CONTRACTIVE PROJECTIONS IN ORLICZ SEQUENCE SPACES

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We characterize norm-one complemented subspaces of Orlicz sequence spaces ℓ_M equipped with either Luxemburg or Orlicz norm, provided that the Orlicz function M is sufficiently smooth and sufficiently different from the square function. We measure smoothness of M using AC^1 and AC^2 classes introduced by Maleev and Troyanski in 1991, and the condition for M to be different from a square function is essentially a requirement that the second derivative M'' of M cannot have a finite nonzero limit at zero. This paper treats the real case; the complex case follows from previously known results.

1. Introduction

The study of norm-one projections and their ranges (one-complemented subspaces) has been an important topic of the isometric Banach space theory since the inception of the field. Contractive projections were also investigated from the approximation theory point of view, as part of the study of minimal projections, that is, projections onto the given subspace with the smallest possible norm (cf. [5, 14]). They are also closely related to the metric projections or nearest point mappings, and they are a natural extension of the notion of orthogonal projections from Hilbert spaces to general Banach spaces. Despite a great amount of work on contractive projections (cf. the survey [19]), not much is known about them in Orlicz spaces.

In Lebesgue spaces L_p and ℓ_p, $1 \leq p < \infty$, a subspace Y is one-complemented if and only if Y is isometrically isomorphic to an L_p-space of appropriate dimension (see [1, 6]). This result has no analogue in other spaces. Lindberg [10] demonstrated that there exist classes of Orlicz sequence spaces ℓ_M containing one-complemented subspaces which are not even isomorphic to ℓ_M. He showed that for all $1 < a \leq b < \infty$, there exists a reflexive Orlicz sequence space ℓ_M so that for all $p \in [a,b]$, there is a contractive projection from ℓ_M onto a subspace isomorphic to ℓ_p. This implies, in particular, that Orlicz sequence spaces can have continuum isomorphic types of one-complemented subspaces.

On the other hand, one-complemented subspaces of ℓ_p are also characterized as subspaces which are spanned by a family of mutually disjoint elements of ℓ_p (see [2, 11]). Moreover, all known examples of one-complemented subspaces in symmetric Banach

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spaces with one-unconditional bases and sufficiently different from Hilbert spaces are spanned by a family of mutually disjoint vectors. (Note that, since in Hilbert spaces every subspace is one-complemented, it is necessary to include in this context some kind of an assumption about the space being different from Hilbert space.) In particular, the above-mentioned examples of Lindberg of one-complemented subspaces of Orlicz sequence spaces are spanned by mutually disjoint vectors and the norm-one projection is an averaging projection. It was shown in [16] that indeed every one-complemented subspace Y in any complex Banach space X, with a one-unconditional basis (not necessarily symmetric) which does not contain a one-complemented isometric copy of a two-dimensional Hilbert space ℓ_2^2, has to be spanned by a family of disjointly supported elements of X and the norm-one projection from X onto Y has to be the averaging projection. In particular, this holds in complex Orlicz sequence spaces ℓ_M equipped with either the Luxemburg or the Orlicz norm when M is sufficiently different from the square function (cf. Remark 3.5).

In the real case, this statement in its full generality is false (cf. [16]). For real spaces, we only had the following much less satisfactory result describing special one-complemented subspaces of finite codimension in Orlicz sequence spaces ℓ_M.

Theorem 1.1 [17, Theorem 7]. *Let M be an Orlicz function such that M satisfies condition Δ_2 at zero, $M(t) > 0$ for all $t > 0$ and M is not similar to t^2 (it is said that M is similar to t^2 if there exist constants C, t_0 so that $M(t) = Ct^2$ for all $t < t_0$). Let ℓ_M be the Orlicz space equipped with either the Luxemburg or the Orlicz norm and $F \subset \ell_M$ a subspace of finite codimension. If F contains at least one basis vector and F is one-complemented in ℓ_M, then F is spanned by a family of disjointly supported vectors.*

In the present paper, we prove a much stronger result—we eliminate all additional assumptions on F. We show that when M is a sufficiently smooth Orlicz function which satisfies condition Δ_2 and is sufficiently different from the square function, then every one-complemented subspace of the real Orlicz space ℓ_M is spanned by a family of mutually disjoint vectors and every norm-one projection in ℓ_M is an averaging projection (see Theorem 3.3 and Corollary 3.4). Here we measure smoothness of M using AC^1 and AC^2 classes introduced by Maleev and Troyanski [12], and the condition for M to be different from a square function is essentially a requirement that the second derivative M'' of M cannot have a finite nonzero limit at zero.

Moreover, using [16, Theorem 6.1], it follows that if, in addition, ℓ_M is not isomorphic to ℓ_p for any $p \in [1,\infty)$, then every one-complemented subspace of ℓ_M is spanned by a block basis with constant coefficients, that is, by mutually disjoint finitely supported elements v_j in ℓ_M of the form $v_j = \sum_{i\in \operatorname{supp} v_j} \varepsilon_i e_i$, where $\varepsilon_i = \pm 1$ and $(e_i)_i$ is the standard basis of ℓ_M.

Our results are valid in Orlicz spaces equipped with either the Luxemburg or the Orlicz norm. Our method of proof is different from that of [17]; it relies on new results characterizing averaging projections through properties related to and generalizing disjointness-preserving operators [15].

Recently, Jamison et al. [7] obtained (using different techniques) a generalization of Theorem 1.1 in another direction—they characterized one-complemented subspaces of finite codimension in sufficiently smooth Musielak-Orlicz sequence spaces, whose Orlicz function is sufficiently different from the square function.

We follow standard definitions and notations as may be found in [8, 11]. Throughout the paper, unless otherwise noted, all spaces are over $\mathbb{R}$.

2. Preliminary definitions

Below we recall the basic definitions and facts about Orlicz spaces that will be important for the present paper, as may be found, for example, in [3, 8, 20].

We say that a function $M:\mathbb{R}\to[0,\infty)$ is an *Orlicz function* if M is even, continuous, convex, $M(0)=0$, $M(1)=1$, $\lim_{u\to 0}M(u)/u=0$, and $\lim_{u\to\infty}M(u)/u=\infty$. Note that since the Orlicz function M is convex, it has the right derivative M'. Let q be the right inverse of M' (i.e., $q(s)=\sup\{t:M'(t)\le s\}$). Then we define the *complementary function of M* by

$$M^*(v)=\int_0^{|v|}q(s)ds. \tag{2.1}$$

Function M^* is also an Orlicz function.

We say that the Orlicz function M *satisfies the Δ_2 condition near zero* ($M\in\Delta_2$) if there exist constants $k>0$ and $u_0\ge 0$ such that $M(u_0)>0$ and for all u with $|u|\le u_0$,

$$M(2u)\le kM(u). \tag{2.2}$$

Note that $M\in\Delta_2$ does not imply that $M^*\in\Delta_2$.

The Orlicz function M generates the *modular* defined for scalar sequences $x=(x_j)_{j\in\mathbb{N}}$ by

$$\rho_M(x)=\sum_{j=1}^{\infty}M(x_j). \tag{2.3}$$

The *Orlicz sequence space ℓ_M* is the space of sequences x such that there exists $\lambda>0$ with $\rho_M(\lambda x)<\infty$. If $M\in\Delta_2$, then $\ell_M=\{x:\rho_M(\lambda x)<\infty \text{ for all } \lambda\in\mathbb{R}\}$. The Orlicz sequence space ℓ_M is usually equipped with one of the following two equivalent norms:

(1) the *Luxemburg norm* defined by

$$\|x\|_M=\inf\left\{\lambda:\rho_M\left(\frac{x}{\lambda}\right)\le 1\right\}, \tag{2.4}$$

(2) the *Orlicz norm* defined by

$$\|x\|_M^O=\sup\left\{\sum_{j=1}^{\infty}x_jy_j:\rho_{M^*}(y)\le 1\right\}. \tag{2.5}$$

If $M\in\Delta_2$, then these norms are dual to each other in the following sense:

$$\left(\ell_M,\|\cdot\|_M\right)^*=\left(\ell_{M^*},\|\cdot\|_{M^*}^O\right), \tag{2.6}$$

$$\left(\ell_M,\|\cdot\|_M^O\right)^*=\left(\ell_{M^*},\|\cdot\|_{M^*}\right). \tag{2.7}$$

We say that two Orlicz functions M_1 and M_2 are *equivalent at zero* if there exist $u_0 > 0$, $k, l > 0$, and $C > 0$ such that $M_1(u_0) > 0$ and for all u with $|u| \le u_0$,

$$C^{-1}M_2(ku) \le M_1(u) \le CM_2(lu). \tag{2.8}$$

Recall that M_1 and M_2 are equivalent at zero if and only if $\ell_{M_1} = \ell_{M_2}$ (as sets) and the identity mapping is an isomorphism (in either norm) [11]. We note that if $M \in \Delta_2$, then every Orlicz function M_1 equivalent to M also satisfies $M_1 \in \Delta_2$. Krasnosel'skiĭ and Rutickiĭ proved the following characterization of the Δ_2-condition at zero in terms of the right derivative M' of M.

Proposition 2.1 [8, Theorem 4.1]. *Let M be an Orlicz function. Then $M \in \Delta_2$ if and only if there exist constants α and $u_0 \ge 0$ such that for $0 \le u \le u_0$,*

$$\frac{uM'(u)}{M(u)} < \alpha, \tag{2.9}$$

where M' is the right derivative of M. Moreover, if (2.9) is satisfied, then $M(2u) \le 2^\alpha M(u)$ for $0 \le u \le u_0/2$.

In [18], we introduced another condition which on one hand is very similar to (2.9), but on the other hand is in its nature of "smoothness type," as we explain below.

Definition 2.2. Let $M \in \Delta_2$ be a twice differentiable Orlicz function. Function M *satisfies condition Δ_{2+} near zero* if there exist constants $\beta > 0$ and $u_0 \ge 0$ such that for all $u \le u_0$,

$$\frac{uM''(u)}{M'(u)} < \beta. \tag{2.10}$$

As proved in [18], condition Δ_{2+} is of "smoothness type" in the following sense:

(i) for every Orlicz function $M \in \Delta_2$, there exists an equivalent Orlicz function $M_1 \in \Delta_{2+}$; (however, we do not know whether for every $\varepsilon > 0$, it is possible to choose M_1 so that it is $(1+\varepsilon)$-equivalent with M),

(ii) for every Orlicz function $M \in \Delta_{2+}$, there exists an equivalent (even up to an arbitrary $\varepsilon > 0$) Orlicz function $M_1 \notin \Delta_{2+}$.

We say that a Banach space X is *smooth* if every element $x \in X$ has a unique norming functional $x^* \in X^*$, that is, the functional with the property that $\|x^*\|_{X^*}^2 = \|x\|_X^2 = x^*(x)$. If $M \in \Delta_2$, then an Orlicz space ℓ_M (with either norm) is smooth whenever M is differentiable everywhere (cf. [3, 4]). Maleev and Troyanski [12] considered a stronger notion of smoothness in Orlicz spaces which guarantees the differentiability of the norm. We recall the relevant definitions and results.

Definition 2.3 (see [13], cf. [11, page 143]). To every Orlicz function M, the following *Matuszewska-Orlicz index* is associated:

$$\alpha_M^0 = \sup\left\{p : \sup\left\{\frac{M(\lambda t)}{t^p M(\lambda)} : \lambda, t \in (0,1]\right\} < \infty\right\}. \tag{2.11}$$

Definition 2.4 [12]. An Orlicz function M belongs to *the class AC^k at zero* ($M \in AC^k$) if

(i) $\alpha_M^0 > k$,

(ii) $M^{(k)}$ is absolutely continuous in every finite interval,

(iii) $t^{k+1}|M^{(k+1)}(t)| \leq cM(ct)$ a.e. in $[0,\infty)$ for some $c > 0$.

Definition 2.5. Let X, Y be Banach spaces. The function $\varphi : X \to Y$ is said to be *k-times differentiable at $x \in X$* if for every j, $1 \leq j \leq k$, there exists a continuous symmetric j-linear form $T_x^j : X \times \cdots \times X = X^{(j)} \to Y$ so that

$$\varphi(x+\alpha y) = \varphi(x) + \sum_{j=1}^{k} \alpha^j T_x^j(y,\dots,y) + \sigma_x(|\alpha|^k) \tag{2.12}$$

uniformly on y from the unit sphere $S(X)$ of X.

For an open set $V \subset X$, $\varphi \in F^k(V,Y)$ means that φ is k-times differentiable at every point of V. If (2.12) is fulfilled uniformly on x over a set $W \subset V$, we will say that φ is *k-times uniformly differentiable over W* and will write $\varphi \in UF^k(W,Y)$. We say that X is UF^k-smooth if the norm in X belongs to $UF^k(S(X),\mathbb{R})$.

Maleev and Troyanski proved the following results.

Theorem 2.6 [12, Theorem 6]. *Let M be an Orlicz function such that $M \in \Delta_2$ and $M \in AC^k$. Then ℓ_M equipped with the Luxemburg norm is UF^k-smooth.*

Theorem 2.7 [12, Corollary 10]. *Let M be an Orlicz function with $M \in \Delta_2$. Then for every $k \in \mathbb{N}$ such that $k < \alpha_M^0$, there exists an Orlicz function $\widetilde{M}$ equivalent to M at zero so that $\ell_{\widetilde{M}}$ (with the Luxemburg norm) is UF^k-smooth. (In particular, $\ell_{\widetilde{M}}$ is isomorphic to ℓ_M.)*

It is well known (see, e.g., [3]) that any Orlicz function M can be "smoothed out," that is, for any M, there exists an equivalent Orlicz function M_1 such that M_1 is twice differentiable everywhere, M_1'' is continuous on $\mathbb{R}$, and $M_1''(u) > 0$ for all $u > 0$. Moreover, given any $\varepsilon > 0$, it is possible to choose M_1 so that ℓ_M and ℓ_{M_1} are $(1+\varepsilon)$-isomorphic to each other [3]. However, we do not know whether in Theorem 2.7, it is possible for any $\varepsilon > 0$ to select $\widetilde{M}$ so that ℓ_M and $\ell_{\widetilde{M}}$ are $(1+\varepsilon)$-isomorphic or not.

Recall that a Banach lattice X is called *strictly monotone* if $\|x+y\| > \|x\|$ for all $x, y \geq 0$, $y \neq 0$, in X. An Orlicz space ℓ_M with the Luxemburg norm is strictly monotone whenever $M \in \Delta_2$ (cf. [9]).

Next, we recall some facts about contractive projections and about disjointness in Orlicz spaces that we will need.

If X is a Banach space with a one-unconditional basis $\{e_n\}_{n\in\mathbb{N}}$ and $x = \sum_{n\in\mathbb{N}} x_n e_n \in X$, then the *support of x* is defined as $\operatorname{supp} x = \{n \in \mathbb{N} : x_n \neq 0\}$. We say that elements $x, y \in X$ are *disjoint* if $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$. We say that a projection P on a Banach space with a one-unconditional basis is an *averaging projection* if there exist mutually disjoint elements $\{u_j\}_{j\in J}$ in X and functionals $\{u_j^*\}_{j\in J}$ in X^* so that $u_j^*(u_k) = 0$ if $j \neq k$, $u_j^*(u_j) = 1$ for all $j \in J$, and for each $f \in X$,

$$Pf = \sum_{j\in J} u_j^*(f) u_j. \tag{2.13}$$

In [15], we introduced two abstract conditions relevant for the study of averaging projections. Namely, if X is a Banach space with a one-unconditional basis and $P : X \to X$ is a linear operator on X, we say that the operator P is *semi-band preserving* if for all $f, g \in X$,

$$\operatorname{supp}(Pf) \cap \operatorname{supp}(g) = \varnothing \Longrightarrow \operatorname{supp}(Pf) \cap \operatorname{supp}(Pg) = \varnothing, \tag{2.14}$$

and we say that P is *semi-containment preserving* if for all $f, g \in X$,

$$\operatorname{supp} g \subset \operatorname{supp} Pf \Longrightarrow \operatorname{supp} Pg \subset \operatorname{supp} Pf. \tag{2.15}$$

It is clear that all averaging projections are both semi-band preserving and semi-containment preserving. In [15], we proved that in certain situations, these conditions characterize averaging projections among contractive projections. Namely, we have the following theorem.

Theorem 2.8 [15]. *Let X be a purely atomic strictly monotone Banach lattice and let $P : X \to X$ be a norm-one projection which is semi-band preserving or semi-containment preserving. Then P is an averaging projection.*

Next, we recall conditions which partially describe disjointness and containment of supports of elements in Orlicz spaces.

Proposition 2.9 [18, Proposition 3.1]. *Let M be an Orlicz function with $M \in \Delta_{2+}$ and such that M'' is continuous on $[0,\infty)$, $M''(0) = 0$, and $M''(t) > 0$ for all $t > 0$. Let $f, g \in \ell_M$ and $N(\alpha) = \|f + \alpha g\|_M$. Then*

(a) *if f, g are disjoint and $\operatorname{card}(\operatorname{supp} g) < \infty$, then $N'(0) = 0$ and $N''(\alpha) \to 0$, as $\alpha \to 0$ along a subset of $[0,1]$ of full measure;*
(b) *if $N'(0) = 0$ and $N''(\alpha) \to 0$, as $\alpha \to 0$ along a subset of $[0,1]$ of full measure, then f, g are disjoint.*

Proposition 2.10 [18, Proposition 4.1]. *Let M be an Orlicz function with $M \in \Delta_{2+}$ and such that M'' is continuous on $(0,\infty)$ with $\lim_{t\to 0} M''(t) = \infty$. Let $f, g \in \ell_M$ with $f, g \neq 0$ and $N(\alpha) = \|f + \alpha g\|_M$. Then*

(a) *if $\operatorname{card}(\operatorname{supp} g \setminus \operatorname{supp} f) > 0$, then $N''(\alpha) \to \infty$, as $\alpha \to 0$ along a subset of $[0,1]$ of full measure;*
(b) *if g is simple (i.e., $\operatorname{card}(\operatorname{supp} g) < \infty$) and $\operatorname{supp} g \subset \operatorname{supp} f$, then there exist a subset E of $[0,1]$ of full measure and $C > 0$ such that for all $\alpha \in E$, $N''(\alpha) \le C$.*

Remark 2.11. We stress that the theorems in [18] are proven for Orlicz spaces equipped with the Luxemburg norm, and the analogs of most of the results from [18] are false in Orlicz spaces equipped with the Orlicz norm.

Remark 2.12. We note here that all the theorems in [18] were formulated and proved for Orlicz function spaces L_M, where M is an Orlicz function satisfying conditions Δ_2 and Δ_{2+} near infinity. However, to adapt to the case of Orlicz sequence spaces ℓ_M, where M is an Orlicz function satisfying conditions Δ_2 and Δ_{2+} near zero, the proofs require only very minor changes, if any. We omit the details, which are routine but require cumbersome notation.

Finally we recall a result from [16] which describes the form of two-dimensional one-complemented subspaces of Orlicz sequence spaces when the two spanning elements have disjoint supports. This result will allow us to give a very detailed description of one-complemented subspaces of any dimension of Orlicz sequence spaces.

Theorem 2.13 [16, Theorem 6.1]. *Let M be an Orlicz function with $M \in \Delta_2$, let ℓ_M be a (real or complex) Orlicz sequence space equipped with either the Luxemburg or the Orlicz norm, and let $x, y \in \ell_M$ be disjoint norm-one elements such that* $\operatorname{span}\{x, y\}$ *is one-complemented in ℓ_M. Then one of the following three possibilities holds:*

(1) $\operatorname{card}(\operatorname{supp} x) < \infty$ *and* $|x_i| = |x_j|$ *for all* $i, j \in \operatorname{supp} x$;
(2) *there exists* $p \in [1, \infty)$ *such that* $M(t) = Ct^p$ *for all* $t \leq \|x\|_\infty$; *or*
(3) *there exists* $p \in [1, \infty)$ *and constants* $C_1, C_2 \geq 0$ *such that* $C_2 t^p \leq M(t) \leq C_1 t^p$ *for all* $t \leq \|x\|_\infty$ *and there exists* $\gamma > 0$ *such that, for all* $j \in \operatorname{supp} x$,

$$|x_j| \in \{\gamma^m \cdot \|x\|_\infty : m \in \mathbb{Z}\}. \tag{2.16}$$

We note that it follows from Theorem 2.13 that if the Orlicz space ℓ_M is not isomorphic to ℓ_p for any $p \in [1, \infty)$, then the possibility (1) has to hold. Hence every one-complemented disjointly supported subspace of any dimension of ℓ_M needs to be spanned by mutually disjoint finitely supported elements v_j in ℓ_M of the form $v_j = \sum_{i \in \operatorname{supp} v_j} \varepsilon_i e_i$, where $\varepsilon_i = \pm 1$ and $(e_i)_i$ is the standard basis of ℓ_M.

3. Main results

We start from a key technical lemma which will allow us to apply Propositions 2.9 and 2.10 to study whether contractive projections in Orlicz sequence spaces are semi-band preserving or semi-containment preserving.

Lemma 3.1. *Let $\varphi, \psi : \mathbb{R} \to [0, \infty)$ be convex functions, differentiable everywhere, and such that $\varphi(0) = \psi(0)$ and $\varphi(\alpha) \leq \psi(\alpha)$ for all $\alpha \in \mathbb{R}$. Then*

(i) $\psi'(0) = \varphi'(0)$;
(ii) *if* $\varphi''(0)$, $\psi''(0)$ *exist and* $\psi''(0) = 0$, *then* $\varphi''(0) = 0$;
(iii) *suppose that φ' and ψ' are absolutely continuous on $[0, 1]$ and that $\varphi''(\alpha) \to \infty$, as $\alpha \to 0$ along a subset of $[0, 1]$ of full measure. Then for every $C > 0$,*

$$\mu\big(\{\alpha \in [0, 1] : \psi''(\alpha) \text{ exists and } \psi''(\alpha) \leq C\}\big) < 1. \tag{3.1}$$

Proof. To prove (i), observe that, since $\varphi(0) = \psi(0)$, we have for all $\alpha \in \mathbb{R}$, $\varphi(\alpha) - \varphi(0) \leq \psi(\alpha) - \psi(0)$. Thus for $\alpha > 0$,

$$\frac{\varphi(\alpha) - \varphi(0)}{\alpha} \leq \frac{\psi(\alpha) - \psi(0)}{\alpha}, \tag{3.2}$$

and for $\alpha < 0$,

$$\frac{\varphi(\alpha)-\varphi(0)}{\alpha} \geq \frac{\psi(\alpha)-\psi(0)}{\alpha}. \tag{3.3}$$

Since $\varphi'(0)$ and $\psi'(0)$ exist, we have, by (3.2),

$$\varphi'(0) = \lim_{\alpha\to 0^+}\frac{\varphi(\alpha)-\varphi(0)}{\alpha} \leq \lim_{\alpha\to 0^+}\frac{\psi(\alpha)-\psi(0)}{\alpha} = \psi'(0), \tag{3.4}$$

and, by (3.3),

$$\varphi'(0) = \lim_{\alpha\to 0^-}\frac{\varphi(\alpha)-\varphi(0)}{\alpha} \geq \lim_{\alpha\to 0^-}\frac{\psi(\alpha)-\psi(0)}{\alpha} = \psi'(0). \tag{3.5}$$

Thus $\varphi'(0) = \psi'(0)$ and (i) is proved.

To prove (ii), we consider the set $A = \{\alpha > 0 : \varphi'(\alpha) = \psi'(\alpha)\}$.

If $\inf\{\alpha \in A\} = 0$, then there exists a sequence $\{\alpha_n\}_{n=1}^{\infty} \subset A$ so that $\lim_{n\to\infty}\alpha_n = 0$. Since $\varphi''(0)$ and $\psi''(0)$ exist, and by (i), we obtain

$$\varphi''(0) = \lim_{n\to\infty}\frac{\varphi'(\alpha_n)-\varphi'(0)}{\alpha_n} = \lim_{n\to\infty}\frac{\psi'(\alpha_n)-\psi'(0)}{\alpha_n} = \psi''(0) = 0, \tag{3.6}$$

so (ii) is proved.

If $\inf\{\alpha \in A\} > 0$ (this includes the case that $A = \varnothing$ and then we say $\inf\{\alpha \in A\} = \infty > 0$), then there exists ε, $0 < \varepsilon < \inf\{\alpha \in A\}$, so that $\varphi'(\alpha) \neq \psi'(\alpha)$ for all $\alpha \in (0,\varepsilon)$.

Let $h = \psi - \varphi$. Then $h(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$, $h(0) = 0$, and $h'(\alpha) \neq 0$ for all $\alpha \in (0,\varepsilon)$. Since h' satisfies the Darboux property, we get either

$$h'(\alpha) > 0 \quad \forall \alpha \in (0,\varepsilon), \tag{3.7}$$

or

$$h'(\alpha) < 0 \quad \forall \alpha \in (0,\varepsilon). \tag{3.8}$$

But $h(0) = 0$ and $h(\varepsilon) \geq 0$, so by the mean value theorem, there exists $\alpha_0 \in (0,\varepsilon)$ so that $h'(\alpha_0) = h(\varepsilon)/\varepsilon \geq 0$. Thus (3.7) has to hold. This implies that since $h''(0)$ exists, $h''(0) \geq 0$. This means that $0 = \psi''(0) \geq \varphi''(0)$. Since φ is convex, we also get $\varphi''(0) \geq 0$. Thus $\varphi''(0) = 0$ and (ii) is proved.

To prove (iii), we denote $E_1 = \{\alpha \in [0,1] : \varphi''(\alpha) \text{ exists}\}$. Since φ and ψ are convex, $\mu(E_1) = 1$. Without loss of generality, we can also assume that

$$\varphi''(\alpha) \longrightarrow \infty \quad \text{as } \alpha \longrightarrow 0,\ \alpha \in E_1. \tag{3.9}$$

Suppose, for contradiction, that there exists $C > 0$ so that the set $E_2 = \{\alpha \in [0,1] : \psi''(\alpha) \text{ exists and } \psi''(\alpha) \leq C\}$ has full measure. Let $E = E_1 \cap E_2$. By (3.9), there exists $\varepsilon > 0$ so that

$$\varphi''(\alpha) > C \quad \text{for every } \alpha \in E \cap (0,\varepsilon). \tag{3.10}$$

Now consider the set $A = \{\alpha > 0 : \varphi'(\alpha) = \psi'(\alpha)\}$ similarly as we did in the proof of (ii). If $\inf\{\alpha \in A\} = 0$, then there exist $\alpha_1, \alpha_2 \in (0, \varepsilon)$ so that $\alpha_1 \neq \alpha_2$ and

$$\varphi'(\alpha_1) = \psi'(\alpha_1), \qquad \varphi'(\alpha_2) = \psi'(\alpha_2). \tag{3.11}$$

But, since φ' is absolutely continuous on $[0,1]$, and by (3.10), we also have

$$\varphi'(\alpha_1) - \varphi'(\alpha_2) = \int_{\alpha_1}^{\alpha_2} \varphi''(\alpha)d\alpha = \int_{[\alpha_1,\alpha_2]\cap E} \varphi''(\alpha)d\alpha > C(\alpha_1 - \alpha_2). \tag{3.12}$$

On the other hand, by the absolute continuity of ψ' on $[0,1]$ and the definition of E_2, we have

$$\psi'(\alpha_1) - \psi'(\alpha_2) = \int_{\alpha_1}^{\alpha_2} \psi''(\alpha)d\alpha = \int_{[\alpha_1,\alpha_2]\cap E} \psi''(\alpha)d\alpha \leq C(\alpha_1 - \alpha_2). \tag{3.13}$$

This is a contradiction since (3.11) implies that $\varphi'(\alpha_1) - \varphi'(\alpha_2) = \psi'(\alpha_1) - \psi'(\alpha_2)$.

Now we consider the case that $\inf\{\alpha \in A\} \neq 0$, that is, $\inf\{\alpha \in A\} > 0$ (this, as in (ii), includes the possibility that $A = \varnothing$ in which case we say that $\inf\{\alpha \in A\} = \infty$). We showed in the proof of (ii) (cf. (3.7)) that in this case, there exists ε_1, $0 < \varepsilon_1 < \inf\{\alpha \in A\}$, so that

$$\psi'(\alpha) > \varphi'(\alpha) \quad \forall \alpha \in (0, \varepsilon_1). \tag{3.14}$$

By (i), $\varphi'(0) = \psi'(0)$. Let $\alpha_0 \in (0, \varepsilon) \cap (0, \varepsilon_1)$. Then, similarly as in the previous case, since φ' is absolutely continuous on $[0,1]$, by (3.10), we obtain

$$\varphi'(\alpha_0) - \varphi'(0) = \int_0^{\alpha_0} \varphi''(\alpha)d\alpha = \int_{[0,\alpha_0]\cap E} \varphi''(\alpha)d\alpha > C\alpha_0. \tag{3.15}$$

On the other hand, again by the absolute continuity of ψ' and the definition of E_2,

$$\psi'(\alpha_0) - \psi'(0) = \int_0^{\alpha_0} \psi''(\alpha)d\alpha = \int_{[0,\alpha_0]\cap E_2} \psi''(\alpha)d\alpha \leq C\alpha_0. \tag{3.16}$$

Thus $\psi'(\alpha_0) < \varphi'(\alpha_0)$, which contradicts (3.14) and ends the proof of (iii). □

We are now ready for our main results.

Theorem 3.2. *Let M be an Orlicz function with $M \in \Delta_{2+}$ and let P be a contractive projection on the real Orlicz sequence space ℓ_M equipped with the Luxemburg norm. Then the* *following hold:*

(a) *if* $M \in AC^2$, $M''(0) = 0$, *and* $M''(t) > 0$ *for all* $t > 0$, *then* P *is semi-band preserving;*
(b) *if* $M \in AC^1$ *near zero,* M'' *is continuous on* $(0, \infty)$, *and* $\lim_{t \to 0} M''(t) = \infty$, *then* P *is semi-containment preserving.*

Proof. Since bounded functions with finite supports are linearly dense in ℓ_M, to show that P is semi-band preserving or semi-containment preserving, respectively, it is enough to verify that (2.14) or (2.15), respectively, are satisfied with the additional assumption that g is a bounded function and $\operatorname{card}(\operatorname{supp} g) < \infty$.

For any functions $f, g \in \ell_M$, we define

$$\psi(\alpha) = \|Pf + \alpha g\|_M, \qquad \varphi(\alpha) = \|Pf + \alpha Pg\|_M \tag{3.17}$$

for all $\alpha \in \mathbb{R}$. Then φ and ψ are convex functions and $\psi(0) = \|Pf\| = \varphi(0)$. Moreover, by Theorem 2.6, in both cases (a) and (b), φ and ψ are differentiable everywhere. Since P is a contractive projection, we also get $\varphi(\alpha) \le \psi(\alpha)$ for all $\alpha \in \mathbb{R}$.

Now to prove (a), assume that $\operatorname{card}(\operatorname{supp} g) < \infty$ and $\operatorname{supp}(g) \cap \operatorname{supp}(Pf) = \emptyset$. Since $M \in AC^2$, by Theorem 2.6, $\varphi''(0)$ and $\psi''(0)$ exist. By Proposition 2.9(a), we get $\psi'(0) = 0$ and $\psi''(0) = 0$. Hence by Lemma 3.1(i) and (ii), $\varphi'(0) = 0$ and $\varphi''(0) = 0$. Thus, by Proposition 2.9(b), we get that Pf and Pg have disjoint supports, which proves that P is semi-band preserving.

To prove (b), assume, for contradiction, that there exist $f, g \in \ell_M$ so that $\operatorname{card}(\operatorname{supp} g) < \infty$, $\operatorname{supp}(g) \subseteq \operatorname{supp}(Pf)$, and $\operatorname{supp}(Pg) \nsubseteq \operatorname{supp}(Pf)$. Note that since $M \in AC^1$, by Theorem 2.6, functions φ and ψ are differentiable everywhere, φ', ψ' are absolutely continuous, and φ'', ψ'' exist almost everywhere. Further, by Proposition 2.10(b), there exists a subset E of $[0,1]$ of full measure and $C_0 > 0$ such that for all $\alpha \in E$,

$$\psi''(\alpha) \le C_0. \tag{3.18}$$

On the other hand, by Proposition 2.10(b), $\psi''(\alpha) \to \infty$, as $\alpha \to 0$ along a subset of $[0,1]$ of full measure. Hence, by Lemma 3.1(iii) for every $C > 0$, (3.1) holds. This contradicts (3.18) and ends the proof of part (b). □

As a consequence, we obtain the characterization of contractive projections in Orlicz sequence spaces.

Theorem 3.3. *Let* M *be an Orlicz function with* $M \in \Delta_{2+}$ *and satisfying one of the following two conditions:*

(i) $M \in AC^2$, $M''(0) = 0$, *and* $M''(t) > 0$ *for all* $t > 0$; *or*
(ii) $M \in AC^1$, M'' *is continuous on* $(0, \infty)$, *and* $\lim_{t \to 0} M''(t) = \infty$.

Let P *be a contractive projection on the real Orlicz sequence space* ℓ_M *equipped with the Luxemburg norm. Then* P *is an averaging projection, that is, there exist mutually disjoint elements* $\{u_j\}_{j \in J}$ *in* ℓ_M *and functionals* $\{u_j^*\}_{j \in J}$ *in* $(\ell_M)^*$ *so that* $u_j^*(u_k) = 0$ *if* $j \neq k$, $u_j^*(u_j) = 1$ *for all* $j \in J$, *and for each* $f \in \ell_M$, *(2.13) holds. Moreover, one of the following* *three possibilities holds:*

(1) $\operatorname{card}(\operatorname{supp} u_j) < \infty$ *for each* $j \in J$, *and* $|(u_j)_k| = |(u_j)_l|$ *for each* $k, l \in \operatorname{supp}(u_j)$, $j \in J$. *(Here* $u_j = \sum_{k \in \operatorname{supp} u_j} (u_j)_k e_k$*);*

(2) *there exist* $p \in (1,\infty) \setminus \{2\}$ *and* $C \in \mathbb{R}$ *so that* $M(t) = Ct^p$ *for all* $t \leq \sup_{j\in J} \|u_j\|_\infty (\leq \infty)$*; or*

(3) *there exist* $p \in (1,\infty) \setminus \{2\}$ *and constants* $C_1, C_2, \gamma > 0$ *so that* $C_2 t^p \leq M(t) \leq C_1 t^p$ *for all* $t \leq \sup_{j\in J} \|u_j\|_\infty (\leq \infty)$, $\|u_j\|_\infty < \infty$ *for all* $j \in J$, *and for all* $j \in J$ *and* $k \in \operatorname{supp}(u_j)$,

$$|(u_j)_k| \in \{\gamma^m \cdot \|u_j\|_\infty : m \in \mathbb{Z}\}. \tag{3.19}$$

Proof. Note first that either condition (i) or (ii) implies that ℓ_M is smooth. Since $M \in \Delta_2$, ℓ_M is strictly monotone. Hence the fact that P is an averaging projection follows immediately from Theorems 2.8 and 3.2.

The moreover part follows directly from Theorem 2.13. Indeed, since the elements $\{u_j\}_{j\in J}$ are mutually disjoint, for any $j_1, j_2 \in J$ and any $f \in \ell_M$, we have

$$\|u_{j_1}^*(f)u_{j_1} + u_{j_2}^*(f)u_{j_2}\| \leq \left\| \sum_{j\in J} u_j^*(f)u_j \right\| = \|Pf\| \leq \|f\|. \tag{3.20}$$

Thus the projection $Q : \ell_M \to \operatorname{span}\{u_{j_1}, u_{j_2}\}$ defined by $Qf = u_{j_1}^*(f)u_{j_1} + u_{j_2}^*(f)u_{j_2}$ has $\|Q\| = 1$. Thus, by Theorem 2.13, conditions (1), (2), and (3) in the statement of Theorem 3.3 are satisfied. □

By duality, we also obtain the description of contractive projections in real Orlicz sequence spaces equipped with the Orlicz norm.

Corollary 3.4. *Let* M *be an Orlicz function with* $M \in \Delta_2$, $M^* \in \Delta_{2+}$, *satisfying one of the following two conditions:*

(i*) $M^* \in AC^2$, M'' *is continuous on* $(0,\infty)$, $M''(t) > 0$ *for all* $t > 0$, *and* $\lim_{t\to 0} M''(t) = \infty$*; or*

(ii*) $M^* \in AC^1$, M'' *is continuous on* $[0,\infty)$, $M''(t) > 0$ *for all* $t > 0$, *and* $M''(0) = 0$.

Let P *be a contractive projection on the real Orlicz sequence space* ℓ_M *equipped with the Orlicz norm. Then* P *has the form described in Theorem 3.3.*

Proof. This follows from Theorem 3.3 by duality. Indeed, since $M \in \Delta_2$, by (2.7) we have $(\ell_M, \|\cdot\|_M^O)^* = (\ell_{M^*}, \|\cdot\|_{M^*})$ and the dual projection P^* is contractive in ℓ_{M^*} equipped with the Luxemburg norm. Further, either of the conditions (i*) or (ii*) implies that M^* is smooth, so the only thing that needs to be verified is that condition (i*) implies that M^* satisfies condition (i) and condition (ii*) implies that M^* satisfies condition (ii) of Theorem 3.3.

For that, note that by the definition of the complementary function M^* and since in either case (i*) or (ii*), $M''(t) > 0$ for $t > 0$, we have, for all $t > 0$,

$$(M^*)''(t) = \frac{1}{M''\left((M^*)'(t)\right)}. \tag{3.21}$$

Since M'' and $(M^*)'$ are both continuous on $(0,\infty)$ in either case (i*) or (ii*), we conclude that also $(M^*)''$ is continuous on $(0,\infty)$ and $(M^*)''(t) > 0$ for all $t > 0$.

Moreover, since $\lim_{t\to 0}(M^*)'(t) = (M^*)'(0) = 0$, we have in case (i*)

$$\lim_{t\to 0}(M^*)''(t) = \lim_{t\to 0}\frac{1}{M''((M^*)'(t))} = \lim_{s\to 0}\frac{1}{M''(s)} = 0. \tag{3.22}$$

It is not difficult to check that this implies that $(M^*)''(0) = 0$. Therefore, condition (i) is implied by (i*).

Similarly, in case (ii*), we have

$$\lim_{t\to 0}(M^*)''(t) = \lim_{t\to 0}\frac{1}{M''((M^*)'(t))} = \lim_{s\to 0}\frac{1}{M''(s)} = \infty. \tag{3.23}$$

So condition (ii) is implied by (ii*).

Hence, by Theorem 3.3, in either case (i*) or (ii*), P^*, and thus also P, have the form (2.13) and the conditions (1), (2), and (3) from Theorem 3.3 hold. □

Remark 3.5. We do not know whether the assumption about smoothness of M is necessary for Theorem 3.3 and Corollary 3.4 to hold or not. We suspect that, similarly as in the complex case, smoothness of M should not be necessary.

However it is clear that some assumption about a behavior of M'' near zero is necessary. Indeed in [16, Example 3], we showed that if $a \in (\sqrt{2/3}, 1)$ and

$$M_a(t) = \begin{cases} t^2 & \text{if } 0 \le t \le a, \\ (1+a)t - a & \text{if } a \le t \le 1, \end{cases} \tag{3.24}$$

then the real or complex four-dimensional Orlicz space $\ell^4_{M_a}$ equipped with either the Luxemburg or the Orlicz norm contains a two-dimensional one-complemented isometric copy of ℓ_2^2 which cannot be spanned by a family of disjoint vectors from $\ell^4_{M_a}$. It is not difficult to adjust this example so that if a is any positive number, then the real or complex Orlicz space ℓ_{M_a} (of infinite dimension) contains a two-dimensional one-complemented isometric copy of ℓ_2^2 which cannot be spanned by a family of disjoint vectors from ℓ_{M_a}.

It would be interesting to characterize what condition on M is equivalent to the fact that ℓ_M (complex or real) does not contain a two-dimensional one-complemented isometric copy of ℓ_2^2 (which cannot be spanned by a family of disjoint vectors from ℓ_M). Either of the conditions (i), (ii), (i*), or (ii*) is clearly sufficient, but they all involve smoothness. We conjecture that the right condition is that for all $a > 0$, the function $M(t)/t^2$ is not constant on the interval $(0,a)$.

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