A REMARK ON THE APPROXIMATE FIXED-POINT PROPERTY

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We give an example of an unbounded, convex, and closed set C in the Hilbert space l^2 with the following two properties: (i) C has the approximate fixed-point property for nonexpansive mappings, (ii) C is not contained in a block for every orthogonal basis in l^2.

1. Introduction

In [6], Goebel and the author observed that some unbounded sets in Hilbert spaces have the approximate fixed-point property for nonexpansive mappings. Namely, they proved that every closed convex set C, which is contained in a block, has the approximate fixed-point property for nonexpansive mappings (AFPP). This result was extended by Ray [14] to all linearly bounded subsets of l_p, $1 < p < \infty$. Next, he proved that a closed convex subset C of a real Hilbert space has the fixed-point property for nonexpansive mappings if and only if it is bounded [15]. The first result of Ray [14] was generalized by Reich [16] (for other results of this type see [1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19]). Reich [16] proved the following remarkable theorem: a closed, convex subset of a reflexive Banach space has the AFPP if and only if it is linearly bounded. Next, Shafrir [18] introduced the notion of a directionally bounded set. Using this concept, he proved two important theorems [18].

(1) A convex subset C of a Banach space X has the AFPP if and only if C is directionally bounded.

(2) For a Banach space X, the following two conditions are equivalent: (i) X is reflexive; (ii) every closed, convex, and linearly bounded subset C of X is directionally bounded.

Therefore, the following statements are equivalent: (a) X is reflexive; (b) a closed, convex subset C of X has the AFPP if and only if C is linearly bounded. This result is strictly connected with the above-mentioned Reich theorem [16].

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Now, it is worth to note that, recently, there is a return to study the AFPP First, Espínola and Kirk [3] published a paper about the AFPP in the product spaces. They proved that the product space $D = (M \times C)_\infty$ has the AFPP for nonexpansive mappings whenever M is a metric space which has the AFPP for such mappings and C is a bounded, convex subset of a Banach space. Next, Wiśnicki wrote a paper about a common approximate fixed-point sequence for two commuting nonexpansive mappings (see [20] for details). Therefore, the author decided to publish an example of a set which is closely related to the AFPP Namely, it is obvious that every blockable set in l^2 is linearly bounded, but there are linearly bounded sets in l^2 which are not contained in any block with respect to an arbitrary basis. This was mentioned in [6] but never published. The aim of this paper is to show the construction of such a set.

2. Preliminaries

Throughout this paper, l^2 is real, $\langle \cdot, \cdot \rangle$ denotes the scalar product in l^2, and $\{e_n\}$ is the standard basis in l^2.

For any nonempty set $K \subset l^2$, the closed convex hull of K is denoted by $\operatorname{conv} K$.

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive if for each $x, y \in C$,

$$||T(x) - T(y)|| \leq \|x - y\|. \tag{2.1}$$

A convex subset C of a Banach space X has the approximate fixed-point property (AFPP) if each nonexpansive $T : C \to C$ satisfies

$$\inf \{||x - T(x)|| : x \in C\} = 0. \tag{2.2}$$

It is obvious that bounded convex sets always have the AFPP.

A set $K \subset l^2$ is said to be a block in the orthogonal basis $\{\tilde{e}_n\}$ if K is of the form

$$K = \{x \in l^2 : |\langle x, \tilde{e}_n \rangle| \leq M_n,\ n = 1, 2, \dots\}, \tag{2.3}$$

where $\{M_n\}$ is a sequence of positive reals.

The set $C \subset l^2$ is called a block set if there exists a block $K \subset l^2$ such that C is a subset of K.

A subset C of a Banach space X is linearly bounded if C has bounded intersections with all lines in X.

3. The construction

Let $\{k_n\}_{n=2}^{\infty}$ and $\{l_n\}_{n=2}^{\infty}$ be two sequences of positive reals such that

$$\sum_{n=2}^{\infty} \frac{k_n}{l_n} < +\infty, \quad \lim_n k_n = +\infty. \tag{3.1}$$

For example, we may take $k_n = n$ and $l_n = n^3$ for $n = 2, 3, \ldots$. Next, we set

$$a_n = k_n e_1 + l_n e_n, \qquad b_n = -k_n e_1 + l_n e_n, \tag{3.2}$$

for $n = 2, 3, \ldots$, and finally,

$$C = \operatorname{conv}\{x \in l^2 : \exists n \geq 2(x = a_n \vee x = b_n)\}. \tag{3.3}$$

Theorem 3.1. *If*

$$x = \sum_{n=1}^{\infty} c_n e_n = c_1 e_1 + \sum_{n=2}^{\infty} d_n l_n e_n = c_1 e_1 + \bar{x} \tag{3.4}$$

is an element of the set C, then

$$d_n \geq 0 \tag{3.5}$$

for $n = 2, 3, \ldots,$

$$\sum_{n=2}^{\infty} d_n \leq 1, \tag{3.6}$$

and there exist sequences $\{\alpha_n\}_{n=2}^{\infty}$ *and* $\{\beta_n\}_{n=2}^{\infty}$ *such that*

$$c_1 = \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n), \quad \alpha_n, \beta_n \geq 0,\ \alpha_n + \beta_n = d_n, \tag{3.7}$$

for $n = 2, 3, \ldots$. *Additionally, there exists a positive constant* $M_{\bar{x}}$ *such that*

$$0 \leq (\alpha_n + \beta_n) k_n = d_n k_n \leq M_{\bar{x}} \frac{k_n}{l_n} \tag{3.8}$$

for $n = 2, 3, \ldots$.

Proof. Set

$$\bar{x} = \sum_{n=2}^{\infty} c_n e_n = \sum_{n=2}^{\infty} d_n l_n e_n. \tag{3.9}$$

Observe that, there exists a sequence $\{x_j\}_{j=1}^{\infty}$ such that

$$x = \lim_j x_j \tag{3.10}$$

with

$$\begin{aligned} x_j &= \sum_{n=2}^{\infty} (\alpha_{nj} a_n + \beta_{nj} b_n) \\ &= \sum_{n=2}^{\infty} (\alpha_{nj} k_n - \beta_{nj} k_n) e_1 + \sum_{n=2}^{\infty} (\alpha_{nj} l_n + \beta_{nj} l_n) e_n \\ &= \sum_{n=2}^{\infty} (\alpha_{nj} k_n - \beta_{nj} k_n) e_1 + \bar{x}_j \in C, \end{aligned} \tag{3.11}$$

where

$$\bar{x}_j = \sum_{n=2}^{\infty} (\alpha_{nj} l_n + \beta_{nj} l_n) e_n, \quad \alpha_{nj}, \beta_{nj} \geq 0, \ \sum_{n=2}^{\infty} (\alpha_{nj} + \beta_{nj}) = 1. \tag{3.12}$$

Without loss of generality, we can assume that $\{\alpha_{nj}\}_{j=1}^{\infty}$ and $\{\beta_{nj}\}_{j=1}^{\infty}$ tend to α_n and β_n, respectively, for $n = 2, 3, \ldots$. Hence, we have

$$c_1 = \sum_{n=2}^{m} (\alpha_n k_n - \beta_n k_n) + \lim_j \sum_{n=m+1}^{\infty} (\alpha_{nj} k_n - \beta_{nj} k_n) \tag{3.13}$$

for each $m \geq 2$. On the other hand,

$$\bar{x} = \lim_j \bar{x}_j = \lim_j \sum_{n=2}^{\infty} (\alpha_{nj} l_n + \beta_{nj} l_n) e_n \tag{3.14}$$

and, therefore, there exists a constant $0 < M_{\bar{x}} < +\infty$ such that

$$\alpha_{nj} l_n + \beta_{nj} l_n \leq M_{\bar{x}} \tag{3.15}$$

for all $n \geq 2$ and $j \in \mathbb{N}$. This implies that

$$\begin{gathered} 0 \leq \alpha_{nj} k_n + \beta_{nj} k_n = (\alpha_{nj} l_n + \beta_{nj} l_n) \frac{k_n}{l_n} \leq M_{\bar{x}} \frac{k_n}{l_n}, \\ 0 \leq (\alpha_n + \beta_n) k_n = d_n k_n \leq M_{\bar{x}} \frac{k_n}{l_n}, \end{gathered} \tag{3.16}$$

for all j, n, and finally,

$$\begin{aligned} \sup_j \left| \sum_{n=m+1}^{\infty} (\alpha_{nj} k_n - \beta_{nj} k_n) \right| &\leq \sup_j \sum_{n=m+1}^{\infty} (\alpha_{nj} k_n + \beta_{nj} k_n) \\ &\leq \sum_{n=m+1}^{\infty} M_{\bar{x}} \frac{k_n}{l_n} = M_{\bar{x}} \sum_{n=m+1}^{\infty} \frac{k_n}{l_n} \xrightarrow{m \to \infty} 0. \end{aligned} \tag{3.17}$$

Combining (3.13) with (3.17), we conclude that

$$c_1 = \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n). \tag{3.18}$$

This completes the proof. □

THEOREM 3.2. *The set C is linearly bounded but is not a block set in any orthogonal basis in l^2.*

Proof. First, we show that C is not a block set in any orthogonal basis,

$$\{\tilde{e}_i\}_{i=1}^{\infty} = \left\{ \sum_{n=1}^{\infty} c_{in} e_n \right\}_{i=1}^{\infty} \tag{3.19}$$

in l^2. Indeed, there exists i_0 such that $c_{i_0 1} \neq 0$. Since we have

$$\max\left(\left| \langle a_n, \tilde{e}_{i_0} \rangle \right|, \left| \langle b_n, \tilde{e}_{i_0} \rangle \right| \right) = k_n \left| c_{i_0 1} \right| + l_n \left| c_{i_0 n} \right| \tag{3.20}$$

for every $n \geq 2$, these two facts imply that

$$\sup\left\{ \left| \langle x, \tilde{e}_{i_0} \rangle \right| : x \in C \right\} = +\infty. \tag{3.21}$$

Therefore, C is not a block set in $\{\tilde{e}_i\}_{i=1}^{\infty}$.

Now, we prove that the set C is linearly bounded. We begin with the following simple observation:

$$\sup\left\{ \left| \langle x, e_n \rangle \right| : x \in C \right\} \leq l_n \tag{3.22}$$

for $n = 2, 3, \ldots$. Next, if $x \in C$ is of the form

$$x = \sum_{n=1}^{\infty} c_n e_n = c_1 e_1 + \sum_{n=2}^{\infty} d_n l_n e_n = c_1 e_1 + \bar{x}, \tag{3.23}$$

then, by Theorem 3.1, we see that

$$d_n \geq 0 \tag{3.24}$$

for $n = 2, 3, \ldots$,

$$\sum_{n=2}^{\infty} d_n \leq 1, \tag{3.25}$$

and there exist sequences $\{\alpha_n\}_{n=2}^{\infty}$ and $\{\beta_n\}_{n=2}^{\infty}$ such that

$$c_1 = \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n), \qquad \alpha_n, \beta_n \geq 0,\ \alpha_n + \beta_n = d_n, \tag{3.26}$$

for $n = 2, 3, \ldots$. Additionally, there exists a positive constant $M_{\bar{x}}$ such that

$$0 \leq (\alpha_n + \beta_n) k_n = d_n k_n \leq M_{\bar{x}} \frac{k_n}{l_n} \tag{3.27}$$

for $n = 2, 3, \ldots$. Hence, we obtain

$$|c_1| = \left| \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n) \right| \leq \sum_{n=2}^{\infty} (\alpha_n + \beta_n) k_n \leq M_{\bar{x}} \sum_{n=2}^{\infty} \frac{k_n}{l_n}. \tag{3.28}$$

Then, it follows from (3.22) and (3.28) that an intersection of C with any line $\{y + tv : t \in \mathbb{R}\}$, where $y, v \in l^2$ and $v \neq 0$, is either empty or bounded which completes the proof. □

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