

# FIXED POINTS OF ASYMPTOTICALLY REGULAR NONEXPANSIVE MAPPINGS ON NONCONVEX SETS

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It is shown that if  $X$  is a Banach space and  $C$  is a union of finitely many non-empty, pairwise disjoint, closed, and connected subsets  $\{C_i : 1 \leq i \leq n\}$  of  $X$ , and each  $C_i$  has the fixed-point property (FPP) for asymptotically regular nonexpansive mappings, then any asymptotically regular nonexpansive self-mapping of  $C$  has a fixed point. We also generalize the Goebel-Schöneberg theorem to some Banach spaces with Opial's property.

## 1. Introduction

The fixed-point property for nonexpansive self-mappings of nonconvex sets has been studied by many authors (see, e.g., [3, 7, 24, 25, 26, 28, 32, 41, 47, 50]). Our first theorem ([Theorem 2.3](#)) improves upon results proved by Smarzewski [50] and Hong and Huang [25].

In [24], Goebel and Schöneberg proved the following result. If  $C$  is a non-empty bounded subset of a Hilbert space such that for any  $x \in \text{conv } C$  there is a unique  $y \in C$  satisfying  $\|x - y\| = \text{dist}(x, C)$ , then  $C$  has the fixed-point property (FPP) for nonexpansive mappings. In [Section 3](#), we generalize this result to some Banach spaces with Opial's property.

## 2. Main result

All Banach spaces considered in this paper are real. We begin by recalling the definition of an asymptotically regular mapping. The concept of asymptotic regularity is due to Browder and Petryshyn [3].

*Definition 2.1.* Let  $X$  be a Banach space and  $C$  its nonempty subset. A mapping  $T : C \rightarrow C$  is said to be asymptotically regular if for any  $x \in C$ ,

$$\lim_{n \rightarrow \infty} \|T^n x - T^{n+1} x\| = 0. \quad (2.1)$$

In 1976, Ishikawa [27] generalized Krasnosel'skiĭ's method of successive approximations [37] (see also [18, 48]). Namely, he obtained, as a special case of a more general iteration scheme, that for any nonexpansive mapping  $T$  on a closed, bounded, and convex domain  $C$  and for  $0 \leq \alpha < 1$ , the averaged mapping  $T_\alpha = \alpha I + (1 - \alpha)T$  is asymptotically regular. The same result was, independently, obtained by Edelstein and O'Brien [19] who showed that the convergence is uniform on  $C$ . In [20, 34], general theorems are obtained which unify both the Ishikawa and the Edelstein-O'Brien results.

Asymptotic regularity is a main assumption in many results in metric fixed point theory for nonexpansive mappings (see [3, 21, 23, 35, 36] and the references therein).

We also need the notion of a nonexpansive mapping [21].

*Definition 2.2.* Let  $X$  be a Banach space and  $C$  its nonempty subset. A mapping  $T : C \rightarrow C$  is nonexpansive if for any  $x, y \in C$ ,

$$\|Tx - Ty\| \leq \|x - y\|. \tag{2.2}$$

Now we show the main result of this section.

**THEOREM 2.3.** *Let  $X$  be a Banach space and let  $C = \bigcup_{i=1}^n C_i$ ,  $n \geq 2$ , be a union of nonempty, pairwise disjoint, closed, and connected subsets  $C_i$  of  $X$ , and let each  $C_i$  have the fixed-point property for asymptotically regular nonexpansive self-mappings. If  $T : C \rightarrow C$  is asymptotically regular and nonexpansive, then  $T$  has a fixed point in  $C$ .*

*Proof.* If there exists  $1 \leq i \leq n$  such that  $C_i$  is  $T$ -invariant, then  $T$  has a fixed point in  $C_i$ . In the other case, since each  $C_i$  is nonempty, connected, and closed, there exists a  $T$ -cycle. So, without loss of generality, we can assume that  $C_1, C_2, \dots, C_n$  form the  $T$ -cycle, that is,  $T(C_i) \subset C_{i+1}$  for  $i = 1, 2, \dots, n - 1$  and  $T(C_n) \subset C_1$ . Now taking  $T^n : C_1 \rightarrow C_1$  and its fixed point  $x_1$ , we get a cycle

$$x_1 \xrightarrow{T} x_2 \xrightarrow{T} \dots \xrightarrow{T} x_n \xrightarrow{T} x_{n+1} = x_1. \tag{2.3}$$

Additionally, nonexpansiveness of  $T$  gives us

$$\|x_i - x_{i-1}\| = c > 0 \quad \text{for } i = 2, 3, \dots, n + 1. \tag{2.4}$$

But this is impossible since  $\|T^{m+1}x - T^m x\| \xrightarrow{m} 0$ . □

Observe that there exist disjoint sets  $C_1$  and  $C_2$  with the fixed-point property for nonexpansive mappings and such that  $\text{dist}(C_1, C_2) = 0$ .

*Example 2.4.* In  $l_1$  we define  $C_1$  and  $C_2$  as follows:

$$\begin{aligned} C_1 &= \operatorname{conv} \left\{ e_1, \left(1 + \frac{1}{2}\right)e_2, \left(1 + \frac{1}{3}\right)e_3, \dots \right\}, \\ C_2 &= \operatorname{conv} \left\{ \frac{1}{2}e_1, \left(1 - \frac{1}{3}\right)e_2, \left(1 - \frac{1}{4}\right)e_3, \dots \right\}, \end{aligned} \quad (2.5)$$

where  $\operatorname{conv} C$  denotes the convex closed hull of  $C$ . Clearly,  $\operatorname{dist}(C_1, C_2) = 0$ . Moreover, these sets have the fixed-point property for nonexpansive mappings [22].

Recall also that there exist connected and nonconvex subsets of a Hilbert space which have the fixed point property for nonexpansive self-mappings [24].

[Theorem 2.3](#) can be generalized. To see this, recall the following definition.

*Definition 2.5* (see [33]). Let  $C$  be a nonempty subset of a Banach space  $X$ . A mapping  $T : C \rightarrow C$  is said to be asymptotically nonexpansive provided  $T$  is continuous and

$$\limsup_{n \rightarrow \infty} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \quad \forall x, y \in C. \quad (2.6)$$

*Remark 2.6.* [Theorem 2.3](#) continues to hold when the two occurrences of “nonexpansive” in its statement are replaced by “asymptotically nonexpansive.”

To formulate some corollaries, a few definitions and facts will be needed. Firmly nonexpansive mappings were introduced by Bruck [4].

*Definition 2.7* (see [4]). Let  $C$  be a nonempty subset of a Banach space  $X$ , and let  $\lambda \in (0, 1)$ . A mapping  $T : C \rightarrow X$  is said to be  $\lambda$ -firmly nonexpansive if

$$\|Tx - Ty\| \leq \|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\| \quad \forall x, y \in C. \quad (2.7)$$

If a mapping  $T$  is  $\lambda$ -firmly nonexpansive for every  $\lambda \in [0, 1]$ , then it is called firmly nonexpansive.

In [6], Bruck and Reich proved that, in a uniformly convex Banach space, every  $\lambda$ -firmly nonexpansive self-mapping  $T$  of a nonempty bounded subset  $C$  of  $X$  is asymptotically regular. Therefore, the following theorem due to Hong and Huang [25] is a simple consequence of [Theorem 2.3](#).

**THEOREM 2.8.** *Let  $X$  be a uniformly convex Banach space, and let  $C = \bigcup_{i=1}^n C_i$  be a union of nonempty, pairwise disjoint, bounded, closed, and convex subsets  $C_i$  of  $X$ . If  $T : C \rightarrow C$  is  $\lambda$ -firmly nonexpansive with some  $\lambda \in (0, 1)$ , then  $T$  has a fixed point in  $C$ .*

Hong and Huang [25] observed that their theorem is equivalent to the earlier fixed-point theorem due to Smarzewski [50]. Hence his theorem, which we quote below, is also a corollary of [Theorem 2.3](#).

**THEOREM 2.9.** *Let  $X$  be a uniformly convex Banach space, and let  $C = \bigcup_{i=1}^n C_i$  be a union of nonempty, bounded, closed, and convex subsets  $C_i$  of  $X$ . If  $T : C \rightarrow C$  is  $\lambda$ -firmly nonexpansive with some  $\lambda \in (0, 1)$ , then  $T$  has a fixed point in  $C$ .*

*Remark 2.10.* Theorem 2.1 of [26] is also an immediate consequence of our result.

*Remark 2.11.* In [28], it was shown that if  $X$  is a strictly convex Banach space and  $C$  is a union of finitely many nonempty, pairwise disjoint, closed, and connected subsets  $\{C_i : 1 \leq i \leq n\}$  of  $X$ , and each  $C_i$  has the fixed-point property for nonexpansive mappings, then any  $\lambda$ -firmly nonexpansive self-mapping of  $C$  has a fixed point.

*Remark 2.12.* In [46], Reich and Shafrir proved that if  $C$  is a nonempty bounded subset of a Banach space  $X$  and  $T : C \rightarrow C$  is firmly nonexpansive, then  $T$  is asymptotically regular (see also [13, 14, 15]). Therefore, we also have the following result: if  $X$  is a Banach space and  $C$  is a union of finitely many nonempty, pairwise disjoint, bounded, closed, and connected subsets  $\{C_i : 1 \leq i \leq n\}$  of  $X$ , and each  $C_i$  has the fixed point property for nonexpansive mappings, then each firmly nonexpansive self-mapping of  $C$  has a fixed point.

### 3. An extension of the Goebel-Schöneberg theorem

First we recall the notion of the Opial property [43].

*Definition 3.1.* A Banach space  $X$  satisfies Opial's property with respect to the weak topology if whenever a sequence  $\{x_n\}$  in  $X$  converges weakly to  $x$ , then for  $y \neq x$ ,

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|. \quad (3.1)$$

In [49], Sims gave a characterization of the Opial property in terms of support mappings (see also [11]).

The Opial property and its modifications and generalizations (see [8, 9, 32, 44, 45]) have many applications in problems of weak convergence of a sequence, either of iterates  $\{T^n(x)\}$  of a nonexpansive (asymptotically nonexpansive) mapping or averaging iterates (see, e.g., [1, 5, 7, 10, 16, 17, 23, 38, 39, 40, 43]). This property is also a crucial assumption in theorems about the behavior of some products of nonexpansive mappings [16, 17] and in ergodic results (see the literature given in [29, 30]). In other papers dealing with the ergodic theory, a Banach space  $X$  is assumed to have the Opial property for nets. It is, then, natural to ask, under what assumptions sequences can be replaced by nets in a given

condition concerning weakly convergent sequences. In [31], we can find general results in this direction and their applications in the theory of weak convergence of almost orbits of semigroups of mappings.

All  $l^p$  spaces with  $1 < p < \infty$  have Opial's property with respect to the weak topology. Even more can be said about the weakly convergent sequences.

PROPOSITION 3.2 (see [22, 42]). *In  $l^p$  with  $1 < p < \infty$ , if a sequence  $\{x_n\}$  tends weakly to  $x$ , then for  $y \in l^p$ ,*

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^p = \limsup_{n \rightarrow \infty} \|x_n - x\|^p + \|x - y\|^p. \tag{3.2}$$

In [24], the authors stated the following theorem.

THEOREM 3.3. *Let  $X$  be a Hilbert space and let  $C$  be a nonempty bounded subset of  $X$  which is Chebyshev with respect to its convex closure (i.e., for any  $x \in \text{conv } C$ , there is a unique  $y \in C$  such that  $\|x - y\| = \text{dist}(x, C)$ ). Then,  $C$  has the fixed-point property for nonexpansive mappings.*

In the case of  $l^p$  ( $1 < p < \infty$ ) spaces, we have the following version of the above theorem.

THEOREM 3.4. *Let  $C$  be a nonempty bounded subset of  $l^p$  ( $1 < p < \infty$ ) which is Chebyshev with respect to its convex closure. Then,  $C$  has the fixed point property for asymptotically regular nonexpansive mappings.*

*Proof.* Let  $T : C \rightarrow C$  be an asymptotically regular and nonexpansive mapping. We choose  $x_0 \in C$  and a subsequence  $\{T^{m_i}(x_0)\}$  of  $\{T^n(x_0)\}$  which tends weakly to  $x \in \text{conv } C$ . Hence, there is a unique  $y \in C$  such that  $\|x - y\| = \text{dist}(x, C)$ . Then, we have

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|T^{m_i}(x_0) - y\|^p &= \limsup_{i \rightarrow \infty} \|T^{m_i}(x_0) - x\|^p + \|x - y\|^p \\ &\leq \limsup_{i \rightarrow \infty} \|T^{m_i}(x_0) - x\|^p + \|x - Ty\|^p \\ &= \limsup_{i \rightarrow \infty} \|T^{m_i}(x_0) - Ty\|^p. \end{aligned} \tag{3.3}$$

On the other hand, by the asymptotic regularity and nonexpansiveness of  $T$ , we get

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|T^{m_i}(x_0) - Ty\|^p &= \limsup_{i \rightarrow \infty} \|T^{m_i+1}(x_0) - Ty\|^p \\ &\leq \limsup_{i \rightarrow \infty} \|T^{m_i}(x_0) - y\|^p. \end{aligned} \tag{3.4}$$

Hence, by the uniqueness of the nearest point in  $C$  to  $x$ , we obtain  $Ty = y$ .  $\square$

It is worth noting that there exist Banach spaces furnished with linear topologies  $\tau$  which satisfy a condition, with respect to  $\tau$ -convergent sequences, similar to that in [Proposition 3.2](#). These spaces have the Opial property with respect to the topology  $\tau$  (see [[32](#), [45](#)]). Consider the following three examples of such Banach spaces (for other examples see [[32](#), [45](#)]).

*Example 3.5* (see [[22](#), [42](#)]). Let  $X = l^1$  and let  $\{x_n\}$  be a  $\sigma(l^1, c_0)$ -convergent to  $x \in l^1$ . Then, for each  $y \in l^1$ ,

$$\limsup_{n \rightarrow \infty} \|x_n - y\| = \limsup_{n \rightarrow \infty} \|x_n - x\| + \|x - y\|. \quad (3.5)$$

*Example 3.6* (see [[2](#)]). Let  $\Omega$  be a measure space with measure  $\mu$  and  $1 \leq p < \infty$ . If  $\{f_n\}$  is a bounded sequence in  $L^p(\Omega)$  converging almost everywhere to  $f \in L^p(\Omega)$ , then

$$\limsup_{n \rightarrow \infty} \|f_n - g\|^p = \limsup_{n \rightarrow \infty} \|f_n - f\|^p + \|f - g\|^p \quad (3.6)$$

for every  $g \in L^p(\Omega)$ .

*Example 3.7* (see [[45](#)]). Let  $1 < p < \infty$  and  $\{X_k\}$  a sequence of Banach spaces with the Schur property [[12](#)], and let

$$X = \left( \sum_{k=1}^{\infty} X_k \right)_{l^p}, \quad (3.7)$$

that is,  $X$  is the space of all sequences  $x = \{x(k)\}$  such that  $x(k) \in X_k$  for every  $k$  and

$$\|x\| = \left( \sum_{k=1}^{\infty} \|x(k)\|_{X_k}^p \right)^{1/p} < \infty. \quad (3.8)$$

If  $\{x_n\}$  is a sequence in  $X$ , weakly convergent to  $x \in X$ , then, for each  $y \in X$ ,

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^p = \limsup_{n \rightarrow \infty} \|x_n - x\|^p + \|x - y\|^p. \quad (3.9)$$

It is easy to observe that the following theorem holds.

**THEOREM 3.8.** *Let  $X$  be one of the spaces from the above examples with a suitable topology  $\tau$ . Let  $C$  be a nonempty, bounded subset of  $X$  which is Chebyshev with respect to its convex closure. If  $\text{conv} C$  is  $\tau$ -sequentially compact, then  $C$  has the fixed point property for asymptotically regular nonexpansive mappings.*

*Remark 3.9.* The above theorem is also valid for some Banach spaces with  $L(\rho, \tau)$  property. This property has its origin in [42] (see also [32, 45]).

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### References

- [1] J. B. Baillon, R. E. Bruck, and S. Reich, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), no. 1, 1–9.
- [2] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.
- [3] F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 571–575.
- [4] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341–355.
- [5] R. E. Bruck, T. Kuczumow, and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math. **65** (1993), no. 2, 169–179.
- [6] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), no. 4, 459–470.
- [7] M. Budzyńska, W. Kaczor, and M. Koter-Mórgowska, *Asymptotic normal structure, semi-Opial property and fixed points*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **50** (1996), 33–41.
- [8] M. Budzyńska, W. Kaczor, M. Koter-Mórgowska, and T. Kuczumow, *Asymptotic normal structure and the semi-Opial property*, Nonlinear Anal. **30** (1997), no. 6, 3505–3515.
- [9] M. Budzyńska, T. Kuczumow, and S. Reich, *Uniform asymptotic normal structure, the uniform semi-Opial property and fixed points of asymptotically regular uniformly Lipschitzian semigroups. I*, Abstr. Appl. Anal. **3** (1998), no. 1-2, 133–151.
- [10] ———, *Uniform asymptotic normal structure, the uniform semi-Opial property, and fixed points of asymptotically regular uniformly Lipschitzian semigroups. II*, Abstr. Appl. Anal. **3** (1998), no. 3-4, 247–263.
- [11] T. Dalby and B. Sims, *Duality map characterisations for Opial conditions*, Bull. Austral. Math. Soc. **53** (1996), no. 3, 413–417.
- [12] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984.
- [13] B. Djafari Rouhani, *Asymptotic behaviour of unbounded nonexpansive sequences in Banach spaces*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 951–956.
- [14] ———, *Asymptotic behaviour of firmly nonexpansive sequences*, Proc. Amer. Math. Soc. **123** (1995), no. 3, 771–777.
- [15] B. Djafari Rouhani and W. A. Kirk, *Asymptotic properties of nonexpansive iterations in reflexive spaces*, J. Math. Anal. Appl. **236** (1999), no. 2, 281–289.
- [16] J. M. Dye, T. Kuczumow, P.-K. Lin, and S. Reich, *Random products of nonexpansive mappings in spaces with the Opial property*, Banach Spaces (Mérida, 1992),

- Contemp. Math., vol. 144, American Mathematical Society, Rhode Island, 1993, pp. 87–93.
- [17] ———, *Convergence of unrestricted products of nonexpansive mappings in spaces with the Opial property*, *Nonlinear Anal.* **26** (1996), no. 4, 767–773.
- [18] M. Edelstein, *A remark on a theorem of M. A. Krasnoselski*, *Amer. Math. Monthly* **73** (1966), 509–510.
- [19] M. Edelstein and R. C. O'Brien, *Nonexpansive mappings, asymptotic regularity and successive approximations*, *J. London Math. Soc.* (2) **17** (1978), no. 3, 547–554.
- [20] K. Goebel and W. A. Kirk, *Iteration processes for nonexpansive mappings*, *Topological Methods in Nonlinear Functional Analysis* (Toronto, Ont., 1982) (S. P. Singh and S. Thomeier, eds.), *Contemp. Math.*, vol. 21, American Mathematical Society, Rhode Island, 1983, pp. 115–123.
- [21] ———, *Topics in Metric Fixed Point Theory*, *Cambridge Studies in Advanced Mathematics*, vol. 28, Cambridge University Press, Cambridge, 1990.
- [22] K. Goebel and T. Kuczumow, *Irregular convex sets with fixed-point property for nonexpansive mappings*, *Colloq. Math.* **40** (1978/79), no. 2, 259–264.
- [23] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, *Monographs and Textbooks in Pure and Applied Mathematics*, vol. 83, Marcel Dekker, New York, 1984.
- [24] K. Goebel and R. Schöneberg, *Moons, bridges, birds ... and nonexpansive mappings in Hilbert space*, *Bull. Austral. Math. Soc.* **17** (1977), no. 3, 463–466.
- [25] Y.-M. Hong and Y.-Y. Huang, *On  $\lambda$ -firmly nonexpansive mappings in nonconvex sets*, *Bull. Inst. Math. Acad. Sinica* **21** (1993), no. 1, 35–42.
- [26] ———, *Fixed point theorems for nonexpansive mappings in nonconvex sets*, *Southeast Asian Bull. Math.* **18** (1994), no. 2, 29–33.
- [27] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, *Proc. Amer. Math. Soc.* **59** (1976), no. 1, 65–71.
- [28] W. Kaczor, *Fixed points of  $\lambda$ -firmly nonexpansive mappings on nonconvex sets*, *Nonlinear Anal.* **47** (2001), 2787–2792.
- [29] W. Kaczor, T. Kuczumow, and S. Reich, *A mean ergodic theorem for nonlinear semigroups which are asymptotically nonexpansive in the intermediate sense*, *J. Math. Anal. Appl.* **246** (2000), no. 1, 1–27.
- [30] ———, *A mean ergodic theorem for mappings which are asymptotically nonexpansive in the intermediate sense*, *Nonlinear Anal.* **47** (2001), 2731–2742.
- [31] W. Kaczor and S. Prus, *Asymptotical smoothness and its applications to weak convergence of almost orbits of semigroups*, preprint, 2002.
- [32] M. A. Khamsi, *On uniform Opial condition and uniform Kadec-Klee property in Banach and metric spaces*, *Nonlinear Anal.* **26** (1996), no. 10, 1733–1748.
- [33] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, *Israel J. Math.* **17** (1974), 339–346.
- [34] ———, *Krasnosel'skii's iteration process in hyperbolic space*, *Numer. Funct. Anal. Optim.* **4** (1981/82), no. 4, 371–381.
- [35] ———, *Nonexpansive mappings and asymptotic regularity*, *Nonlinear Anal.* **40** (2000), no. 1–8, Ser. A: Theory Methods, 323–332.
- [36] W. A. Kirk and B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.
- [37] M. A. Krasnosel'skiĭ, *Two remarks on the method of successive approximations*, *Uspehi Mat. Nauk (N.S.)* **10** (1955), no. 1(63), 123–127 (Russian).



- [38] T. Kuczumow, *Weak convergence theorems for nonexpansive mappings and semigroups in Banach spaces with Opial's property*, Proc. Amer. Math. Soc. **93** (1985), no. 3, 430–432.
- [39] ———, *Opial's modulus and fixed points of semigroups of mappings*, Proc. Amer. Math. Soc. **127** (1999), no. 9, 2671–2678.
- [40] T. Kuczumow and S. Reich, *An application of Opial's modulus to the fixed point theory of semigroups of Lipschitzian mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **51** (1997), no. 2, 185–192.
- [41] T. Kuczumow, S. Reich, and A. Stachura, *Minimal displacement of points under holomorphic mappings and fixed point properties for unions of convex sets*, Trans. Amer. Math. Soc. **343** (1994), no. 2, 575–586.
- [42] T.-C. Lim, *Asymptotic centers and nonexpansive mappings in conjugate Banach spaces*, Pacific J. Math. **90** (1980), no. 1, 135–143.
- [43] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.
- [44] S. Prus, *Banach spaces with the uniform Opial property*, Nonlinear Anal. **18** (1992), no. 8, 697–704.
- [45] ———, *Geometrical background of metric fixed point theory*, Handbook of Metric Fixed Point Theory (W. A. Kirk and B. Sims, eds.), Kluwer Academic Publishers, Dordrecht, 2001, pp. 93–132.
- [46] S. Reich and I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc. **101** (1987), no. 2, 246–250.
- [47] J. Reinermann and R. Schöneberg, *Some results and problems in the fixed point theory for nonexpansive and pseudocontractive mappings in Hilbert-space*, Fixed Point Theory and Its Applications (Proc. Sem., Dalhousie Univ., Halifax, N.S., 1975), Academic Press, New York, 1976, pp. 187–196.
- [48] H. Schaefer, *Über die Methode sukzessiver Approximationen*, Jber. Deutsch. Math.-Verein. **59** (1957), no. Abt. 1, 131–140 (German).
- [49] B. Sims, *A support map characterization of the Opial conditions*, Miniconference on Linear Analysis and Function Spaces (Canberra, 1984), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 9, Austral. Nat. Univ., Canberra, 1985, pp. 259–264.
- [50] R. Smarzewski, *On firmly nonexpansive mappings*, Proc. Amer. Math. Soc. **113** (1991), no. 3, 723–725.

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