ASYMPTOTIC FORMULAS AND CRITICAL EXPONENTS FOR TWO-PARAMETER NONLINEAR EIGENVALUE PROBLEMS

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We study the nonlinear two-parameter problem $-u''(x) + \lambda u(x)^q = \mu u(x)^p$, $u(x) > 0, x \in (0,1), u(0) = u(1) = 0$. Here, $1 < q < p$ are constants and $\lambda, \mu > 0$ are parameters. We establish precise asymptotic formulas with exact second term for variational eigencurve $\mu(\lambda)$ as $\lambda \to \infty$. We emphasize that the critical case concerning the decaying rate of the second term is $p = (3q-1)/2$ and this kind of criticality is new for two-parameter problems.

1. Introduction

We consider the following nonlinear two-parameter problem:

$$\begin{gathered} -u''(x) + \lambda u(x)^q = \mu u(x)^p, \quad x \in I = (0,1), \\ u(x) > 0, \quad x \in I, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < q < p$ and $\lambda, \mu > 0$ are parameters.

The purpose of this paper is to establish the asymptotic formulas for the eigencurve $\mu = \mu(\lambda)$ with the exact second term as $\lambda \to \infty$ by using a variational method. We also establish the critical relationship between p and q from a viewpoint of the decaying rate of the second term of $\mu(\lambda)$.

The study of two-parameter eigenvalue problems began with the oscillation theory and has been investigated by many authors. We refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein. One of the main problems in this area is to analyze the structure of the solution set $\{(\lambda, \mu, u)\}$ of (1.1), and the effective approach to this problem is to study the structure of the set $S_{\lambda,\mu} := \{(\lambda, \mu, \|u\|_{p+1})\} \subset \mathbb{R}^3$ for large λ. In Shibata [7], by using a standard variational framework (see Section 2), the variational eigencurve $\mu = \mu(\lambda)$ was defined to

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analyze $S_{\lambda,\mu}$ and the following asymptotic formula for $\mu(\lambda)$ as $\lambda \to \infty$ was established:

$$\mu(\lambda) = C_1\lambda^{(p+3)/(2p-q+3)} + o\big(\lambda^{(p+3)/(2p-q+3)}\big), \tag{1.2}$$

where

$$\begin{aligned} C_1 &= \left(\frac{(p+1)(q+3)}{(p+3)(q+1)}\frac{1}{\gamma^{p+1}}\frac{2}{p-q}\sqrt{\frac{\pi(q+1)}{2}}\left(\frac{p+1}{q+1}\right)^{(q+3)/2(p-q)} \right. \\ &\quad \left. \times \frac{\Gamma((q+3)/2(p-q))}{\Gamma((p+3)/2(p-q))}\right)^{2(p-q)/(2p-q+3)}, \\ \Gamma(r) &= \int_0^\infty y^{r-1}e^{-y}\,dy \quad (r>0). \end{aligned} \tag{1.3}$$

By this formula, we understood the first term of $\mu(\lambda)$ as $\lambda \to \infty$. However, the remainder estimate of $\mu(\lambda)$ has not been obtained. The purpose here is to obtain *the exact second term* of $\mu(\lambda)$ as $\lambda \to \infty$. We emphasize that the second term depends deeply on the relationship between p and q, and the critical case is $p = (3q-1)/2$. More precisely, if $p = (3q-1)/2$, then the asymptotic behavior of the second term of $\mu(\lambda)$ is completely different from that of the case where $p \neq (3q-1)/2$. As far as we know, this kind of criticality is new for two-parameter problems and great interest by itself. Finally, it should be mentioned that the asymptotic behavior of such eigencurve is also effected by the variational framework (cf. [6, 8]).

2. Main results

We explain notations before stating our results. Let $H_0^1(I)$ be the usual real Sobolev space. Let $\|u\|_r$ denote the usual L^r-norm. For $u \in H_0^1(I)$,

$$\begin{aligned} E_\lambda(u) &:= \frac{1}{2}\|u'\|_2^2 + \frac{1}{q+1}\lambda\|u\|_{q+1}^{q+1}, \\ M_\gamma &:= \{u \in H_0^1(I) : \|u\|_{p+1} = \gamma\}, \end{aligned} \tag{2.1}$$

where $\gamma > 0$ is a *fixed constant*. For a given $\lambda > 0$, we call $\mu(\lambda)$ the variational eigenvalue when the following conditions are satisfied:

$$\begin{aligned} (\lambda,\mu(\lambda),u_\lambda) &\in \mathbb{R}_+ \times \mathbb{R}_+ \times M_\gamma \quad \text{satisfies (1.1)}, \\ E_\lambda(u_\lambda) &= \inf_{u\in M_\gamma} E_\lambda(u). \end{aligned} \tag{2.2}$$

Then $\mu(\lambda)$ is obtained as a Lagrange multiplier and is represented explicitly as follows:

$$\mu(\lambda) = \frac{\|u_\lambda'\|_2^2 + \lambda\|u_\lambda\|_{q+1}^{q+1}}{\gamma^{p+1}}. \tag{2.3}$$

Indeed, multiply the equation in (1.1) by u_λ. Then integration by parts yields

$$||u_\lambda'||_2^2 + \lambda ||u_\lambda||_{q+1}^{q+1} = \mu(\lambda)||u_\lambda||_{p+1}^{p+1} = \mu(\lambda)\gamma^{p+1}. \tag{2.4}$$

This implies (2.3). The existence of $\mu(\lambda)$ for a given $\lambda > 0$ is ensured in [7, Theorem 2.1] and $\mu(\lambda)$ is continuous for $\lambda > 0$ (cf. [7, Theorem 2.2]). Finally, let

$$\begin{aligned}
K_1 &:= \left(\sqrt{2}\left(\frac{q+1}{p+1}\right)^{(q-1)/(2(p-q))} \frac{\Gamma(1/(q+1))\Gamma((q-1)/2(q+1))}{\sqrt{\pi(q+1)}} \right.\\
&\qquad \left. \times C_1^{(q-1)/(2(p-q))}\right)^{2(q+1)/(q-1)},\\
K_2 &:= \frac{1}{2}\int_0^1 \frac{s^{(2p-3q-1)/2}(1-s^{p+1})}{(1-s^{p-q})^{3/2}}ds, \\
K_3 &:= \frac{2^{2(p+2)/(q+1)}}{q+1}\int_0^1 \frac{y^{(2p-2q+2)/(q+1)}}{(1+y)^{2(p+2)/(q+1)}(1-y)^{(2p-2q+2)/(q+1)}}dy,\\
J_0 &= \frac{\sqrt{\pi}}{p-q}\frac{q+3}{p+3}\frac{\Gamma((q+3)/2(p-q))}{\Gamma((p+3)/2(p-q))}.
\end{aligned} \tag{2.5}$$

Now, we state our results.

Theorem 2.1. (1) *Assume $p > (3q-1)/2$. Then the following asymptotic formula holds as $\lambda \to \infty$:*

$$\mu(\lambda) = C_1\lambda^{(p+3)/(2p-q+3)}\{1 + C_2(1+o(1))\lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))}\}, \tag{2.6}$$

where

$$C_2 = K_1\left(1 - \frac{2(p-q)K_2}{(2p-q+3)J_0}\right). \tag{2.7}$$

(2) *Assume $p < (3q-1)/2$. Then as $\lambda \to \infty$,*

$$\mu(\lambda) = C_1\lambda^{(p+3)/(2p-q+3)}\{1 - C_3(1+o(1))\lambda^{-(p+1)/(q-1)}\}, \tag{2.8}$$

where

$$C_3 = \frac{2(p-q)}{(2p-q+3)J_0}K_3K_1^{(2p-q+3)/(2(q+1))}. \tag{2.9}$$

(3) *Assume $p = (3q-1)/2$. Then as $\lambda \to \infty$,*

$$\mu(\lambda) = C_1\lambda^{(p+3)/(2p-q+3)}\{1 - C_4(1+o(1))\lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))}\log\lambda\}, \tag{2.10}$$

where

$$C_4 = \frac{2(p-q)(p+1)}{(q-1)(2p-q+3)^2 J_0} K_1. \tag{2.11}$$

We briefly explain the idea of the proof. Put

$$\begin{gathered} \nu(\lambda) = \lambda^{(p-1)/2(p-q)} \mu(\lambda)^{(1-q)/2(p-q)}, \\ w_\lambda(t) = \left(\frac{\mu(\lambda)}{\lambda}\right)^{1/(p-q)} u_\lambda(x), \quad t = \nu(\lambda)\left(x - \frac{1}{2}\right). \end{gathered} \tag{2.12}$$

Then it follows from (1.1) that w_λ satisfies

$$\begin{gathered} -w_\lambda''(t) = w_\lambda(t)^p - w_\lambda(t)^q, \quad t \in I_{\nu(\lambda)} := \left(-\frac{1}{2}\nu(\lambda), \frac{1}{2}\nu(\lambda)\right), \\ w_\lambda(t) > 0, \quad t \in I_{\nu(\lambda)}, \\ w_\lambda\left(\pm \frac{1}{2}\nu(\lambda)\right) = 0. \end{gathered} \tag{2.13}$$

Then by [7, Lemma 5.1],

$$\nu(\lambda) \longrightarrow \infty \tag{2.14}$$

as $\lambda \to \infty$. Put $z_\lambda = w_\lambda / \|w_\lambda\|_\infty$. Then it is easy to see from (2.3) that

$$\begin{aligned} \mu(\lambda) &= \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} \left(\|w_\lambda'\|_2^2 + \|w_\lambda\|_{q+1}^{q+1}\right)}{\gamma^{p+1}} \\ &= \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} \|w_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}} \\ &= \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} \|w_\lambda\|_\infty^{p+1} \|z_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}}. \end{aligned} \tag{2.15}$$

Therefore, it is crucial to study the asymptotic behavior of $\|w_\lambda\|_\infty$ and $\|z_\lambda\|_{p+1}$ as $\lambda \to \infty$.

3. Asymptotic behavior of $\|w_\lambda\|_\infty$

In this section, we study the asymptotic behavior of $\|w_\lambda\|_\infty$ as $\lambda \to \infty$. We put

$$\|w_\lambda\|_\infty = \left(\frac{p+1}{q+1}(1+\epsilon(\lambda))\right)^{1/(p-q)}. \tag{3.1}$$

Then by [7, (5.10), Lemma 5.2], we know that $\epsilon(\lambda) > 0$ and $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$.

Lemma 3.1. *The following equality holds for $\lambda > 0$:*

$$\nu(\lambda) = \sqrt{2(q+1)}\left(\frac{p+1}{q+1}(1+\epsilon(\lambda))\right)^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)), \tag{3.2}$$

where

$$\begin{gathered} L(\epsilon) = \int_0^1 \frac{1}{m(\epsilon,s)}ds, \\ m(\epsilon,s) = \sqrt{s^{q+1} - s^{p+1} + \epsilon(1-s^{p+1})} \quad (\epsilon > 0). \end{gathered} \tag{3.3}$$

Proof. Multiply the equation in (2.13) by w'_λ. Then for $t \in I_{\nu(\lambda)}$,

$$w''_\lambda(t)w'_\lambda(t) + w_\lambda(t)^p w'_\lambda(t) - w_\lambda(t)^q w'_\lambda(t) = 0, \tag{3.4}$$

which implies that

$$\frac{d}{dt}\left(\frac{1}{2}(w'_\lambda(t))^2 + \frac{1}{p+1}w_\lambda(t)^{p+1} - \frac{1}{q+1}w_\lambda(t)^{q+1}\right) = 0. \tag{3.5}$$

We know that $w_\lambda(0) = \|w_\lambda\|_\infty$ and $w'_\lambda(0) = 0$ since $u_\lambda(1/2) = \|u_\lambda\|_\infty$ and $u'_\lambda(1/2) = 0$. Then put $t = 0$ to obtain

$$\frac{1}{2}w'_\lambda(t)^2 + \frac{1}{p+1}w_\lambda(t)^{p+1} - \frac{1}{q+1}w_\lambda(t)^{q+1} \equiv \frac{1}{p+1}||w_\lambda||_\infty^{p+1} - \frac{1}{q+1}||w_\lambda||_\infty^{q+1}. \tag{3.6}$$

Note that $w'_\lambda(t) < 0$ for $t \in (0, \nu(\lambda)/2)$ since $u'_\lambda(x) < 0$ for $x \in (1/2, 1)$. Then it follows from this and (3.1) that for $t \in (0, \nu(\lambda)/2)$,

$$\begin{aligned} -z'_\lambda(t) &= ||w_\lambda||_\infty^{(q-1)/2}\sqrt{\frac{2}{q+1}}\sqrt{z_\lambda(t)^{q+1} - z_\lambda(t)^{p+1} + \epsilon(\lambda)(1 - z_\lambda(t)^{p+1})} \\ &= ||w_\lambda||_\infty^{(q-1)/2}\sqrt{\frac{2}{q+1}}m(\epsilon(\lambda), z_\lambda(t)). \end{aligned} \tag{3.7}$$

Put $s = z_\lambda$. Then (3.1) and (3.7) yield

$$\begin{aligned} \frac{\nu(\lambda)}{2} &= \int_0^{\nu(\lambda)/2} \frac{-z'_\lambda(t)}{\sqrt{2/(q+1)}||w_\lambda||_\infty^{(q-1)/2}m(\epsilon(\lambda), z_\lambda(t))}dt \\ &= \sqrt{\frac{q+1}{2}}\left(\frac{p+1}{q+1}(1+\epsilon(\lambda))\right)^{-(q-1)/(2(p-q))}\int_0^1 \frac{1}{m(\epsilon(\lambda), s)}ds. \end{aligned} \tag{3.8}$$

This implies (3.2). □

In order to study the asymptotic behavior of $\epsilon(\lambda)$ as $\lambda \to \infty$, we investigate the asymptotic behavior of $L(\epsilon)$ as $\epsilon \to 0$.

Lemma 3.2. *For* $0 < \epsilon \ll 1$,

$$L(\epsilon) = \frac{\Gamma(1/(q+1))\Gamma((q-1)/2(q+1))}{(q+1)\sqrt{\pi}}\epsilon^{-(q-1)/(2(q+1))} + o(\epsilon^{-(q-1)/(2(q+1))}). \tag{3.9}$$

Proof. Put

$$L_1(\epsilon) := L(\epsilon) - \int_0^1 \frac{1}{\sqrt{s^{q+1}+\epsilon}}ds. \tag{3.10}$$

Put $s = \epsilon^{1/(q+1)}\tan^{2/(q+1)}\theta$. Then

$$\begin{aligned}
&\int_0^1 \frac{1}{\sqrt{s^{q+1}+\epsilon}}ds \\
&= \frac{2}{q+1}\epsilon^{-(q-1)/(2(q+1))}\int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \sin^{-(q-1)/(q+1)}\theta\cos^{-2/(q+1)}\theta\, d\theta \\
&= \frac{2}{q+1}(1+o(1))\epsilon^{-(q-1)/(2(q+1))}\int_0^{\pi/2} \sin^{-(q-1)/(q+1)}\theta\cos^{-2/(q+1)}\theta\, d\theta \\
&= \frac{1}{q+1}(1+o(1))\epsilon^{-(q-1)/(2(q+1))}B\left(\frac{1}{q+1}, \frac{q-1}{2(q+1)}\right) \\
&= \frac{1}{q+1}(1+o(1))\epsilon^{-(q-1)/(2(q+1))}\frac{\Gamma(1/(q+1))\Gamma((q-1)/2(q+1))}{\Gamma(1/2)} \\
&= \frac{1}{q+1}(1+o(1))\epsilon^{-(q-1)/(2(q+1))}\frac{\Gamma(1/(q+1))\Gamma((q-1)/2(q+1))}{\sqrt{\pi}}.
\end{aligned} \tag{3.11}$$

We use here the formula

$$2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (m,n>0), \tag{3.12}$$

where $B(m,n)$ is the beta function. Next, we calculate $L_1(\epsilon)$. Note that for $0 \le s \le 1$,

$$m(\epsilon,s) = \sqrt{s^{q+1}(1-s^{p-q}) + \epsilon(1-s^{p+1})} \ge \sqrt{(s^{q+1}+\epsilon)(1-s^{p-q})}. \tag{3.13}$$

By this, we obtain

$$
\begin{aligned}
&|L_1(\epsilon)|\\
&\quad = \int_0^1 \frac{(1+\epsilon)s^{p+1}}{m(\epsilon,s)\sqrt{s^{q+1}+\epsilon}\left(m(\epsilon,s)+\sqrt{s^{q+1}+\epsilon}\right)}ds\\
&\quad \le \int_0^1 \frac{(1+\epsilon)s^{p+1}}{\sqrt{(s^{q+1}+\epsilon)(1-s^{p-q})}\sqrt{s^{q+1}+\epsilon}\left(\sqrt{(s^{q+1}+\epsilon)(1-s^{p-q})}+\sqrt{s^{q+1}+\epsilon}\right)}ds\\
&\quad \le (1+\epsilon)\int_0^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}\left(1+\sqrt{1-s^{p-q}}\right)}ds\\
&\quad \le 2\int_0^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}}ds\\
&\quad = 2\int_0^\delta \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}}ds+2\int_\delta^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}}ds\\
&\quad := I+II,
\end{aligned}
\tag{3.14}
$$

where $0<\delta \ll 1$ is a fixed constant. Let $C_{j,\delta}>0$ $(j=1,2,\dots)$ be constants depending only on δ. Put $s=\sin^{2/(p-q)}\theta$. Then

$$
\begin{aligned}
II &\le \frac{2}{\delta^{3(q+1)/2}}\int_\delta^1 \frac{1}{\sqrt{1-s^{p-q}}}ds\\
&= \frac{2}{\delta^{3(q+1)/2}}\frac{2}{p-q}\int_{\sin^{-1}\delta^{(p-q)/2}}^1 \sin^{(2+q-p)/(p-q)}\theta\, d\theta\\
&\le C_{1,\delta}.
\end{aligned}
\tag{3.15}
$$

Moreover, put $s=\epsilon^{1/(q+1)}t$. Then for $0<\epsilon\ll 1$,

$$
\begin{aligned}
I &\le \frac{2}{\sqrt{1-\delta^{p-q}}}\int_0^\delta \frac{\epsilon^{(p+1)/(q+1)}t^{p+1}}{\epsilon^{3/2}(t^{q+1}+1)^{3/2}}\epsilon^{1/(q+1)}\,dt\\
&\le 2\frac{\delta^{p+1}}{\sqrt{1-\delta^{p-q}}}\epsilon^{(2p-3q+1)/(2(q+1))}=o\left(\epsilon^{-(q-1)/(2(q+1))}\right).
\end{aligned}
\tag{3.16}
$$

By (3.14), (3.15), and (3.16), we have

$$
|L_1(\epsilon)| = o\left(\epsilon^{-(q-1)/(2(q+1))}\right).
\tag{3.17}
$$

By this, (3.10), and (3.11), we obtain (3.9). □

Now, we study the asymptotic behavior of $\epsilon(\lambda)$ as $\lambda\to\infty$.

Lemma 3.3. *As $\lambda\to\infty$,*

$$
\epsilon(\lambda)=K_1\left(1+o(1)\right)\lambda^{-2(p+1)(q+1)/((q-1)(2p-q+3))}.
\tag{3.18}
$$

Proof. By (1.2) and (2.12), we have

$$\begin{aligned}\nu(\lambda) &= \lambda^{(p-1)/(2(p-q))}\mu(\lambda)^{(1-q)/(2(p-q))} \\ &= \lambda^{(p-1)/(2(p-q))}\left(C_1\lambda^{(p+3)/(2p-q+3)}\right)^{(1-q)/(2(p-q))}(1+o(1)) \\ &= C_1^{(1-q)/(2(p-q))}(1+o(1))\lambda^{(p+1)/(2p-q+3)}.\end{aligned} \tag{3.19}$$

On the other hand, by Lemmas 3.1 and 3.2 and Taylor expansion, we have

$$\begin{aligned}\nu(\lambda) &= \sqrt{2(q+1)}\left(\frac{p+1}{q+1}\right)^{-(q-1)/(2(p-q))}(1+\epsilon(\lambda))^{-(q-1)/(2(p-q))}L(\epsilon(\lambda)) \\ &= \sqrt{2(q+1)}\left(\frac{p+1}{q+1}\right)^{-(q-1)/(2(p-q))}\left(1-\frac{q-1}{2(p-q)}\epsilon(\lambda)+o(\epsilon(\lambda))\right) \\ &\quad\times\left(\frac{\Gamma(1/(q+1))\Gamma((q-1)/2(q+1))}{(q+1)\sqrt{\pi}}\epsilon(\lambda)^{-(q-1)/(2(q+1))}\right. \\ &\qquad\left.+o\left(\epsilon(\lambda)^{-(q-1)/(2(q+1))}\right)\right) \\ &= \sqrt{2}\left(\frac{p+1}{q+1}\right)^{-(q-1)/(2(p-q))}\frac{\Gamma(1/(q+1))\Gamma((q-1)/2(q+1))}{\sqrt{\pi(q+1)}} \\ &\quad\times\epsilon(\lambda)^{-(q-1)/(2(q+1))}(1+o(1)).\end{aligned} \tag{3.20}$$

By this and (3.19), we obtain (3.18). □

4. Asymptotic behavior of $\|z_\lambda\|_{p+1}$

In this section, we calculate $\|z_\lambda\|_{p+1}$. Note that $z_\lambda(t)=z_\lambda(-t)$ for $t\in I_{\nu(\lambda)}$. Then by (3.7) and putting $s=z_\lambda(t)$, we have

$$\begin{aligned}\|z_\lambda\|_{p+1}^{p+1} &= 2\int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1}\,dt \\ &= 2\int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1}\frac{-z_\lambda'(t)}{\|w_\lambda\|_\infty^{(q-1)/2}\sqrt{2/(q+1)}m(\epsilon(\lambda),z_\lambda(t))}dt \\ &= \frac{\sqrt{2(q+1)}}{\|w_\lambda\|_\infty^{(q-1)/2}}J(\epsilon(\lambda)),\end{aligned} \tag{4.1}$$

where

$$J(\epsilon):=\int_0^1\frac{s^{p+1}}{m(\epsilon,s)}ds \quad (\epsilon>0). \tag{4.2}$$

Therefore, we study the precise asymptotics of $J(\epsilon)$ as $\epsilon \to 0$. Put $s = \sin^{2/(p-q)}\theta$. Then as $\epsilon \to 0$,

$$\begin{aligned} J(\epsilon) \longrightarrow J(0) &= \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{1-s^{p-q}}}ds \\ &= \frac{2}{p-q}\int_0^{\pi/2} \sin^{(p+3)/(p-q)}\theta\, d\theta \\ &= \frac{\sqrt{\pi}}{p-q}\frac{q+3}{p+3}\frac{\Gamma\big((q+3)/2(p-q)\big)}{\Gamma\big((p+3)/2(p-q)\big)} = J_0. \end{aligned} \tag{4.3}$$

We use here the formulas

$$\begin{gathered} \int_0^{\pi/2} \sin^r\theta\, d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\big((r+1)/2\big)}{\Gamma(r/2+1)} \quad (r > -1), \\ \Gamma(r+1) = r\Gamma(r). \end{gathered} \tag{4.4}$$

Therefore, put

$$\begin{gathered} J_1(\epsilon) := J(\epsilon) - J_0 := -\epsilon J_2(\epsilon), \\ J_2(\epsilon) := \int_0^1 \frac{s^{p+1}\big(1-s^{p+1}\big)}{m(\epsilon,s)m(0,s)\big(m(\epsilon,s)+m(0,s)\big)}ds. \end{gathered} \tag{4.5}$$

Now, we study the asymptotic behavior of $J_2(\epsilon)$ as $\epsilon \to 0$.

Lemma 4.1. (1) *If $p > (3q-1)/2$, then $J_2(\epsilon) \to K_2$ as $\epsilon \to 0$.*

(2) *If $p < (3q-1)/2$, then as $\epsilon \to 0$,*

$$J_2(\epsilon) = K_3\big(1+o(1)\big)\epsilon^{(2p-3q+1)/(2(q+1))}. \tag{4.6}$$

(3) *If $p = (3q-1)/2$, then as $\epsilon \to 0$,*

$$J_2(\epsilon) = -\frac{1}{2(q+1)}\big(1+o(1)\big)\log\epsilon. \tag{4.7}$$

Proof. (1) Since $p > (3q-1)/2$, we have $(2p-3q-1)/2 > -1$. Therefore, by Lebesgue's convergence theorem, as $\epsilon \to 0$,

$$J_2(\epsilon) \longrightarrow \frac{1}{2}\int_0^1 \frac{s^{(2p-3q-1)/2}\big(1-s^{p+1}\big)}{\big(1-s^{p-q}\big)^{3/2}}ds = K_2. \tag{4.8}$$

(2) We have the following two steps.

Step 1. Assume that $p < (3q-1)/2$. We introduce $J_3(\epsilon)$ to approximate $J_2(\epsilon)$:

$$\begin{aligned}
J_3(\epsilon) &:= \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}\left(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon}\right)}ds \\
&= J_4(\epsilon,\delta)+J_5(\epsilon,\delta) \\
&:= \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}\left(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon}\right)}ds \\
&\quad + \int_\delta^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}\left(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon}\right)}ds,
\end{aligned} \tag{4.9}$$

where $0<\delta \ll 1$ is a fixed small constant. We study the asymptotic behaviors of J_3, J_4, and J_5 as $\epsilon \to 0$. Note that $0<(2p-2q+2)/(q+1)<1$ since $p<(3q-1)/2$. Then put $s=\epsilon^{1/(q+1)}\tan^{2/(q+1)}\theta$ and $y=\tan(\theta/2)$ to obtain

$$\begin{aligned}
J_3(\epsilon) &= \frac{2}{q+1}\epsilon^{(2p-3q+1)/(2(q+1))}\int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \frac{\tan^{(2p-2q+2)/(q+1)}\theta}{1+\sin\theta}d\theta \\
&= \frac{2^{2(p+2)/(q+1)}}{q+1}\epsilon^{(2p-3q+1)/(2(q+1))} \\
&\quad \times \int_0^{\tan(1/2)(\tan^{-1}(1/\sqrt{\epsilon}))} \frac{y^{(2p-2q+2)/(q+1)}}{(1+y)^{2(p+2)/(q+1)}(1-y)^{(2p-2q+2)/(q+1)}}dy \\
&= \frac{2^{2(p+2)/(q+1)}}{q+1}(1+o(1))\epsilon^{(2p-3q+1)/(2(q+1))} \\
&\quad \times \int_0^1 \frac{y^{(2p-2q+2)/(q+1)}}{(1+y)^{2(p+2)/(q+1)}(1-y)^{(2p-2q+2)/(q+1)}}dy \\
&= K_3(1+o(1))\epsilon^{(2p-3q+1)/(2(q+1))}.
\end{aligned} \tag{4.10}$$

Similarly, we obtain

$$\begin{aligned}
J_4(\epsilon,\delta) &= K_3(1+o(1))\epsilon^{(2p-3q+1)/(2(q+1))}, \\
J_5(\epsilon,\delta) &\leq \frac{1}{\delta^{q+1}}.
\end{aligned} \tag{4.11}$$

Since $p<(3q-1)/2$, this along with (4.10) implies that $J_3(\epsilon)/J_4(\epsilon,\delta)\to 1$ as $\epsilon \to 0$ for a fixed δ.

Step 2. We show that as $\epsilon \to 0$,

$$\frac{J_2(\epsilon)}{J_3(\epsilon)} \longrightarrow 1. \tag{4.12}$$

Let an arbitrary $0 < \delta \ll 1$ be fixed. Put

$$\begin{aligned} J_2(\epsilon) &= J_6(\epsilon,\delta)+J_7(\epsilon,\delta) \\ &:= \int_0^\delta \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s)+m(0,s))}ds \\ &\quad + \int_\delta^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s)+m(0,s))}ds. \end{aligned} \tag{4.13}$$

Then for $0 < \epsilon \ll 1$,

$$|J_7(\epsilon,\delta)| \le C_{2,\delta}\int_\delta^1 \frac{1-s^{p+1}}{(1-s^{p-q})^{3/2}}ds \le C_{3,\delta}. \tag{4.14}$$

Moreover, by (3.13), we obtain

$$\begin{aligned} &(1-\delta^{p+1})\int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}\left(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon}\right)}ds \\ &\quad \le J_6(\epsilon,\delta) \le \frac{1}{(1-\delta^{p-q})^{3/2}}\int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}\left(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon}\right)}ds. \end{aligned} \tag{4.15}$$

This implies

$$(1-\delta^{p+1})J_4(\epsilon,\delta) \le J_6(\epsilon,\delta) \le \frac{1}{(1-\delta^{p-q})^{3/2}}J_4(\epsilon,\delta). \tag{4.16}$$

By (4.11), (4.14), and (4.16), we see that $J_7(\epsilon,\delta) = o(J_6(\epsilon,\delta))$ as $\epsilon \to 0$ for a fixed δ since $p < (3q-1)/2$. Then by (4.10), (4.11), (4.13), (4.14), and (4.16),

$$\begin{aligned} (1-\delta^{p+1}) &\le \liminf_{\epsilon\to 0}\frac{J_6(\epsilon,\delta)}{J_4(\epsilon,\delta)} \\ &= \liminf_{\epsilon\to 0}\frac{J_2(\epsilon)}{J_3(\epsilon)} \le \limsup_{\epsilon\to 0}\frac{J_2(\epsilon)}{J_3(\epsilon)} \\ &= \limsup_{\epsilon\to 0}\frac{J_6(\epsilon,\delta)}{J_4(\epsilon,\delta)} \le \frac{1}{(1-\delta^{p-q})^{3/2}}. \end{aligned} \tag{4.17}$$

By letting $\delta \to 0$, we obtain (4.12). Then by (4.10) and (4.12), we obtain (4.6).

(3) If $p = (3q-1)/2$, then by the asymptotic formula

$$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + O\left(\frac{1}{x^3}\right) \quad (x \gg 1), \tag{4.18}$$

Taylor expansion of $\tan x$ at $x = \pi/4$, and (4.10), we obtain

$$\begin{aligned}
J_3(\epsilon) &= \frac{8}{q+1}\int_0^{\tan((1/2)(\tan^{-1}(1/\sqrt{\epsilon})))} \frac{y}{(1+y)^3(1-y)}dy \\
&= \frac{1}{q+1}\int_0^{\tan((1/2)(\tan^{-1}(1/\sqrt{\epsilon})))} \left\{\frac{1}{1-y}+\frac{1}{1+y}+\frac{2}{(1+y)^2}-\frac{4}{(1+y)^3}\right\}dy \\
&= \frac{1}{q+1}\left(-\log\left|1-\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{1}{\sqrt{\epsilon}}\right)\right)\right|+\log 2-\frac{1}{2}+o(1)\right) \\
&= \frac{1}{q+1}\left(-\log\left|\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{\sqrt{\epsilon}}\right)\right)(1+o(1))\right|+\log 2-\frac{1}{2}+o(1)\right) \\
&= \frac{1}{q+1}\left(-\frac{1}{2}(1+o(1))\log\epsilon+\log 2-\frac{1}{2}+o(1)\right).
\end{aligned} \tag{4.19}$$

By this and the same arguments as those in the proof of (2), we obtain (4.7). □

5. Proof of Theorem 2.1

By summing up Lemmas 3.1, 3.2, 3.3, and 4.1, we now prove Theorem 2.1. By (2.15), (3.1), (4.1), and (4.5), we have

$$\begin{aligned}
\mu(\lambda)^{(2p-q+3)/(2(p-q))} &= \frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\lambda^{(p+3)/(2(p-q))}\|w_\lambda\|_\infty^{(2p-q+3)/2}J(\epsilon(\lambda)) \\
&= \frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\lambda^{(p+3)/(2(p-q))}\left(\frac{p+1}{q+1}\right)^{(2p-q+3)/(2(p-q))} \\
&\quad\times(1+\epsilon(\lambda))^{(2p-q+3)/(2(p-q))}(J_0-\epsilon(\lambda)J_2(\epsilon(\lambda))).
\end{aligned} \tag{5.1}$$

Moreover, it is easy to check that

$$\left(\frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\right)^{2(p-q)/(2p-q+3)}\frac{p+1}{q+1}J_0^{2(p-q)/(2p-q+3)} = C_1. \tag{5.2}$$

By this, (5.1), and Taylor expansion, we obtain

$$\begin{aligned}
\mu(\lambda) &= \left(\frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\right)^{2(p-q)/(2p-q+3)}\frac{p+1}{q+1}J_0^{2(p-q)/(2p-q+3)}\lambda^{(p+3)/(2p-q+3)} \\
&\quad\times(1+\epsilon(\lambda))\left(1-\frac{2(p-q)}{(2p-q+3)J_0}\epsilon(\lambda)J_2(\epsilon(\lambda))+o(\epsilon(\lambda)J_2(\epsilon(\lambda)))\right)
\end{aligned}$$

$$
\begin{aligned}
&= C_1\lambda^{(p+3)/(2p-q+3)} \\
&\quad\times\left(1+\epsilon(\lambda)-\frac{2(p-q)}{(2p-q+3)J_0}(1+o(1))\epsilon(\lambda)J_2(\epsilon(\lambda))\right).
\end{aligned}
\tag{5.3}
$$

There are three cases to consider.

Case 5.1. Assume that $p > (3q-1)/2$. Then by Lemmas 3.3 and 4.1(1), we have

$$
\begin{aligned}
\mu(\lambda) &= C_1\lambda^{(p+3)/(2p-q+3)}\left\{1+\left(1-\frac{2(p-q)K_2}{(2p-q+3)J_0}\right)\epsilon(\lambda)+o(\epsilon(\lambda))\right\} \\
&= C_1\lambda^{(p+3)/(2p-q+3)} \\
&\quad\times\left\{1+\left(1-\frac{2(p-q)K_2}{(2p-q+3)J_0}\right)K_1(1+o(1))\lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))}\right\}.
\end{aligned}
\tag{5.4}
$$

This implies (2.6).

Case 5.2. Assume that $p < (3q-1)/2$. Then by Lemma 3.3, (4.6), and (5.3), we have

$$
\begin{aligned}
\mu(\lambda) &= C_1\lambda^{(p+3)/(2p-q+3)} \\
&\quad\times\left\{1-\frac{2(p-q)}{(2p-q+3)J_0}K_3K_1^{(2p-q+3)/(2(q+1))}(1+o(1))\lambda^{-(p+1)/(q-1)}\right\}.
\end{aligned}
\tag{5.5}
$$

This implies (2.8).

Case 5.3. Assume that $p = (3q-1)/2$. Then by Lemma 3.3, we have

$$
\log\epsilon(\lambda) = -\frac{2(p+1)(q+1)}{(q-1)(2p-q+3)}(1+o(1))\log\lambda. \tag{5.6}
$$

This along with (4.7) and (5.3) implies

$$
\begin{aligned}
\mu(\lambda) &= C_1\lambda^{(p+3)/(2p-q+3)} \\
&\quad\times\left\{1-\frac{2(p-q)(p+1)}{(q-1)(2p-q+3)^2J_0}\right. \\
&\quad\left.\times K_1(1+o(1))\lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))}\log\lambda\right\}.
\end{aligned}
\tag{5.7}
$$

This implies (2.10). Thus, the proof is complete.

References

[1] F. V. Atkinson, *Multiparameter spectral theory*, Bull. Amer. Math. Soc. **74** (1968), 1–27.

[2] P. Binding and P. J. Browne, *Asymptotics of eigencurves for second order ordinary differential equations. I*, J. Differential Equations **88** (1990), no. 1, 30–45.

[3] R. S. Cantrell, *Multiparameter bifurcation problems for second order ordinary differential equations*, Rocky Mountain J. Math. **12** (1982), no. 4, 795–806.

[4] M. Faierman, *Two-Parameter Eigenvalue Problems in Ordinary Differential Equations*, Pitman Research Notes in Mathematics Series, vol. 205, Longman Scientific & Technical, Harlow, 1991.

[5] M. Faierman and G. F. Roach, *Eigenfunction expansions associated with a multiparameter system of differential equations*, Differential Integral Equations **2** (1989), no. 1, 45–56.

[6] T. Shibata, *Asymptotic behavior of eigenvalues of two-parameter nonlinear Sturm-Liouville problems*, J. Anal. Math. **66** (1995), 277–294.

[7] ______, *Two-parameter eigenvalue problems in nonlinear second order differential equations*, Results Math. **31** (1997), no. 1-2, 136–147.

[8] ______, *The effect of the variational framework on the spectral asymptotics for two-parameter nonlinear eigenvalue problems*, Math. Nachr. **248/249** (2003), 168–184.

[9] B. D. Sleeman, *The two parameter Sturm-Liouville problem for ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **69** (1971), 139–148.

[10] L. Turyn, *Sturm-Liouville problems with several parameters*, J. Differential Equations **38** (1980), no. 2, 239–259.

[11] H. Volkmer, *On multiparameter theory*, J. Math. Anal. Appl. **86** (1982), no. 1, 44–53.

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