solution.

Using the *ground-state* obtained in the above theorem together with some estimates given in Sections 4 and 5, we establish a second theorem which shows the existence of a nodal solution. For this result, we will need the following hypothesis:

(f_4) there exists $\eta \leq \sigma \leq p^* - 1$ verifying

$$f'(t)t + (1-p)f(t) \geq C|t|^{\sigma-1}t, \quad \eta \leq \sigma \leq p^* - 1. \tag{1.11}$$

Theorem 1.2. *Suppose that f satisfies (f_1), (f_2), (f_3), and (1.11), $p \geq 2$, and the function Q satisfies (1.3) and*

$$Q(x) \geq \bar{Q} + Ce^{-\gamma|x|}, \quad \forall x \in \mathbb{R}^N, \tag{1.12}$$

where C is a positive constant and $\gamma < q/(q+1)$. Then, (1.1) has a nodal solution.

Remark 1.3. In the proof of Theorems 1.1 and 1.2, we used variational methods and adapted some arguments explored by Cao in [6]. These results complete the study made in [6] in the sense that we consider the p-Laplacian operator and a general class of nonlinearity.

2. Technical lemmas

In this section, we state some results necessary for the proof of Theorems 1.1 and 1.2. It is known that, under assumptions (f_1), (f_2), and (f_3), the arguments used in [3] show that (1.8) possesses a *ground-state* solution. About the behavior of the solutions at infinity, we have the following result.

Lemma 2.1. *Any positive solution $\bar{u} \in W^{1,p}(\mathbb{R}^N)$ of problem (1.8) with $p \geq 2$ has the following asymptotic behavior:*

$$\begin{gathered} \lim_{|x|\to\infty} \bar{u}(x) = 0, \\ C_1 e^{-a|x|} \leq \bar{u}(x) \leq C_2 e^{-b|x|} \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{2.1}$$

where $C_1, C_2 > 0$ are positive constants and $0 < b < 1 < a$. Moreover, numbers a, b can be chosen of the form $a = 1 + \delta$ and $b = 1 - \delta$ for $\delta > 0$.

Proof. The proof follows by similar arguments found in [11, Theorem 3.1]. □

Remark 2.2. With the same arguments used in the proof of the above lemma, we can show that all positive weak solutions of (1.1) have exponential decaying.

The next lemma shows an important inequality related to the vectors of $\mathbb{R}^N$, and its proof can be found in [15, Lemma 4.2].

Lemma 2.3. *For all $v,w \in \mathbb{R}^N$ with $N \geq 1$ and $p \geq 2$,*

$$\left(|v|^{p-2}v - |w|^{p-2}w\right)(v-w) \geq |v-w|^p. \tag{2.2}$$

Lemma 2.4. *Let $F \in C^2(\mathbb{R},\mathbb{R}_+)$ be a convex and even function such that $F(0)=0$ and $f(s) = F'(s) \geq 0$ for all $s \in [0,\infty)$. Then, for all $u,v \geq 0$,*

$$\left|F(u-v) - F(u) - F(v)\right| \leq 2\left(f(u)v + f(v)u\right). \tag{2.3}$$

Proof. Indeed, we have two cases to consider. If $v \leq u$, by convexity, we have

$$\frac{F(v)-F(0)}{v-0} \leq f(u), \tag{2.4}$$

that is, $F(v) \leq f(u)v$. On the other hand, since $f' = F'' \geq 0$, we have that f is nondecreasing and consequently

$$\left|F(u-v) - F(u)\right| \leq v\int_0^1 f(u-tv)\,dt \leq vf(u). \tag{2.5}$$

Therefore,

$$\left|F(u-v) - F(u) - F(v)\right| \leq 2vf(u). \tag{2.6}$$

If $u \leq v$, we repeat the above argument to find

$$\left|F(u-v) - F(u) - F(v)\right| \leq 2uf(v). \tag{2.7}$$

From (2.6) and (2.7) the lemma follows. □

Remark 2.5. Notice that, if f satisfies (f_1), (f_2), and (f_3), the primitive F of f verifies the hypothesis from Lemma 2.4.

3. Behavior of the Palais-Smale sequence

In this section, we prove some important lemmas to establish the CGL. The CGL is a key result for the understanding of the behavior of Palais-Smale sequence. We recall that a sequence $(u_n) \subset W^{1,p}(\mathbb{R}^N \setminus \Omega)$ is called a $(PS)_c$ sequence for I, at level $c \in \mathbb{R}$, if

$$I(u_n) \longrightarrow c, \qquad I'(u_n) \longrightarrow 0. \tag{3.1}$$

Lemma 3.1. *Let $B \subseteq \mathbb{R}^N$ be an open set and $g_n : B \to \mathbb{R}$ with $g_n \in L^t(B) \cap L^{p^*}(B)$ $(t \geq p)$, $|g_n|_{L^{p^*}}(B) \leq C$, and $g_n(x) \to 0$ a.e. in B.*

(I) *Suppose that f satisfies (f_1). Then,*

$$\int_B \left|F(g_n + w) - F(g_n) - F(w)\right| = o_n(1), \tag{3.2}$$

for each $w \in L^{\eta+1}(B) \cap L^{q+1}(B)$ where F is the primitive of f.

(II) *Assume that f satisfies (f_1), (f_2), and (f_3). Then,*

$$\int_B |f(g_n+w)-f(g_n)-f(w)|^r = o_n(1), \quad \textit{for } r\in\left(\frac{p}{q},\frac{p^*}{\eta}\right), \tag{3.3}$$

and $w\in L^p(B)\cap L^{p^*}(B)$.

Proof. We will show only (I) because the same arguments can be used in the proof of (II).

We begin remarking that

$$F(g_n+w)-F(g_n)=\int_0^1\left(\frac{d}{dt}F(g_n+tw)\right)dt. \tag{3.4}$$

Then

$$F(g_n+w)-F(g_n)=\int_0^1 f(g_n+tw)w\,dt, \tag{3.5}$$

hence, by (f_1),

$$|F(g_n+w)-F(g_n)|\le\int_0^1[\delta|g_n+tw|^q|w|+c_\delta|g_n+tw|^\eta|w|]dt, \tag{3.6}$$

that is,

$$|F(g_n+w)-F(g_n)|\le(\delta_1|g_n|^q|w|+\delta_1|w|^{q+1}+c_{\delta_1}|g_n|^\eta|w|+c_{\delta 1}|w|^{\eta+1}). \tag{3.7}$$

For each $\epsilon>0$, we obtain using Young's inequality that

$$\begin{aligned}&|F(g_n+w)-F(g_n)-F(w)|\\&\quad\le C[(\epsilon|g_n|^{q+1}+C_\epsilon|w|^{q+1})+(\epsilon|g_n|^{\eta+1}+C_\epsilon|w|^{\eta+1})].\end{aligned} \tag{3.8}$$

We consider the function $G_{\epsilon,n}$ given by

$$G_{\epsilon,n}(x)=\max\left\{|F(g_n+w)-F(g_n)-F(w)|(x)-\epsilon|g_n|^{\eta+1}(x)-\epsilon|g_n|^{q+1}(x),0\right\} \tag{3.9}$$

which satisfies

$$\begin{gathered}G_{\epsilon,n}(x)\longrightarrow 0\quad\text{a.e. in } B,\\ 0\le G_{\epsilon,n}(x)\le C_3|w|^{q+1}+C_4|w|^{\eta+1}\in L^1(B).\end{gathered} \tag{3.10}$$

Therefore, by Lebesgue's theorem, we have

$$\int_B G_{\epsilon,n}(x)dx\longrightarrow 0. \tag{3.11}$$

From the definition of $G_{\epsilon,n}$, it follows that

$$|F(g_n+w)-F(g_n)-F(w)| \leq \epsilon|g_n|^{q+1}+\epsilon|g_n|^{\eta+1}+C_5|G_{\epsilon,n}|. \tag{3.12}$$

Thus, we obtain the following inequality

$$\limsup_{n\to\infty}\int_B |F(g_n+w)-F(g_n)-F(w)| \leq C\epsilon, \tag{3.13}$$

which implies that

$$\int_B |F(g_n+w)-F(g_n)-F(w)| = o_n(1). \tag{3.14}$$

□

The next result can be found in [2].

Lemma 3.2. *Let $B\subseteq\mathbb{R}^N$ be an open set and $g_n : B\to\mathbb{R}^K$ $(K\geq 1)$ with $g_n\in L^p(B)\times\cdots\times L^p(B)$ $(p\geq 2)$, $g_n(x)\to 0$ a.e. in B, and $A(y)=|y|^{p-2}y$ for all $y\in B$. Then, if $|g_n|_{L^p(B)}\leq C$ for all $n\in\mathbb{N}$,*

$$\int_B |A(g_n+w)-A(g_n)-A(w)|^{p/(p-1)}dx = o_n(1) \tag{3.15}$$

for each $w\in L^p(B)\times\cdots\times L^p(B)$ fixed.

Lemma 3.3 (compactness global lemma). *Suppose that f satisfies (f_1), (f_2), and (f_3). Let (u_n) be a sequence in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$ verifying*

$$I(u_n)\longrightarrow c, \qquad I'(u_n)\longrightarrow 0, \tag{3.16}$$

and $u_0\in W^{1,p}(\mathbb{R}^N\setminus\Omega)$ such that $u_n\rightharpoonup u_0$ in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$. Then, either

(a) *$u_n\to u_0$ in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$ or*

(b) *there exists $k\in\mathbb{N}$, $(y_n^j)\in\mathbb{R}^N$ with $|y_n^j|\to\infty$, $j=1,\ldots,k$, and nontrivial solutions $u^1,\ldots,u^k$ of the problem (1.8), such that*

$$\left\|u_n-u_0-\sum_{j=1}^k u^j(\cdot-y_n^j)\right\|\longrightarrow 0, \qquad I(u_n)\longrightarrow I(u_0)+\sum_{j=1}^k I_\infty(u^j). \tag{3.17}$$

Proof. The arguments used in this proof follow the same ideas found in [2, 5]. The sequence (u_n) is bounded, thus there exists $u_0\in W^{1,p}(\mathbb{R}^N\setminus\Omega)$ such that

$$u_n\rightharpoonup u_0 \quad \text{in } W^{1,p}(\mathbb{R}^N\setminus\Omega). \tag{3.18}$$

Adapting arguments found in [1, 9, 10, 15], it follows that $I'(u_0)=0$. Define the function

$$\Psi_m^1(x)=u_m(x)-u_0(x), \quad x\in\mathbb{R}^N\setminus\Omega. \tag{3.19}$$

Then

$$\Psi_m^1 \rightharpoonup 0 \quad \text{in } W^{1,p}(\mathbb{R}^N \setminus \Omega),$$
$$\Psi_m^1(x) \longrightarrow 0 \quad \text{a.e. in } \mathbb{R}^N \setminus \Omega. \tag{3.20}$$

It follows, using Lemmas 2.4 and 3.2, that

$$I(\Psi_m^1) = I(u_m) - I(u_0) + o_n(1), \tag{3.21}$$
$$I'(\Psi_m^1) = o_m(1) \quad \text{in } (W^{1,p}(\mathbb{R}^N \setminus \Omega))'. \tag{3.22}$$

Suppose that

$$\Psi_m^1 \not\to 0 \quad \text{in } W^{1,p}(\mathbb{R}^N \setminus \Omega). \tag{3.23}$$

Consequently, by (f_1), (f_2), and (f_3), there exists $\alpha > 0$ such that

$$I(\Psi_m^1) \geq \alpha > 0. \tag{3.24}$$

Now, we decompose $\mathbb{R}^N$ into N-dimensional unit hypercubes Q_i with vertex having integer coordinates and put

$$d_m = \max_i |\Psi_m^1|_{L^p(U_i)}^p, \tag{3.25}$$

where $U_i = Q_i \cap (\mathbb{R}^N \setminus \Omega)$. From (3.24) and (f_1), (f_2), and (f_3), we find $\gamma > 0$ verifying

$$d_m \geq \gamma > 0. \tag{3.26}$$

Fix y_m^1 the center of hypercube Q_i in which

$$|\Psi_m^1|_{L^p(U_i)}^p = d_m \geq \gamma > 0. \tag{3.27}$$

It follows from Sobolev imbeddings and the last equality that $\{y_m^1\}$ is unbounded, that is,

$$|y_m^1| \longrightarrow \infty. \tag{3.28}$$

Let

$$z_m(x) = \Psi_m^1(x + y_m^1), \quad x \in D_m^1 = \{x - y_m^1 : x \in \mathbb{R}^N \setminus \Omega\}. \tag{3.29}$$

From boundedness of $\{u_n\}$, there exists $u^1 \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ with

$$z_m \rightharpoonup u^1 \quad \text{in } W_{\text{loc}}^{1,p}(\mathbb{R}^N). \tag{3.30}$$

Using (3.22) and the fact that $D_m^1 \to \mathbb{R}^N$, we conclude that u^1 is a nontrivial solution of (1.8). Define

$$\Psi_m^2(x) = \Psi_m^1(x + y_m^1) - u^1(x). \tag{3.31}$$

If $\|\Psi_m^2(\cdot - y_m^1)\| \to 0$, the theorem is finished, otherwise for the contrary case, we repeat the arguments and we will find $u^1, u^2, \dots, u^k$ nontrivial solutions for (1.8) and sequences (y_m^j) with $|y_m^j| \to \infty$ such that

$$\left\| u_m - u_0 - \sum_{j=1}^{k} u^j(\cdot - y_m^j) \right\|^p = o_m(1),$$
$$I(\Psi_m^j(\cdot - y_m^j)) = I(u_m) - I(u_0) - \sum_{j=1}^{k} I_\infty(u^j) + o_n(1). \tag{3.32}$$

Notice that there exists $\xi > 0$ verifying

$$I_\infty(u) \geq \xi \quad \forall u \in \Upsilon, \tag{3.33}$$

where

$$\Upsilon = \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid I'_\infty(u)u = 0\}. \tag{3.34}$$

Inequality (3.33) along with (3.32) tell us that the iteration must finish at some index $k \in \mathbb{N}$. This completes the proof of this lemma. □

Corollary 3.4. *The functional I satisfies $(PS)_c$ condition for all*

$$0 < c < c_\infty, \tag{3.35}$$

where c_∞ is the mountain pass level of the energy functional associated to (1.8).

4. Existence of ground-state solution

In this section, we will prove the existence of a positive ground-state solution for the functional I. To this end, we suppose that $f(t) = 0$ as $t \leq 0$. The first lemma is related to the mountain pass geometry, and its proof uses well-known arguments.

Lemma 4.1. *The functional I verifies the mountain pass geometry, that is,*

(i) *there exists $r, \rho > 0$ such that $I(u) \geq r$, $\|u\| = \rho$,*
(ii) *there exists $e \in B_\rho^c(0)$ such that $I(e) < 0$.*

Using a version of mountain pass theorem without Palais-Smale condition (see [14, Theorem 1.15]) and (f_3), there exists $(u_n) \subset W^{1,p}(\mathbb{R}^N \setminus \Omega)$ satisfying

$$I(u_n) \longrightarrow c_1, \quad I'(u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{4.1}$$

where

$$c_1 = \inf\left\{ \sup_{t \geq 0} I(tu);\ u \in W^{1,p}(\mathbb{R}^N \setminus \Omega) \setminus \{0\} \right\}. \tag{4.2}$$

The next result establishes a relation between the levels c_1 and c_∞.

Proposition 4.2. *Assume that Q satisfies (1.3) and (1.10). Then*

$$0 < c_1 < c_\infty. \tag{4.3}$$

Proof. Let $\bar{u}$ be a ground-state solution of problem (1.8) and define $u_n(x) = \bar{u}(x - x_n)$, $x_n = (0,\dots,n)$. By the characterization of c_1, given in (4.2), we have

$$c_1 \le \max_{t\ge 0} I(tu_n). \tag{4.4}$$

Let $\gamma_n \in (0,\infty)$ such that

$$I(\gamma_n u_n) = \max_{t\ge 0} I(tu_n), \tag{4.5}$$

then we have

$$\begin{aligned} c_1 &\le I(\gamma_n u_n) \\ &= \frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega} \left(|\gamma_n \nabla u_n|^p + |\gamma_n u_n|^p\right) - \int_{\mathbb{R}^N\setminus\Omega} Q(x)F(\gamma_n u_n) \\ &= I_\infty(\gamma_n u_n) - \frac{1}{p} t_n \gamma_n^p + \int_\Omega \bar{Q}F(\gamma_n u_n) + \int_{\mathbb{R}^N\setminus\Omega} (\bar{Q} - Q)F(\gamma_n u_n), \end{aligned} \tag{4.6}$$

where

$$t_n = \int_\Omega \left(|\nabla u_n|^p + |u_n|^p\right). \tag{4.7}$$

Now, notice that $I(\gamma_n u_n) = \max_{t\ge 0} I(tu_n)$ if and only if

$$\int_{\mathbb{R}^N\setminus\Omega} \left(|\nabla u_n|^p + |u_n|^p\right) = \int_{\mathbb{R}^N\setminus\Omega} Q(x)\frac{f(\gamma_n u_n)}{(\gamma_n u_n)^{p-1}} u_n^p. \tag{4.8}$$

It is not difficult to see that (γ_n) is bounded and therefore $\gamma_n \to \gamma_o$ for some subsequence still denoted by (γ_n). We claim that $\gamma_o = 1$. In fact, since $|x_n| \to \infty$, it follows from (4.8) that

$$\int_{\mathbb{R}^N} \left(|\nabla \bar{u}|^p + |\bar{u}|^p\right) = \int_{\mathbb{R}^N} \bar{Q}\frac{f(\gamma_o \bar{u})}{(\gamma_o \bar{u})^{p-1}} \bar{u}^p. \tag{4.9}$$

Since $\bar{u}$ is a ground-state, we get

$$\int_{\mathbb{R}^N} \bar{Q}\frac{f(\bar{u})}{(\bar{u})^{p-1}} \bar{u}^p = \int_{\mathbb{R}^N} \bar{Q}\frac{f(\gamma_o \bar{u})}{(\gamma_o \bar{u})^{p-1}} \bar{u}^p. \tag{4.10}$$

Therefore, by (f_3), we have that $\gamma_o = 1$.

From (f_1), we obtain

$$c_1 \le I_\infty(\bar{u}) - t_n\left(\frac{\gamma_n^p}{p} - O(\epsilon)\right) + s_n, \tag{4.11}$$

where

$$s_n = C_1 \int_{\Omega} |u_n|^{\eta+1} + \int_{\mathbb{R}^N \setminus \Omega} (\bar{Q} - Q) F(\gamma_n u_n). \tag{4.12}$$

We claim that

$$\frac{s_n}{t_n} \longrightarrow 0. \tag{4.13}$$

Indeed, by Lemma 2.1, we have

$$\begin{gathered} t_n = \int_{\Omega} (|\nabla u_n|^p + |u_n|^p) \geq \int_{\Omega} |u_n|^p \geq C_2 e^{-pan}, \\ \int_{\Omega} |u_n|^{\eta+1} \leq C_3 e^{-bn(\eta+1)}. \end{gathered} \tag{4.14}$$

Fix $r_n \in (0, n)$ and observe that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Omega} (\bar{Q} - Q) F(\gamma_n u_n) = & \int_{(\mathbb{R}^N \setminus \Omega) \cap \{|x| > r_n\}} (\bar{Q} - Q) F(\gamma_n u_n) \\ & + \int_{(\mathbb{R}^N \setminus \Omega) \cap \{|x| \leq r_n\}} (\bar{Q} - Q) F(\gamma_n u_n). \end{aligned} \tag{4.15}$$

On the other hand, by (1.10), it follows that

$$\int_{(\mathbb{R}^N \setminus \Omega) \cap \{|x| > r_n\}} (\bar{Q} - Q) F(\gamma_n u_n) \leq C_4 e^{-m r_n}, \tag{4.16}$$

and by condition (f_1), we have

$$\begin{aligned} & \int_{(\mathbb{R}^N \setminus \Omega) \cap \{|x| \leq r_n\}} (\bar{Q} - Q) F(\gamma_n u_n) \\ & \quad \leq C_5 \epsilon \int_{(\mathbb{R}^N \setminus \Omega) \cap \{|x| \leq r_n\}} |u_n|^{q+1} + C_6 \int_{(\mathbb{R}^N \setminus \Omega) \cap \{|x| \leq r_n\}} |u_n|^{\eta+1} \\ & \quad \leq C_7 n^N e^{-(n - r_n)(q+1)b}. \end{aligned} \tag{4.17}$$

Consequently, using the estimates obtained,

$$\frac{s_n}{t_n} \leq C_8 \left\{ \frac{e^{pan}}{e^{bn(\eta+1)}} + \frac{e^{pna}}{e^{m r_n}} + \frac{e^{pan} n^N}{e^{(n - r_n)(q+1)b}} \right\}. \tag{4.18}$$

Since $a/b \to 1$ as $\delta \to 0$ (see Lemma 2.1), there exists $\epsilon > 0$ such that

$$m > \frac{pab(q+1)}{b(q+1) - p(a+\epsilon)}. \tag{4.19}$$

Choosing $r_n = n(1 - p(a+\epsilon)/b(q+1))$, we obtain $s_n/t_n \to 0$ and hence $c_1 < c_\infty$. □

Proof of Theorem 1.1. It follows from Corollary 3.4 and mountain pass theorem (see Ambrosetti and Rabinowitz [4]) that I has a critical point u_1 in the level c_1. We claim that u_1 is nonnegative. Indeed, we know that $I'(u_1)u_1{}^- = 0$, thus

$$0 = |\nabla u_1^-|_p^p + |u_1^-|_p^p = \|u_1^-\|^p. \tag{4.20}$$

Hence $u_1^- = 0$. Using the strong maximum principle, we have $u_1 > 0$ in $\mathbb{R}^N \setminus \Omega$. Thus, we conclude that u_1 is a ground-state solution. □

5. Existence of nodal solution

In this section, we will show that there is a solution of (1.1) that changes sign. Here, we adapt for our case some arguments explored by Cerami et al. [7] (see also Cao [6] and Noussair and Wei [13]). We start with some notations. Consider the closed set

$$\mathcal{M} := \{u \in W^{1,p}(\mathbb{R}^N \setminus \Omega) \mid u^\pm \not\equiv 0,\ I'(u^\pm)u^\pm = 0\}. \tag{5.1}$$

Using well-known arguments, we can show that there exists a constant $\mu_1 > 0$ verifying

$$\int_{\mathbb{R}^N \setminus \Omega} |u^\pm|^{\eta+1} > \mu_1 \quad \forall u \in \mathcal{M}. \tag{5.2}$$

Consider the real number

$$\hat{c} = \inf_{u \in \mathcal{M}} I(u). \tag{5.3}$$

Lemma 5.1. *There exists a sequence $(u_n) \subset \mathcal{M}$ satisfying*

$$I(u_n) \longrightarrow \hat{c}, \qquad I'(u_n) \longrightarrow 0. \tag{5.4}$$

Proof. It is easy to verify that I is bounded from below on $\mathcal{M}$. Hence we may apply the Ekeland variational principle to obtain a minimizing sequence $\{u_n\} \subset \mathcal{M}$ for $\hat{c}$ satisfying

$$\hat{c} \le I(u_n) \le \hat{c} + \frac{1}{n}, \tag{5.5}$$

$$I(v) \ge I(u_n) - \frac{1}{n}\|v - u_n\| \quad \forall v \in \overline{\mathcal{M}}. \tag{5.6}$$

Using standard arguments, we have that u_n is bounded. We claim that

$$I'(u_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.7}$$

To this end, for each $\varphi \in W^{1,p}(\mathbb{R}^N \setminus \Omega)$ and $n \in \mathbb{N}$, we introduce the functions $h_n^i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1,2$, given by

$$\begin{aligned} h_n^1(t,s,l) &= \int_{\mathbb{R}^N \setminus \Omega} |\nabla(u_n + t\varphi + su_n^+ + lu_n^-)^+|^p + |(u_n + t\varphi + su_n^+ + lu_n^-)^+|^p \\ &\quad - \int_{\mathbb{R}^N \setminus \Omega} f((u_n + t\varphi + su_n^+ + lu_n^-)^+)((u_n + t\varphi + su_n^+ + lu_n^-)^+), \\ h_n^2(t,s,l) &= \int_{\mathbb{R}^N \setminus \Omega} |\nabla(u_n + t\varphi + su_n^+ + lu_n^-)^-|^p + |(u_n + t\varphi + su_n^+ + lu_n^-)^-|^p \\ &\quad - \int_{\mathbb{R}^N \setminus \Omega} f((u_n + t\varphi + su_n^+ + lu_n^-)^-)((u_n + t\varphi + su_n^+ + lu_n^-)^-). \end{aligned} \tag{5.8}$$

Note that the functions h_n^i, $i = 1,2$, are of class C^1 and $h_n^i(0,0,0) = 0$, $(\partial h_n^1/\partial l)(0,0,0) = 0$, $(\partial h_n^2/\partial s)(0,0,0) = 0$, and

$$\begin{aligned} \left(\frac{\partial h_n^1}{\partial s}\right)(0,0,0) &= p\int_{\mathbb{R}^N \setminus \Omega} (|\nabla u_n^+|^p + (u_n^+)^p) \\ &\quad - \int_{\mathbb{R}^N \setminus \Omega} f'(u_n^+)(u_n^+)^2 + f(u_n^+)(u_n^+), \end{aligned} \tag{5.9}$$

thus

$$\left(\frac{\partial h_n^1}{\partial s}\right)(0,0,0) = -\int_{\mathbb{R}^N \setminus \Omega} f'(u_n^+)(u_n^+)^2 + (1-p)f(u_n^+)(u_n^+). \tag{5.10}$$

Since $u_n \in \mathcal{M}$, from condition (1.11), there exists $C > 0$ verifying

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega} f'(u_n^+)(u_n^+)^2 + (1-p)f(u_n^+)(u_n^+) > C \tag{5.11}$$

which implies that

$$\left(\frac{\partial h_n^1}{\partial s}\right)(0,0,0) < -C_1 \quad \forall n \geq n_o \tag{5.12}$$

for some positive constant C_1. Using similar arguments, we have

$$\left(\frac{\partial h_n^2}{\partial l}\right)(0,0,0) < -C_1 \quad \forall n \geq n_o. \tag{5.13}$$

Therefore there are, by the implicit function theorem, functions $s_n(t)$, $l_n(t)$ of class C^1 defined on some interval $(-\delta_n, \delta_n)$, $\delta_n > 0$, such that $s_n(0) = l_n(0) = 0$, and

$$h_n^i(t, s_n(t), l_m(t)) = 0, \quad t \in (-\delta_n, \delta_n),\ i = 1,2. \tag{5.14}$$

This shows that for $t \in (-\delta_n, \delta_n)$,

$$v_n = u_n + t\varphi + s_n(t)u_n^+ + l_n(t)u_n^- \in \mathcal{M}. \tag{5.15}$$

Furthermore,

$$|s_n'(0)| \le C, \qquad |l_n'(0)| \le C \tag{5.16}$$

for some constant C, independent of n. In fact,

$$s_n'(0) = \left(\frac{\partial h_n^1}{\partial t}\right)(0,0,0)\Big/\left(\left(\frac{\partial h_n^1}{\partial s}\right)(0,0,0)\right), \tag{5.17}$$

thus

$$s_n'(0) = -\frac{p\int_{\mathbb{R}^N\setminus\Omega}|\nabla u_n^+|^{p-2}\nabla u_n^+\nabla\varphi - \int_{\mathbb{R}^N\setminus\Omega}(f'(u_n^+)u_n^+ + f(u_n^+))\varphi}{\int_{\mathbb{R}^N\setminus\Omega} f'(u_n^+)(u_n^+)^2 + (1-p)f(u_n^+)u_n^+}. \tag{5.18}$$

From the boundedness of u_n in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$ and (5.11), it follows that $\{s_n'(0)\}$ is bounded. A similar argument can be applied for the sequence $\{l_n'(0)\}$ to conclude that it is also bounded.

From (5.6), we have

$$\begin{aligned} &I(u_n + t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-) - I(u_n) \\ &\quad \ge -\frac{1}{n}\|t\varphi + s_n(t)u_n^+ + l_n(t)u_n^-\| \quad \forall t\in(-\delta_n,\delta_n) \end{aligned} \tag{5.19}$$

which implies that

$$I'(u_n)\varphi \ge -\frac{1}{n}\|\varphi\| - \frac{C}{n}. \tag{5.20}$$

Then, for all $\varphi\in W^{1,p}(\mathbb{R}^N\setminus\Omega)$ with $\|\varphi\|\le 1$, we get

$$I'(u_n)\varphi \ge -\frac{C_2}{n}, \tag{5.21}$$

hence

$$\|I_n'(u_n)\| \longrightarrow 0. \tag{5.22}$$

□

Proposition 5.2. *Suppose that Q satisfies (1.3), (1.10), and (1.12). Then*

$$0 < \hat{c} < c_1 + c_\infty. \tag{5.23}$$

Proof. Let $\bar{u}$ be a ground-state of (1.8). Define $\bar{u}_n(x) = \bar{u}(x - x_n)$ and $u_n = \alpha u_1 - \beta\bar{u}_n$, where u_1 is a positive ground-state of (1.1), $x_n = (0,\dots 0,n)$, $\alpha,\beta > 0$. Consider the functions

$$\begin{aligned} h^\pm(\alpha,\beta,n) = &\int_{\mathbb{R}^N\setminus\Omega}|\nabla(\alpha u_1 - \beta\bar{u}_n)^\pm|^p + |(\alpha u_1 - \beta\bar{u}_n)^\pm|^p \\ &- \int_{\mathbb{R}^N\setminus\Omega} Qf((\alpha u_1 - \beta\bar{u}_n)^\pm)(\alpha u_1 - \beta\bar{u}_n)^\pm. \end{aligned} \tag{5.24}$$

Since

$$\int_{\mathbb{R}^N\setminus\Omega}(|\nabla u_1|^p+u_1^p)-\int_{\mathbb{R}^N\setminus\Omega}Qf(u_1)u_1=0, \tag{5.25}$$

by (f_3),

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega}\left(\left|\frac{1}{p}\nabla u_1\right|^p+\left|\frac{1}{p}u_1\right|^p\right)-\int_{\mathbb{R}^N\setminus\Omega}Qf\left(\frac{1}{p}u_1\right)\frac{1}{p}u_1\\
&\qquad=\int_{\mathbb{R}^N\setminus\Omega}Q\left(\frac{f(u_1)}{(u_1)^{p-1}}-\frac{f((1/p)u_1)}{((1/p)u_1)^{p-1}}\right)\left(\frac{u_1}{p}\right)^p>0,\\
&\int_{\mathbb{R}^N\setminus\Omega}(|p\nabla u_1|^p+|pu_1|^p)-\int_{\mathbb{R}^N\setminus\Omega}Qf(pu_1)pu_1\\
&\qquad=\int_{\mathbb{R}^N\setminus\Omega}Q\left(\frac{f(u_1)}{(u_1)^{p-1}}-\frac{f(pu_1)}{(pu_1)^{p-1}}\right)(pu_1)^p<0.
\end{aligned} \tag{5.26}$$

Thus, for n large enough, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega}\left(\left|\frac{1}{p}\nabla \bar{u}_n\right|^p+\left|\frac{1}{p}\bar{u}_n\right|^p\right)-\int_{\mathbb{R}^N\setminus\Omega}Q(x)f\left(\frac{1}{p}\bar{u}_n\right)\frac{1}{p}\bar{u}_u>0,\\
&\int_{\mathbb{R}^N\setminus\Omega}(|p\nabla \bar{u}_n|^p+|p\bar{u}_n|^p)-\int_{\mathbb{R}^N\setminus\Omega}Q(x)f(p\bar{u}_n)p\bar{u}_n<0.
\end{aligned} \tag{5.27}$$

Since $\bar{u}(x)\to 0$ as $|x|\to\infty$, there exists $n_o>0$ such that

$$h^+\left(\frac{1}{p},\beta,n\right)>0,\qquad h^+(p,\beta,n)<0, \tag{5.28}$$

for $n\geq n_o$ and $\beta\in[1/p,p]$. Now, for all $\alpha\in[1/p,p]$, we have

$$h^-\left(\alpha,\frac{1}{p},n\right)>0,\qquad h^-(\alpha,p,n)<0. \tag{5.29}$$

By the mean value theorem (see [12]), we have α^*,β^* such that $1/p\leq\alpha^*,\beta^*\leq p$,

$$h^\pm(\alpha^*,\beta^*,n)=0\quad\text{for } n\geq n_o, \tag{5.30}$$

that is,

$$\alpha^*u_1-\beta^*\bar{u}_n\in\mathcal{M}\quad\text{for } n\geq n_o. \tag{5.31}$$

Hence, we only need to verify that

$$\sup_{1/p\leq\alpha,\beta\leq p} I(\alpha u_1-\beta\bar{u}_n)<c_1+c_\infty\quad\text{for } n\geq n_o. \tag{5.32}$$

Indeed, since

$$I(\alpha u_1-\beta\bar{u}_n)=\frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega}|\nabla\alpha u_1-\beta\nabla\bar{u}_n|^p+|\alpha u_1-\beta\bar{u}_n|^p \\ -\int_{\mathbb{R}^N\setminus\Omega}Q(x)F(\alpha u_1-\beta\bar{u}_n), \tag{5.33}$$

from Lemmas 2.3 and 2.4, we get

$$I(\alpha u_1-\beta\bar{u}_n)\le I_1+I_2-I_3, \tag{5.34}$$

where

$$\begin{aligned} I_1&=\frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega}(|\nabla(\alpha u_1)|^{p-2}\nabla(\alpha u_1)-|\nabla(\beta\bar{u}_n)|^{p-2}\nabla(\beta\bar{u}_n))(\nabla(\alpha u_1)-\nabla(\beta\bar{u}_n)),\\ I_2&=\frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega}(|\alpha u_1|^{p-2}\alpha u_1-|\beta\bar{u}_n|^{p-2}\beta\bar{u}_n)(\alpha u_1-\beta\bar{u}_n),\\ I_3&=\int_{\mathbb{R}^N\setminus\Omega}QF(\alpha u_1)+\int_{\mathbb{R}^N\setminus\Omega}QF(\beta\bar{u}_n)-2\int_{\mathbb{R}^N\setminus\Omega}f(\alpha u_1)\beta\bar{u}_n+\alpha u_1f(\beta\bar{u}_n). \end{aligned} \tag{5.35}$$

Since u_1 is a solution of (1.1) and $\bar{u}_n$ is related with a ground-state of (1.8), we have

$$I(\alpha u_1-\beta\bar{u}_n)\le I(\alpha u_1)+I_\infty(\beta\bar{u}_n)-\int_{\mathbb{R}^N\setminus\Omega}(Q-\bar{Q})F(\beta\bar{u}_n) \\ +C_1\int_{\mathbb{R}^N\setminus\Omega}(f(u_1)\bar{u}_n+u_1f(\bar{u}_n))+\int_\Omega\bar{Q}F(\beta\bar{u}_n). \tag{5.36}$$

Therefore, we conclude that

$$\sup_{1/p\le\alpha,\beta\le p}I(\alpha u_1-\beta\bar{u}_n)\le\sup_{\alpha\ge0}I(\alpha u_1)+\sup_{\beta\ge0}I_\infty(\beta\bar{u}_n)-\int_{\mathbb{R}^N\setminus\Omega}(Q-\bar{Q})F\left(\frac{1}{p}\bar{u}_n\right) \\ +C_1\int_{\mathbb{R}^N\setminus\Omega}(f(\alpha u_1)\beta\bar{u}_n+\alpha u_1f(\beta\bar{u}_n))+\int_\Omega\bar{Q}F(p\bar{u}_n). \tag{5.37}$$

Now, by (1.12), we obtain

$$\int_{\mathbb{R}^N\setminus\Omega}(Q-\bar{Q})F(\bar{u}_n)\ge Ce^{-\gamma n}, \tag{5.38}$$

and, by (f_1), we get

$$\int_\Omega\bar{Q}F(\bar{u}_n)\le\epsilon e^{-nb(q+1)}+C_2e^{-nb(\eta+1)}\le Ce^{-nb(q+1)}. \tag{5.39}$$

On the other hand, one has

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Omega} f(u_1)\bar{u}_n &\leq C\bigg(\int_{(\mathbb{R}^N\setminus\Omega)\cap\{|x|<(1/(q+1))n\}}+\int_{(\mathbb{R}^N\setminus\Omega)\cap\{|x|\geq(1/(q+1))n\}}\bigg)|u_1|^q|\bar{u}_n|\\ &\quad+C\bigg(\int_{(\mathbb{R}^N\setminus\Omega)\cap\{|x|<(1/(\eta+1))n\}}+\int_{(\mathbb{R}^N\setminus\Omega)\cap\{|x|\geq(1/(\eta+1))n\}}\bigg)|u_1|^\eta|\bar{u}_n|\\ &\leq C_1e^{-(q/(q+1))bn}+C_2e^{-bn(q/(q+1))}+C_3e^{-(\eta/(\eta+1))bn}+C_2e^{-bn(\eta/(\eta+1))}\\ &\leq Ce^{-nb(q/(q+1))},\end{aligned} \tag{5.40}$$

$$\int_{\mathbb{R}^N\setminus\Omega} u_1 f(\bar{u}_n)\leq Ce^{-nb(q/(q+1))}. \tag{5.41}$$

Recalling that $\gamma<q/(q+1)$, and substituting (5.38), (5.39), and (5.40) in (5.37), with a and b near 1, we have for n large enough that

$$\sup_{1/p\leq\alpha,\beta\leq p} I(\alpha u_1-\beta\bar{u}_n)<\sup_{\alpha\geq 0} I(\alpha u_1)+\sup_{\beta\geq 0} I_\infty(\beta\bar{u}_n)=c_1+c_\infty. \tag{5.42}$$

Thus

$$\hat{c}<c_1+c_\infty, \tag{5.43}$$

which proves the proposition. □

As an immediate consequence of Lemma 3.3 and the last proposition, we get the following lemma.

Lemma 5.3. *Let $(u_n)\subset\mathcal{M}$ be the sequence obtained in Lemma 5.1. Then (u_n) has a subsequence converging strongly in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$.*

Proof. It is easy to see that (u_n) is bounded in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$. Denote by u the weak limit of (u_n) in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$. Thus, either $u_n\to u$ in $W^{1,p}(\mathbb{R}^N\setminus\Omega)$ or there exist k functions u^j with $1\leq j\leq k$ satisfying Lemma 3.3. It is clear that $k\leq 1$. Suppose that $u\equiv 0$. Since $c_2>0$, we have $k=1$ and

$$u_n^1\longrightarrow u^1(\cdot-y_n^1)\quad\text{in } W^{1,p}(\mathbb{R}^N\setminus\Omega). \tag{5.44}$$

On the other hand, since $u_n\in\mathcal{M}$ and $|y_m^1|\to\infty$, we obtain

$$\int_{\mathbb{R}^N}|(u^1)^\pm|^{\eta+1}dx\geq\frac{\mu}{2}>0. \tag{5.45}$$

So, we can conclude that

$$\hat{c}=I_\infty(u^1)\geq 2c_\infty, \tag{5.46}$$

which is a contradiction. Thus $u \not\equiv 0$. If (u_n) does not converge strongly to u, then $u^1 \not\equiv 0$. Hence,

$$\hat{c} \geq I(u) + I_\infty(u^1) \geq c_1 + c_\infty, \tag{5.47}$$

which contradicts the inequality $\hat{c} < c_1 + c_\infty$. Hence, there is no k and (u_n) is strongly convergent to u in $W^{1,p}(\mathbb{R}^N \setminus \Omega)$. □

Proof of Theorem 1.2. By Lemma 5.3, there exists $u \in \mathcal{M}$ such that

$$I(u) = \hat{c}, \qquad I'(u) = 0, \tag{5.48}$$

hence, u is a nodal solution of (1.1). □

Acknowledgments

The authors would like to thank the anonymous referee for valuable commentaries and suggestions made in a very detailed report. The second author was supported partially by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). The third author was supported by Fundação de Amparo a Pesquisa do Estado de Minas Gerais (FAPEMIG).

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