APPROXIMATION STRUCTURES AND APPLICATIONS TO EVOLUTION EQUATIONS

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We discuss various properties of the nonlinear A-proper operators as well as a generalized Leray-Schauder principle. Also, a method of approximating arbitrary continuous operators by A-proper mappings is described. We construct, via appropriate Browder-Petryshyn approximation schemes, approximative solutions for linear evolution equations in Banach spaces.

1. Introduction

The purpose of this paper is to describe some approximation structures for nonlinear operators in Banach spaces as well as a number of applications to evolution equations. In Sections 2 and 3, we prove results concerning the nonlinear A-proper operators, such as the positive decomposition, the A-properness of the Dirac mass operator, and a method of approximating arbitrary continuous operators by A-proper mappings. We also prove a generalized Leray-Schauder Principle via the A-proper mapping theory.

The third section is devoted to the existence of approximative solutions for linear evolution equations in Banach spaces. This is done via the same approximation schemes as in the case of A-proper operators. Our results complement somehow the classical results on C_0-semigroups and show what happens beyond the standard hypotheses.

Given a separable Banach space E with Schauder basis, we construct (via an approximation scheme) a linear operator A on E such that the differential equation

$$\frac{du}{dt} = Au, \quad u(0) = u_0, \tag{1.1}$$

admits a sequence $(u_n)_n$ of C^∞ solutions for each $u_0 \in E$, $u_0 \neq 0$, such that $(u_n)_n$

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converges uniformly to a C^∞ function u and $(du_n/dt - Au_n)_n$ converges uniformly to 0, but u is not a solution.

A complementary phenomenon is described in Theorem 4.2. For $E = C([0,1] \times [0,1])$, one shows the existence of a wildly discontinuous linear operator $A : E \to E$, for which the initial value problem

$$\frac{du}{dt} = Au + f, \quad t \in [0,T], \qquad u(0) = u_0, \tag{1.2}$$

has a generalized solution whatever are $f \in C^1([0,T],E)$, $u_0 \in E$, and $T > 0$.

2. A-properness via approximation schemes

Let X and Y be two separable Banach spaces and let

$$\Gamma = (\{X_n\},\{Y_n\},\{P_n\},\{Q_n\}) \tag{2.1}$$

be an approximation scheme, where $X_n \subset X$, $Y_n \subset Y$ are linear subspaces with $\dim X_n = \dim Y_n < \infty$, $P_n : X_n \to X$ are the canonical isometries, and $Q_n : Y \to Y_n$ are continuous operators such that $Q_n y \to y$ ($y \in Y$).

An operator $T : X \to Y$ is named *A-proper* with respect to Γ provided that the operators $T_n = Q_n T P_n$ are continuous, and any bounded sequence $\{x_k;\, x_k \in X_{n_k}\}$ such that $T_{n_k} x_k \to f$, where $f \in Y$, has a subsequence $\{x_{k_j}\}$ so that $x_{k_j} \to x_0$ and $Tx_0 = f$.

Definition 2.1. The scheme Γ is called *of type* (**C**) if and only if it satisfies the following conditions:

(a) $X_n \subset X_{n+1}$; $Y_n \subset Y_{n+1}$ for any n;
(b) $Q_n \in L(Y)$ are projectors such that $R(Q_n) = Y_n$;
(c) $m \le n$ implies $Q_m Q_n = Q_m$.

Theorem 2.2. *Suppose that Γ is of type* (**C**)*. Then there exists a continuous noncompact operator $S : X \to Y$, with bounded $R(S)$, such that for any A-proper operator $T : X \to Y$ which is uniformly continuous on bounded subsets, the sum operator $T + S$ is A-proper, too.*

Proof. Let $\delta \in (0,1)$. We may construct the sequences $\{x_i\} \subset X$ and $\{z_i\} \subset Y$ such that

$$\begin{gathered} \|x_i\| = \|z_i\| = 1, \quad x_i \in X_i,\ z_i \in Y_i, \\ d(x_i, X_j) \ge \delta, \quad d(z_i, Y_j) \ge \delta \quad (\text{for } j < i). \end{gathered} \tag{2.2}$$

We define $y_i \in Y$ by

$$y_i = z_i - Q_{i-1} z_i. \tag{2.3}$$

Then

$$\begin{gathered}\sup\|y_i\| \le 1+\sup\|Q_i\| < +\infty,\\ y_i \in Y_i, \quad d(y_i, Y_j) \ge \delta \quad (j<i),\\ Q_m y_k = \begin{cases} y_k, & \text{if } m \ge k,\\ 0, & \text{if } m<k.\end{cases}\end{gathered} \tag{2.4}$$

Now, choose $\mu \in (0, \delta/2)$ and the sequence $\{\varepsilon_n\} \subset (0,\mu]$ such that $\varepsilon_n \to 0$. We define the functions $\{\varphi_i\} \subset C(X,\mathbb{R})$ by

$$\varphi_i(x) = \max\left(0, 1-\varepsilon_i^{-1}\|x-x_i\|\right) \quad (x \in X). \tag{2.5}$$

Clearly,

$$i \ne j \Longrightarrow \operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \varnothing \tag{2.6}$$

and $0 \le \varphi_i(x) \le 1$ for all $i \in \mathbb{N}^*$ and all $x \in X$.

We introduce the operator $S: X \to Y$ defined by

$$S(x) = \sum_{i=1}^{\infty} \varphi_i(x) y_{i+1}. \tag{2.7}$$

Then

$$\sup_{x\in X}\|S(x)\| \le \sup_n \|y_n\| < +\infty. \tag{2.8}$$

We claim that S is continuous. Indeed, let $u_0 \in X$. We have the following two alternatives.

(a) We have $\|u_0 - x_i\| > \varepsilon_i$ for any $i \in \mathbb{N}^*$. We claim that there exists $\lambda > 0$ such that $u \in B(u_0,\lambda)$ implies

$$\|u-x_i\| > \varepsilon_i \quad (i \in \mathbb{N}^*). \tag{2.9}$$

Indeed, if this is not true, then we can find $n \in \mathbb{N}^*$ and $u_1 \in B(u_0,\alpha)$ such that $\|u_1 - x_n\| \le \varepsilon_n$, where $\alpha \in (0, \delta/2 - \mu)$. We now choose a $\beta > 0$ with $\beta < \min\{\alpha, \|u_0 - x_n\| - \varepsilon_n\}$. Then there exist $u_2 \in B(u_0,\beta)$ and $m \in \mathbb{N}^*$ such that $\|u_2 - x_m\| \le \varepsilon_m$.

We obtain

$$\|u_0 - x_n\| \le \alpha + \varepsilon_n, \qquad \|u_0 - x_m\| \le \beta + \varepsilon_m, \tag{2.10}$$

hence

$$\|x_m - x_n\| \le \alpha + \beta + \varepsilon_m + \varepsilon_n \le 2\mu + 2\alpha < \delta, \tag{2.11}$$

and thus $m = n$. Then we have

$$||u_0 - x_n|| \leq \beta + \varepsilon_n < ||u_0 - x_n||, \tag{2.12}$$

a contradiction. Consequently, there exists $\lambda > 0$ such that

$$\varphi_i(u) = 0 \quad \forall i \in \mathbb{N}^*,\ u \in B(u_0, \lambda), \tag{2.13}$$

so that $S|_{B(u_0,\lambda)} = 0$ and S is continuous at u_0.

(b) There exists a (unique) $j \in \mathbb{N}^*$ such that $\|u_0 - x_j\| \leq \varepsilon_j$.

Let $\gamma \in (0, \delta - 2\mu)$. Then

$$S(u) = \varphi_j(u) y_{j+1} \quad \text{for } u \in \overline{B}(u_0, \gamma), \tag{2.14}$$

so that S is continuous at u_0.

Now, let $T : X \to Y$ be an A-proper operator which is uniformly continuous on bounded subsets. We claim that the operator $T + S$ is A-proper too. Let $\{n_k\} \subset \mathbb{N}^*$ be such that $n_k \to \infty$ and let $\{w_{n_k};\ w_{n_k} \in X_{n_k}\}$ be a bounded sequence such that

$$T_{n_k} w_{n_k} + S_{n_k} w_{n_k} \longrightarrow f \tag{2.15}$$

for some $f \in Y$.

Notice that

$$\varphi_m | X_{n_k} = 0 \quad (m > n_k), \tag{2.16}$$

so that

$$S(x) = \sum_{i=1}^{n_k} \varphi_i(x) y_{i+1} \quad \text{on } X_{n_k}, \tag{2.17}$$

which yields

$$S_{n_k}(x) = \sum_{i=1}^{k-1} \varphi_i(x) y_{i+1} \tag{2.18}$$

because $Q_{n_k} y_{n_k+1} = 0$.

Then, for each $k \in \mathbb{N}^*$, we may choose an $i_k \in \mathbb{N}^*$, $1 \leq i_k \leq n_k - 1$, such that

$$S_{n_k}(w_{n_k}) = \varphi_{i_k}(w_{n_k}) y_{i_k+1}. \tag{2.19}$$

The following two cases are possible:

(1) there exist $\{k_l\} \subset \mathbb{N}^*$, $k_l \to \infty$ as $l \to \infty$, and $\mathbb{N}^*$ such that $i_{k_l} < m$ $(l \geq 1)$. Then we obtain

$$S_{n_{k(l)}}(w_{n_{k(l)}}) \in \bar{B}(0, \sup ||y_i||) \cap \operatorname{sp}\{y_j;\ j = \overline{2, m}\}. \tag{2.20}$$

Now, since T is A-proper, it follows that $\{w_{n_{k(l)}}\}_l$ has no convergent subsequence and, since $T+S$ is continuous, it will be A-proper too;

(2) $i_k \to \infty$ $(k \to \infty)$. Now, if there exists a sequence $\{k_j\} \subset \mathbb{N}^*$ such that $k_j \to +\infty$ $(j \to +\infty)$ and $\varphi_{i_{k(j)}}(w_{n_{k(j)}}) > 0$ $(j \geq 1)$, then we obtain

$$||w_{n_{k(j)}} - x_{i_{k(j)}}|| < \varepsilon_{i_{k(j)}} \longrightarrow 0 \quad (j \longrightarrow \infty). \tag{2.21}$$

We also have

$$S_{n_{k(j)}}(w_{n_{k(j)}}) = \varphi_{i_{k(j)}}(w_{n_{k(j)}})\, y_{i_{k(j)}+1}, \tag{2.22}$$

hence

$$S_{i_{k(j)}}(w_{n_{k(j)}}) = \varphi_{i_{k(j)}}(w_{n_{k(j)}})\, Q_{i_{k(j)}} y_{i_{k(j)}+1} = 0. \tag{2.23}$$

Now, since T is uniformly continuous on bounded subsets, we have

$$||Tw_{n_{k(j)}} - Tx_{i_{k(j)}}|| \longrightarrow 0. \tag{2.24}$$

We also have

$$Q_{i_{k(j)}}[T_{n_{k(j)}} w_{n_{k(j)}} + S_{n_{k(j)}} w_{n_{k(j)}}] \longrightarrow f, \tag{2.25}$$

and, from (2.23), $T_{i_{k(j)}} w_{n_{k(j)}} \to f$ and, from (2.24), $T_{i_{k(j)}} x_{i_{k(j)}} \to f$, which is impossible because T is A-proper and $\{x_n\}$ has no convergent subsequence.

Then there exists $k_0 \in \mathbb{N}^*$ such that $k \geq k_0$ implies $\varphi_{i_k}(w_{n_k}) = 0$. Consequently, $S_{n_k}(w_{n_k}) = 0$, so that

$$T_{n_k}(w_{n_k}) \longrightarrow f. \tag{2.26}$$

Thus, $\{w_{n_k}\}$ has convergent subsequences and, since $T+S$ is continuous, it will be A-proper too.

Finally, we remark that $Sx_i = y_{i+1}$ $(i \in \mathbb{N}^*)$, whence $S(\bar{B}(0,1))$ is not relatively compact. $\square$

Theorem 2.3. *Let E be a separable Banach space such that E^* is endowed with a sequence of finite-dimensional subspaces $\{X_n\}_{n \geq 1}$, with the following properties:*

(a) *for every n, there is a projection $\Pi_n \in L(E^*)$ with $R(\Pi_n) = X_n$;*

(b) *$\Pi_n x \to x$ for all $x \in E^*$;*

(c) *there exists an (algebraic-topological) isomorphism into $U : E^* \to C(S_{E^*})^*$, where S_{E^*} denotes the closed unit ball of E^*, endowed with the weak-star topology.*

Then the operator $S_{E^*} \to C(S_{E^*})^*$, $x \to \delta_x$ (the Dirac mass), is A-proper from $S_{E^*} \subset E^*$ to $C(S_{E^*})^*$.

Proof. For every $n \in \mathbb{N}^*$, we choose $\mu_n \in U(X_n)$ with $\|\mu_n\| = n$. We define the finite-dimensional subspace $Y_n \subset C(S_{E^*})^*$ by $Y_n = U(X_n)$. Then, clearly, $\dim Y_n = \dim X_n < +\infty$. We also introduce the sequence of continuous operators $Q_n : C(S_{E^*})^* \to Y_n$ given by

$$Q_n \nu = U\Pi_n j^* \nu + \operatorname{dist}(\nu, \{\varepsilon_x;\ x \in S_{E^*}\})\mu_n, \tag{2.27}$$

where $j : E \to C(S_{E^*})$ is the canonical isometrical embedding.

We claim that the operator $T : S_{E^*} \subset E^* \to C(S_{E^*})^*$, $Tx = \varepsilon_x$, is A-proper with respect to the approximation scheme

$$\Gamma = (\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\}), \tag{2.28}$$

where $P_n : X_n \to E^*$ are the canonical isometries.

Clearly, if $x \in S_{E^*} \cap X_n$, then we have $Q_n Tx = Ux$, hence the operator $Q_n T|_{S_{E^*} \cap X_n}$ is continuous.

Now, let $\{x_m;\ x_m \in S_{E^*} \cap X_m\}$ be a sequence such that

$$||Q_m Tx_m - Q_m \mu|| \longrightarrow 0 \tag{2.29}$$

for some $\mu \in C(S_{E^*})^*$. Then we obtain

$$\begin{gathered} ||Ux_m - U\Pi_m j^* \mu + \operatorname{dist}(\mu, \{\varepsilon_x;\ x \in S_{E^*}\})\mu_m|| \longrightarrow 0, \\ \operatorname{dist}(\mu, \{\varepsilon_x;\ x \in S_{E^*}\}) \cdot m \le \|U\| \Big(1 + \sup_m ||\Pi_m|| \cdot \|\mu\|\Big). \end{gathered} \tag{2.30}$$

Therefore, there is an $x \in S_{E^*}$ such that $\mu = \delta_x$. It follows that

$$||Ux_m - U\Pi_m x|| \longrightarrow 0, \tag{2.31}$$

thus

$$x_m \longrightarrow x \tag{2.32}$$

and clearly $Tx = \mu$, which completes the proof. □

3. The generalized Leray-Schauder principle

Let X be a Banach space and let D be an open bounded subset of X with $0 \in D$. We say that $\Gamma \subset D$ is a *pseudoboundary* of D if there exist a Banach space Y, an open bounded subset $D_* \subset Y$, and a continuous A-proper operator $A : \overline{D_*} \to \overline{D}$ with $\operatorname{Deg}(A, D_*, 0) \neq \{0\}$ such that $\Gamma = A(\partial D_*)$.

Theorem 3.1. *Let $K : \overline{D} \to X$ be a compact operator. Suppose that there exists a pseudoboundary Γ of D such that*

$$Kx \neq \lambda x \quad (\lambda > 1;\ x \in \Gamma). \tag{3.1}$$

Then K has at least one fixed point.

Proof. We consider the A-proper homotopy $H : [0,1] \times \overline{D_*} \to X$ given by

$$H(t,z) = Az - tKAz. \tag{3.2}$$

Then $H(t,z) = 0$ with $t \in (0,1)$ and $z \in \partial D_*$ implies $Kx = \lambda x$ with $\lambda = t^{-1}$ and $x = Az \in \Gamma$, a contradiction. Clearly, $0 \notin H(\{0\} \times \partial D_*)$. If $H(1,z) = 0$ with $z \in \partial D_*$, we obtain that Az is a fixed point of K. Finally, if $0 \notin H([0,1] \times \partial D_*)$, then

$$\begin{aligned} \operatorname{Deg}(A - KA, D_*, 0) &= \operatorname{Deg}(H(1,\cdot), D_*, 0) = \operatorname{Deg}(H(0,\cdot), D_*, 0) \\ &= \operatorname{Deg}(A, D_*, 0) \neq \{0\}, \end{aligned} \tag{3.3}$$

which assures the existence of a $z \in D_*$ such that $(A - KA)(z) = 0$. Consequently, $Az \in \overline{D}$ is a fixed point of K and the theorem is proved. □

In what follows, X will be a Banach space and $\Delta \subset X$ will be an open bounded subset satisfying the following property:

($\mathcal{P}$) there are a finite-dimensional subspace $V \subset X$, a point $p \in \overline{\Delta_V} = \overline{\Delta} \cap V$, and a sequence of linear subspaces $X_n \subset V$ such that
- (a) the sequence $\overline{\Delta_n} = \overline{\Delta} \cap X_n$ converges to $\{p\}$ in the hyperspace $2^{\overline{\Delta_V}}$ of $\overline{\Delta_V}$ (see, e.g., [3]),
- (b) $\overline{\Delta_m} \cap \overline{\Delta_n} = \varnothing$ $(m \neq n)$,
- (c) $p \notin \overline{\Delta_n}$ $(n \in \mathbb{N}^*)$.

We have the following approximation result.

Theorem 3.2. *For each continuous operator $T : \overline{\Delta} \subset X \to \ell_2$, there exists a sequence of operators $\{T_n;\ T_n : \overline{\Delta} \to \ell_2\}$ that are A-proper with respect to various approximation schemes for the pair (X, ℓ_2) such that*

$$\sup\{\|T_n x - Tx\|;\ x \in \overline{\Delta}\} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.4}$$

Proof. Let $Q := T(\overline{\Delta_V})$. Since T is continuous and $\overline{\Delta_V}$ is compact, so is Q. We take $\varepsilon > 0$. We denote by $\{e_n\}_{n\geq 1}$ the standard orthonormal basis of ℓ_2 and by $\{\Pi_n\}_{n\geq 1}$ the associated sequence of orthoprojectors. Then there exists $k \in \mathbb{N}^*$ such that

$$\sup_{x\in Q} \|\Pi_k x - x\| < \frac{\varepsilon}{2}. \tag{3.5}$$

Now, we define the operator $\tilde{T} : \overline{\Delta} \to \ell_2$ given by the formula

$$\tilde{T}x = \begin{cases} Tx, & \text{if } x \in \overline{\Delta} \setminus \left(\bigcup_{n\geq 1} \overline{\Delta_n} \right) \\ \Pi_k Tx + \dfrac{\varepsilon}{2} e_{k+m}, & \text{if } x \in \overline{\Delta_m}\ (m \in \mathbb{N}^*). \end{cases} \tag{3.6}$$

Then it is clear that

$$\sup_{x\in\overline{\Delta}} \|\tilde{T}x - Tx\| < \varepsilon. \tag{3.7}$$

We observe that, if $x \in \overline{\Delta_m}$ and $y \in \overline{\Delta_n}$ $(m \neq n)$, then

$$\|\tilde{T}x - \tilde{T}y\| \geq \frac{\varepsilon}{\sqrt{2}}. \tag{3.8}$$

We now choose $f \in \ell_2\setminus\{0\}$ and a sequence of subspaces $\{Y_n\}_{n\geq 1}$, $Y_n \subset \ell_2$, such that

$$\dim Y_n = \dim X_n \tag{3.9}$$

and $f \in Y_n$ $(n \in \mathbb{N}^*)$. We then define a sequence of continuous nonlinear operators

$$\{Q_n;\, Q_n : \ell_2 \longrightarrow Y_n\}_{n\geq 1}, \tag{3.10}$$

where

$$Q_n x = \operatorname{dist}(x, \tilde{T}(\overline{\Delta_n})) \cdot \|x - Tp\| \cdot f. \tag{3.11}$$

We claim that the operator $\tilde{T}$ is A-proper with respect to the approximation scheme

$$\Gamma = (\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\}), \tag{3.12}$$

where $P_n : X_n \to X$ are the canonical isometries. Indeed, we observe that if $x \in \overline{\Delta_n}$, then $Q_n\tilde{T}x = 0$. On the other hand, let $\{x_m;\, x \in \overline{\Delta_m}\}$ be a sequence such that

$$||Q_m\tilde{T}x_m - Q_m y|| \longrightarrow 0 \tag{3.13}$$

for some $y \in \ell_2$. It follows that necessarily $y = Tp$ $(= \tilde{T}p)$. Moreover, the property $(\mathcal{P})$ implies that $x_m \to p$. The proof is complete. □

Remark 3.3. The above argument can be easily modified in order to conclude that all the operators T_n are A-proper with respect to the same approximation scheme.

In what follows, BCA denotes the class of all bounded continuous A-proper operators with respect to a given approximation scheme.

Proposition 3.4. *Let X and Y be two Banach spaces, Y being ordered by the cone Y^+ such that* $\operatorname{int} Y^+ \neq \emptyset$. *Let $D \subset X$ be an open bounded nonempty subset. Then, for each operator $A \in \mathrm{BCA}(\overline{D}, Y)$, there exist two operators $A_\pm \in \mathrm{BCA}(\overline{D}, Y^+)$ such that*

$$A = A_+ - A_-, \qquad A_\pm^{-1}(\{0\}) = A^{-1}(\{0\}). \tag{3.14}$$

Proof. Since $\operatorname{int} Y^+ \neq \emptyset$, there exist $y_0 \in Y$ and $r > 0$ such that

$$\overline{B}(y_0, r) \subset Y^+. \tag{3.15}$$

Let $r^* = \min\{r, (1/2)\|y_0\|\}$. Since clearly $y_0 \neq 0$, it follows that $r^* > 0$. Then we define the operators $A_\pm$ by

$$A_\pm x = \frac{1}{2}\left(\pm Ax + \frac{\|Ax\|}{r^*} y_0 \right) \quad (x \in \overline{D}). \tag{3.16}$$

It is easy to see that $A_\pm \in \mathrm{BCA}(\overline{D}, Y)$ because $\lambda A + K \in \mathrm{BCA}(\overline{D}, Y)$ for each $\lambda \in \mathbb{R} \setminus \{0\}$ and $K \in K(\overline{D}, Y)$ (= the space of all compact maps). Moreover, $A_\pm x \geq 0$ $(x \in \overline{D})$ because $\overline{B}(y_0, r^*) \subset Y^+$. Clearly, $A = A_+ - A_-$. Because

$$\frac{1}{2}\left(\frac{\|y_0\|}{r^*} - 1 \right)\|Ax\| \leq \|A_\pm x\| \leq \frac{1}{2}\left(\frac{\|y_0\|}{r^*} + 1 \right)\|Ax\| \quad (x \in \overline{D}), \tag{3.17}$$

we conclude that $A^{-1}(\{0\}) = A_\pm^{-1}(\{0\})$. The proof is done. □

4. Approximative solutions for evolution equations

Our first goal is to show that for every separable Banach space E with Schauder basis there exists a linear operator $A : E \to E$ with the following two properties:

(A1) the problem

$$\frac{du}{dt} = Au \quad (t \geq 0), \tag{4.1}$$

has a solution u of class C^∞ for each choice of the initial datum $u(0) = u_0$ in E;

(A2) for each $u_0 \in E$, $u_0 \neq 0$, there exist u and $\{u_n\} \subset C^\infty(\mathbb{R}^+, E)$ such that

$$\begin{gathered} \sup_{t \geq 0} \|u_n(t) - u(t)\| \longrightarrow 0, \quad u(0) = u_0, \\ \sup_{t \geq 0} \left\| \frac{du_n}{dt}(t) - A(u_n(t)) \right\| \longrightarrow 0, \\ \frac{du}{dt}(t) \neq A(u(t)) \quad (t \geq 0). \end{gathered} \tag{4.2}$$

In order to prove this result, we will need the following construction. Let $\{e_n\}$ be a Schauder basis for E, let $E_0 = \operatorname{span}\{e_n\}$, and let $\{e_n\} \cup \{b_x;\ x \in E_0\}$ be a Hamel basis for E. We define the linear operator $A : E \to E$ by $Ae_n = 0$ $(n \in \mathbb{N}^*)$ and $Ab_x = x$ $(x \in E_0)$. It is easy to see that $A^2 = 0$.

Proof of (A1). For each $u_0 \in E$, the function $u(t) = u_0 + tAu_0$ $(t \geq 0)$ is a C^∞ solution of (4.1) with $u(0) = u_0$.

Proof of (A2). We consider the sequence of linear continuous projections

$$P_n x = \sum_{k=1}^{n} e_k^*(x) e_k \quad (x \in E,\ n \in \mathbb{N}^*), \tag{4.3}$$

where $\{e_n^*\} \subset E^*$ is the associated sequence of coefficient functionals to $\{e_n\}$.

We choose a sequence of integers $\{k_n\} \subset \mathbb{N}^*$, $k_n \to \infty$, such that

$$||P_{k_n} b_{-P_n u_0} - b_{-P_n u_0}|| < n^{-1} \quad (n \in \mathbb{N}^*) \tag{4.4}$$

and consider the functions

$$\begin{gathered} u_n(t) = e^{-t}\left[b_{-P_n u_0} - P_{k_n}\left(b_{-P_n u_0} - u_0\right)\right] \quad (t \geq 0), \\ u(t) = e^{-t} u_0 \quad (t \geq 0). \end{gathered} \tag{4.5}$$

Then, letting $c_n = \|b_{-P_n u_0} - P_{k_n} b_{-P_n u_0} + P_{k_n} u_0 - u_0\| \to 0$, we have

$$\begin{gathered} \sup_{t \geq 0} ||u_n(t) - u(t)|| \leq e^{-t} c_n, \qquad \frac{du_n}{dt}(t) = -u_n(t), \\ A\left(u_n(t)\right) = -e^{-t} P_n u_0, \\ \left|\left|\frac{du_n}{dt}(t) - A\left(u_n(t)\right)\right|\right| = e^{-t}\left|\left| - b_{-P_n u_0} + P_{k_n}\left(b_{-P_n u_0} - u_0\right) + P_n u_0\right|\right| \\ \leq \left|\left| - b_{-P_n u_0} + P_{k_n} b_{-P_n u_0}\right|\right| + \left|\left|P_n u_0 - P_{k_n} u_0\right|\right| \longrightarrow 0, \end{gathered} \tag{4.6}$$

for each $t \geq 0$.

Finally, we suppose that

$$\frac{du}{dt}(t_0) = A\left(u(t_0)\right) \tag{4.7}$$

for some $t_0 \in \mathbb{R}^+$. Then

$$\begin{gathered} -e^{-t_0} u_0 = e^{-t_0} A u, \quad -u_0 = A u_0, \\ -A u_0 = 0, \quad u_0 = 0, \end{gathered} \tag{4.8}$$

a contradiction.

In the remainder of this section, we put $I = [0,1]$, $E = C(I^2)$, and keep fixed a number $T > 0$. It is known that $E \approx C(I)\overline{\otimes}_\varepsilon C(I)$ (the completion of the injective tensor product, see [1]).

We will need the following classical result.

Theorem 4.1 (Miljutin [3]; see also [2]). *Let Q be an uncountable compact metric space. Then $C(Q)$ is linearly isomorphic to $C(I)$.*

We consider the initial value problem

$$\begin{gathered} u(t) = Au(t) + f(t), \quad t \in [0,T], \\ u(0) = u_0, \end{gathered} \tag{4.9}$$

where $A : D(A) = E \to E$ is a linear operator and $f : [0,T] \to E$, $u_0 \in E$ are given.

THEOREM 4.2. *There exists a discontinuous linear operator $A : D(A) = E \to E$, with dense kernel, such that for each $f \in C^1([0,T],E)$ and each $u_0 \in E$ there exist $u, u_1, u_2, \ldots \in C^1([0,T],E)$ such that $A \circ u_k \in C([0,T],E)$ and*

$$\begin{gathered} u_k \longrightarrow u \quad \textit{uniformly}, \\ \frac{du_k}{dt} - Au_k \longrightarrow f \quad \textit{uniformly}, \\ u_k(0) \longrightarrow u_0 \quad \textit{in } E. \end{gathered} \tag{4.10}$$

Proof. Using Miljutin's theorem, we obtain an isomorphism $U : C(I) \to E$. Let $\{b_n\}$ be a normalized Schauder base of $C(I)$. Then there is an uncountable subset Γ of $C(I)$ such that $B = \{b_n\} \cup \Gamma$ is a Hamel basis of $C(I)$. Let $\{c_n\} \subset \Gamma$ be such that $c_m \neq c_n$ $(m \neq n)$. We put $e(t) = 1$ $(t \in I)$. Then we consider the linear operator $T : C(I) \to C(I)$, defined by

$$\begin{gathered} Tc_k = ke \quad (k \geq 1), \\ Tb = 0 \quad (b \in B \setminus \{c_k\}). \end{gathered} \tag{4.11}$$

We also consider the following linear operators:

$$\begin{gathered} S : C(I) \otimes C(I) \longrightarrow C(I) \otimes C(I), \quad S = T \otimes id_{C(I)}, \\ B : C(I) \otimes C(I) \longrightarrow C(I), \qquad B\left(\sum_i f_i \otimes g_i\right)(t) = \sum_i f_i(t) g_i(t), \\ A : E \to E, \quad A = UBSP, \end{gathered} \tag{4.12}$$

where $P : E \to C(I) \otimes_\varepsilon C(I)$ is an algebraic projection.

Now, let $u_0 \in E$ and $f \in C^1([0,T],E)$. Since the linear subspace $F = \operatorname{span}\{b_n\} \otimes_\varepsilon C(I)$ is dense in E (see [1, page 280]), there is a sequence $\{q_n\} \subset F$ such that $q_n \to u_0$. We remark that $F \subset \operatorname{Ker} S$.

We define $\{u_k\} \subset C^1([0,T],E)$ by

$$u_k(t) = q_k - k^{-1} c_k \otimes U^{-1}(f(t)). \tag{4.13}$$

Then, defining $u(t) = u_0$ $(t \in [0,T])$, we have $u_k \to u$ uniformly because

$$\|k^{-1} c_k \otimes U^{-1}(f(t))\| \leq k^{-1} \|U^{-1}\| \|f\|_\infty \longrightarrow 0. \tag{4.14}$$

We also have

$$\frac{du_k}{dt} = -k^{-1} c_k \otimes U^{-1}(f(t)), \tag{4.15}$$

so that $du_k/dt \to 0$ uniformly. On the other hand,

$$\begin{aligned} A(u_k(t)) &= UBSP(u_k(t)) = UBS(u_k(t)) \\ &= -k^{-1}UBS(c_k \otimes U^{-1}(f(t))) \\ &= -UB(e \otimes U^{-1}(f(t))) \\ &= -UU^{-1}(f(t)) \\ &= -f(t). \end{aligned} \tag{4.16}$$

Finally, we remark that A has a dense kernel (since $F \subset \operatorname{Ker} A$), and $A \neq 0$, because

$$A(c_1 \otimes c_1) = UBSP(c_1 \otimes c_1) = UB(e \otimes c_1) = Uc_1 \neq 0. \tag{4.17}$$

Hence A is discontinuous and the proof of our theorem is done. □

References

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