ILL-POSED EQUATIONS WITH TRANSFORMED ARGUMENT

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We discuss the operator transforming the argument of a function in the L^2-setting. Here this operator is unbounded and closed. For the approximate solution of ill-posed equations with closed operators, we present a new view on the Tikhonov regularization.

1. Introduction

In theory and applications, many kinds of equations occur with transformed argument, that is, with a transformation operator T_ρ, defined on function spaces by $T_\rho x = x \circ \rho$. Examples are differential or integral equations with delay, the algebraic approach of Przeworska-Rolewicz [6] with involutions, reflections or rotations, control problems, and so forth.

In the spaces $C(K)$ of continuous functions on a compact set K, the transformation operator T_ρ is completely discussed (see [10]). Here we will consider the question of (approximate) solvability of an equation

$$T_\rho x = y \tag{1.1}$$

in the Hilbert space $L^2(K)$. The transformation operator in general is not continuous and the range is not closed. Equations of type (1.1) are ill posed. We will use Tikhonov regularization for the approximate solution of (1.1). For this, we have to develop a theory of Tikhonov regularization for unbounded operators in Hilbert spaces.

2. The transformation operator

Let $K \subset \mathbb{R}^n$ be a compact subset and $X = L^2(K)$. Let $\rho : K \to K$ be a continuous surjective mapping with the property

(P) the image and the preimage of sets of measure zero are of measure zero.

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Remark 2.1. If ρ is continuously differentiable and the set S_ρ of critical points of ρ is of measure zero, then ρ has property (P). If ρ is continuously differentiable (or Lipschitzian), it maps zero sets into zero sets. In every connecting component of $K \setminus S_\rho$, the map ρ is a diffeomorphism and ρ^{-1} maps zero sets into zero sets.

Let T_ρ be the following transformation operator: for $x \in D(T_\rho)$, we define

$$(T_\rho x)(t) = x(\rho(t)), \tag{2.1}$$

where the domain of T_ρ is

$$D(T_\rho) = \left\{x \in L^2(K) : \int_K |x(\rho(t))|^2 < \infty\right\}. \tag{2.2}$$

Then T_ρ is a linear not necessarily bounded operator.

Theorem 2.2. *If ρ satisfies property (P), then the operator $T_\rho : D(T_\rho) \to L^2(K)$ is well defined, injective, and closed.*

Proof. If two functions x, y differ by a function of measure zero, then

$$T_\rho x - T_\rho y = T_\rho(x - y) = 0 \tag{2.3}$$

by property (P), thus T_ρ is well defined. If $T_\rho x = 0$, then x is equivalent to a zero function, hence $x = 0$ and T_ρ is injective.

Let (x_n) be a sequence in $D(T_\rho)$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Tx_n = y$. Then, there exists a subsequence (x_{n_k}) with the property

$$\lim x_{n_k}(t) = x(t) \quad \text{a.e.}, \qquad \lim x_{n_k}(\rho(t)) = y(t) \quad \text{a.e.} \tag{2.4}$$

The set $M := \{t \in K : \lim x_{n_k}(\rho(t)) \neq x(\rho(t))\}$ is of measure zero since $M = \rho^{-1}(M')$ with $M' := \{s \in K : \lim x_{n_k}(s) \neq x(s)\}$. Since M' is of measure zero and ρ has property (P), the set M is of measure zero; hence

$$\lim x_{n_k}(\rho(t)) = x(\rho(t)) \quad \text{a.e.}, \qquad x(\rho(t)) = y(t) \quad \text{a.e.} \tag{2.5}$$

So we see that $x \in D(T_\rho)$ and $T_\rho x = y$. □

It is easy to see that the characteristic functions of measurable subsets of K belong to $D(T_\rho)$, hence the set of step functions is dense in $D(T_\rho)$. Now we will show that the transformation operator is symmetric only in the trivial case $\rho = \text{id}$, and in opposite to the case of $C(K)$ the transformation operator is not isometric except for $\rho = \text{id}$.

Theorem 2.3. *Let $T_\rho : D(T_\rho) \to X$ be a transformation operator with property (P). Then,*

(a) *T_ρ is symmetric if and only if $\rho = \text{id}$;*
(b) *T_ρ is isometric if and only if $\rho = \text{id}$.*

Proof. If T_ρ is symmetric, then for every $x, y \in D(T_\rho)$ we have $\langle T_\rho x, y\rangle = \langle x, T_\rho y\rangle$. Since the unit function $x = \mathbf{1} \in D(T_\rho)$, we get with $T_\rho x = x$

$$\int_K y(t)dt = \int_K y(\rho(t))dt. \tag{2.6}$$

If there exists a $t_0 \in K$ with $\rho(t_0) \neq t_0$, then there is a nonzero step function y with the value one in an open cube C containing t_0, such that $C \cap \rho(C) = \varnothing$, and the value zero outside of C. Then,

$$\int_K y(t)dt = \operatorname{meas}(K \cap C) \neq 0, \qquad \int_K y(\rho(t))dt = 0. \tag{2.7}$$

Hence $\rho(t) = t$ for all $t \in K$. If T_ρ is isometric, then for every $x, y \in D(T_\rho)$ we have $\langle T_\rho x, T_\rho y\rangle = \langle x, y\rangle$. If we again use the unit function x and the step function y, then we have the desired result. □

3. Tikhonov regularization of equations with closed operators

Here we discuss the convergence and the speed of convergence for equations in Hilbert spaces

$$Tx = y, \tag{3.1}$$

where $T : D(T) \to X_2$ is a densely defined closed operator with $D(T) \subset X_1$. We will see that the results and the proofs are very similar to the case of continuous operators in Hilbert spaces, especially we will choose a different method as in [9]. Our investigations strongly depend on the following result of von Neumann [11].

THEOREM 3.1. *Let $T : D(T) \to X$ be a closed and densely defined operator in a Hilbert space X. Then the operators*

$$\begin{gathered} B = (I + T^*T)^{-1}, \qquad C = T(I + T^*T)^{-1}, \\ A = T^*T(I + T^*T)^{-1} \end{gathered} \tag{3.2}$$

are continuous and bounded by

$$\|B\| \le 1, \qquad \|C\| \le \frac{1}{2}, \qquad \|A\| \le 1. \tag{3.3}$$

Proof. The continuity and the bound of B is shown by von Neumann, also the continuity of C, see also [8]. The equation $A + B = I$ is easy to verify. Since B is positive definite, $\|B\| \le 1$, we get

$$0 \le A = I - B \le I, \tag{3.4}$$

hence $\|A\| \leq 1$. Finally from

$$C^*C = BA = B - B^2, \tag{3.5}$$

we obtain, with $0 \leq B \leq I$,

$$\|BA\| = \sup_{0\leq\beta\leq 1} (\beta - \beta^2) = \frac{1}{4}, \qquad \|C\| \leq \frac{1}{2}. \tag{3.6}$$

□

By easy calculations, we can show the following corollary.

Corollary 3.2. *Let $A_\alpha = T^*T(\alpha I + T^*T)^{-1}$, $B_\alpha = (\alpha I + T^*T)^{-1}$, and $C_\alpha = T(\alpha I + T^*T)^{-1}$. Then for all positive reals α, the operators A_α, B_α, and C_α are continuous with*

$$||A_\alpha|| \leq 1, \qquad ||B_\alpha|| \leq \frac{1}{\alpha}, \qquad ||C_\alpha|| \leq \frac{1}{2\sqrt{\alpha}}. \tag{3.7}$$

If T is not surjective, then equation

$$Tx = y \tag{3.8}$$

is ill posed, also in the case when $y \in \operatorname{Range} T$. In this case, Tikhonov regularization is a widely used method for a stable approximation of the solution of (1.1) (see, e.g., [2]).

For every $\alpha > 0$, we determine the approximation x_α of the solution $\hat{x}$ of (3.8) using the equation

$$(\alpha I + T^*T)x_\alpha = T^*y. \tag{3.9}$$

In the next theorem, we show that x_α converge to $\hat{x}$.

Theorem 3.3. *Let T be injective, densely defined, and closed. Then for every $y \in \operatorname{Range} T$, the elements x_α defined by*

$$x_\alpha = T^*(\alpha I + TT^*)^{-1}y \tag{3.10}$$

converge to the solution $\hat{x}$ of (3.8) if α tends to zero.

Proof. With the notation of Corollary 3.2, we have

$$x_\alpha = C_\alpha^* y = C_\alpha^* T\hat{x} = A_\alpha \hat{x}, \qquad x_\alpha - \hat{x} = \alpha B_\alpha \hat{x}. \tag{3.11}$$

The family of continuous operators αB_α, $\alpha > 0$, is uniformly bounded by one. Since A is injective, $\operatorname{Range} A^\nu$ is dense in X for $0 < \nu \leq 1$. For $\hat{x} \in \operatorname{Range} A^\nu$, we obtain

$$x_\alpha - \hat{x} = \alpha B_\alpha A^\nu \hat{u}. \tag{3.12}$$

Now we estimate

$$x_\alpha - \hat{x} = \alpha B_\alpha A^\nu \hat{u} = \alpha^\nu \left(\frac{T^*T}{\alpha}\right)^\nu \left(I + \frac{T^*T}{\alpha}\right)^{-1} B^\nu \hat{u} \tag{3.13}$$

and (if $\nu < 1$)

$$\begin{aligned} \|x_\alpha - \hat{x}\| &\le \alpha^\nu \nu^\nu (1-\nu)^{1-\nu} \|B^\nu \hat{u}\| \\ &\le \alpha^\nu \nu^\nu (1-\nu)^{1-\nu} \|\hat{u}\| \end{aligned} \tag{3.14}$$

(resp., $\|x_\alpha - \hat{x}\| \le \alpha^\nu \|\hat{u}\| = \alpha \|\hat{u}\|$ if $\nu = 1$). Therefore, we have convergence on a dense set, by the uniform boundedness principle we have convergence for all $\hat{x} \in X$. □

Checking this proof, we see the following corollary.

Corollary 3.4. *If* $\hat{x} \in \operatorname{Range} A^\nu$, $0 \le \nu \le 1$, *then the speed of convergence*

$$\|x_\alpha - \hat{x}\| = \mathbb{O}(\alpha^\nu). \tag{3.15}$$

In the ill-posed cases, that is, if T is not surjective, then (3.8) is not solvable for a set of second category. Let $\{y_\delta, \delta > 0\}$ be a family of elements in X_2 with $\|y - y_\delta\| \le \delta$. Then, we have to discuss the behaviour of $x_{\alpha,\delta}$ defined by

$$x_{\alpha,\delta} = T^* (\alpha I + TT^*)^{-1} y_\delta. \tag{3.16}$$

Theorem 3.5. *Let* $y \in \operatorname{Range}(T)$, $y_\delta \in D(T^*)$, *and* $\|y - y_\delta\| \le \delta$. *Then,*

$$\|x_\alpha - x_{\alpha,\delta}\| \le \frac{\delta}{2\sqrt{\alpha}}. \tag{3.17}$$

Proof. We have

$$\|x_\alpha - x_{\alpha,\delta}\| = \|C_\alpha^* (y - y_\delta)\| \le \|C_\alpha^*\| \cdot \delta \le \frac{\delta}{2\sqrt{\alpha}}. \tag{3.18}$$

□

If we additionally assume $\hat{x} \in \operatorname{Range} A^\nu$, then we have the following corollary.

Corollary 3.6. *Let* $\hat{x} \in \operatorname{Range} A^\nu$, $0 < \nu \le 1$, *then*

$$\|\hat{x} - x_{\alpha,\delta}\| = \mathbb{O}(\delta^{2\nu/(2\nu+1)}). \tag{3.19}$$

Remark 3.7. This speed of convergence is the optimal speed with a priori information. Of course this information is in general not available. But about the choice of the parameter α in the Tikhonov regularization with perturbed data for equations with unbounded operators, some results exist, for example, Cheng and Yamamoto [1], Hegland [3], Ivanov et al. [4], Liskovets [5], and Ramm [7].

4. Computational remarks

Since $D(T_\rho)$ is dense in $L^2(K)$, the adjoint operator T_ρ^* is well defined, but only in the case where ρ is a (simple) diffeomorphism of K, T_ρ^* can be given explicitly. But if in the case when one solves (1.1) by the Galerkin method, this is not a disadvantage: the Galerkin method consists in solving the equation

$$(\alpha I + P_n T_\rho^* T_\rho P_n) x_\alpha^G = P_n T_\rho^* y, \tag{4.1}$$

where x_α^G is contained in a finite-dimensional subspace X_n of X with an orthoprojection $P_n : X \to X_n$. If $u_1, \ldots, u_n$ is an orthonormal basis of X_n, then the Fourier coefficients ξ_j of x_α^G can be determined by the equations

$$\alpha \xi_j + \sum \xi_k \langle T_\rho u_k, T_\rho u_j \rangle = \langle y, T_\rho u_j \rangle \tag{4.2}$$

for $j = 1, 2, \ldots, n$. So the form of the operator T_ρ^* is not necessary.

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