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ASYMPTOTIC BEHAVIOUR AND HOPF BIFURCATION OF A THREE-DIMENSIONAL NONLINEAR AUTONOMOUS SYSTEM

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Abstract. A three-dimensional real nonlinear autonomous system of a concrete type is studied. The Hopf bifurcation is analyzed and the existence of a limit cycle is proved. A positively invariant set, which is globally attractive, is found using a suitable Lyapunov-like function. Corollaries for a cubic system are presented. Also, a two-dimensional nonlinear system is studied as a restricted system. An application in economics to the Kodera's model of inflation is presented. In some sense, the model of inflation is an extension of the dynamic version of the neo-keynesian macroeconomic IS-LM model and the presented results correspond to the results for the IS-LM model.

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1. Introduction

In the present paper we will consider the three-dimensional real dynamical autonomous system

$$\begin{aligned} \dot{x}_1 &= x_1 f(x_1) + b x_2, \\ \dot{x}_2 &= c x_1 + d(x_2 + x_3), \\ \dot{x}_3 &= \mu(x_1 f(x_1) + b x_2) - x_3, \end{aligned} \tag{1.1$_\mu$}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a real parameter and b, c, d are real coefficients satisfying following assumptions:

$$b < 0, \quad c > 0, \quad d < 0. \tag{1.2}$$

The nonlinear function $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative. In Section 5, we will also consider a boundary condition for the function f:

$$\lim_{x\to-\infty} f(x) < -L, \quad \lim_{x\to\infty} f(x) < -L, \tag{1.3}$$

where $L = \frac{bd\mu^2}{2c} + \frac{1}{2} > 0$.

Notice that for $\mu = 0$ the plane $x_3 = 0$ is an invariant set of system (1.1_0). The phase space restricted to this plane corresponds to the phase space of a two-dimensional system

$$\begin{aligned} \dot{x}_1 &= x_1 f(x_1) + b x_2, \\ \dot{x}_2 &= c x_1 + d x_2. \end{aligned} \tag{1.4}$$

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In this sense, the system (1.1_μ) is an extension of the two-dimensional system which have studied in [4] for cubic right-hand sides. The results presented in Section 4 are a generalization of the previously published results for a two-dimensional system.

We will also study a cubic version of the system (1.1_μ):

$$\begin{aligned}\dot{x}_1 &= a_1x_1 + a_2x_1^2 + a_3x_1^3 + bx_2,\\ \dot{x}_2 &= cx_1 + d(x_2 + x_3),\\ \dot{x}_3 &= \mu(a_1x_1 + a_2x_1^2 + a_3x_1^3 + bx_2) - x_3,\end{aligned} \tag{1.5_μ}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a real parameter and a_1, a_2, a_3, b, c, d are real coefficients satisfying following assumptions:

$$a_3 < 0, \quad b < 0, \quad c > 0, \quad d < 0. \tag{1.6}$$

The nonlinear function $f : \mathbb{R} \to \mathbb{R}$ then satisfies (1.3) since

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty}(a_1 + a_2x + a_3x^2) = -\infty,\\ \lim_{x\to\infty} f(x) &= \lim_{x\to\infty}(a_1 + a_2x + a_3x^2) = -\infty.\end{aligned}$$

2. Critical Points of the System (1.1_μ)

The system (1.1_μ) has a trivial solution corresponding to the critical point $P_0 = [0, 0, 0]$.

Any critical point of the system (1.1_μ) has to satisfy the following equations:

$$\begin{aligned}0 &= x_1f(x_1) + bx_2,\\ 0 &= cx_1 + d(x_2 + x_3),\\ 0 &= \mu(x_1f(x_1) + bx_2) - x_3.\end{aligned} \tag{2.1}$$

The first equation of the system (2.1) multiplied by $-\mu$ together with the third equation of (2.1) gives

$$x_3 = 0.$$

Therefore all the critical points of the system (1.1_μ) lie in the plane $x_3 = 0$. The intersection of this plane and the phase space of the system (1.1_μ) reduces the null surfaces to the null curves described by the equations

$$\begin{aligned}0 &= x_1f(x_1) + bx_2,\\ 0 &= cx_1 + dx_2.\end{aligned} \tag{2.2}$$

The x_1-null curve $x_2 = \varphi(x_1)$ of this reduced system is defined for all $x_1 \in \mathbb{R}$ and the function φ satisfies

$$\varphi(x_1) = -\frac{x_1f(x_1)}{b}\,.$$

The x_2-null curve $x_2 = \psi(x_1)$ of the reduced system is defined for all $x_1 \in \mathbb{R}$ and the function ψ satisfies

$$\psi(x_1) = -\frac{c}{d}x_1\,.$$

The critical points of (1.1_μ) are the intersections of these null curves in the plane $x_3 = 0$ and any non-zero critical point $P = [x_1^*, x_2^*, 0]$ has to satisfy

$$f(x_1^*) = \frac{bc}{d}\,. \tag{2.3}$$

3. Linearization of the System (1.1_μ)

Let $P = [x_1^*, x_2^*, 0]$ be any critical point of the system (1.1_μ). We denote

$$\begin{aligned}
h(x_1^*) &= \frac{d\,[x_1 f(x_1)]}{d\,x_1}|_{x_1=x_1^*} = f'(x_1^*)x_1^* + f(x_1^*),\\
A &= \begin{pmatrix} h(x_1^*) & b \\ c & d \end{pmatrix},\\
\operatorname{tr} A &= h(x_1^*) + d,\\
\det A &= h(x_1^*)d - cb,\\
B &= \begin{pmatrix} h(x_1^*) & b & 0 \\ c & d & d \\ \mu h(x_1^*) & \mu b & -1 \end{pmatrix}.
\end{aligned}$$

The matrix A is a matrix of the two-dimensional linearized system (1.4) at the point $[x_1^*, x_2^*]$. We denote its trace by $\operatorname{tr} A$ and its determinant by $\det A$ to simplify further computations. The matrix B is the matrix of the three-dimensional linearized system (1.1_μ).

Assume that B is a regular matrix. The eigenvalues of the matrix B determine the type of the critical point P. Since

$$\begin{aligned}
\det(B - \lambda I) &= \begin{vmatrix} h(x_1^*) - \lambda & b & 0 \\ c & d-\lambda & d \\ \mu h(x_1^*) & \mu b & -1-\lambda \end{vmatrix}\\
&= \begin{vmatrix} h(x_1^*) - \lambda & b & 0 \\ c & d-\lambda & d \\ \mu \lambda & 0 & -1-\lambda \end{vmatrix}\\
&= \begin{vmatrix} \multicolumn{2}{c}{A - \lambda I} & \begin{matrix} 0 \\ d \end{matrix} \\ \mu\lambda & 0 & -1-\lambda \end{vmatrix},
\end{aligned}$$

the characteristic polynomial $p(\lambda)$ of the matrix B can be written as

$$p(\lambda) = (1+\lambda)(\lambda^2 - \operatorname{tr} A\,\lambda + \det A) - \mu b d \lambda, \tag{3.1}$$

where $\operatorname{tr} A$ and $\det A$ at the point x_1^* are independent of μ since the critical point P does not depend on μ. The characteristic polynomial can also be written as

$$p(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0, \tag{3.2}$$

where the coefficients satisfy

$$\begin{aligned} p_0 &= \det A, \\ p_1 &= \det A - \operatorname{tr} A - \mu bd, \\ p_2 &= 1 - \operatorname{tr} A. \end{aligned} \tag{3.3}$$

We denote by $\lambda_1, \lambda_2, \lambda_3$ the roots of the characteristic polynomial $p(\lambda)$ and, using the Newton–Viète relations, we obtain

$$\det A = -\lambda_1 \lambda_2 \lambda_3. \tag{3.4}$$

Hence that the condition $\det A > 0$ is a necessary condition for the asymptotic stability of the constant solution corresponding to the critical point P of the system (1.1_μ).

4. Hopf Bifurcation in the System (1.1_μ)

This section will describe the Hopf bifurcation of the $\mu-$parametric system (1.1_μ) with the assumption (1.2) in the neighbourhood of the trivial solution. The trivial solution corresponds to the critical point $P_0 = [x_1^*, x_2^*, 0] = [0, 0, 0]$ and so the matrices A and B, function h and expresions $\operatorname{tr} A$, $\det A$ denoted in the previous section are now taken at the point $x_1^* = 0$. Hence we have

$$\begin{aligned} \operatorname{tr} A &= f(0) + d, \\ \det A &= f(0)d - cb. \end{aligned}$$

In this section, we use the notation

$$\sigma_1 = \min\left\{1 - d, \frac{cb}{d}\right\}, \quad \sigma_2 = \max\left\{1 - d, \frac{cb}{d}\right\},$$

and define the interval I

$$I = \langle \sigma_1, \sigma_2 \rangle.$$

Since the assumption (1.2) guarantees that σ_1 and σ_2 are positive numbers, I is a subinterval of $\mathbb{R}_+ = (0, \infty)$ that may degenerate to a point in the case that $1 - d = \frac{cb}{d}$.

Theorem 4.1. *Let (1.1_μ) be a three-dimensional μ-parametric system satisfying assumptions (1.2). Let*

$$f(0) \notin I \tag{4.1}$$

hold for the function f. Then

$$\tilde{\mu} = \frac{(f(0) + d)(1 - f(0) - d + f(0)d - cb)}{bd(f(0) + d - 1)} \tag{4.2}$$

is a critical value of the parameter μ and there exists a unique one-parametric system of periodic trajectories in the neighbourhood of the trivial critical point P_0 for some values of the parameter μ close to the critical value $\tilde{\mu}$.

Proof. The system (1.1_μ) has a trivial solution for all $\mu \in \mathbb{R}$. The zero critical point is an izolated critical point and the matrix $B = B(\mu)$ of the linearized system (1.1_μ) at this point $P_0 = [0, 0, 0]$ has three eigenvalues $\lambda_1(\mu)$, $\lambda_2(\mu)$ and $\lambda_3(\mu)$.

According to the Hopf theorem on the existence of a limit cycle (see [1], Ch. 26, p. 406), the Hopf bifurcation occurs for a critical value of the parameter μ, $\mu = \tilde{\mu}$, for which these two conditions are fulfilled at the critical point $P_0 = [0, 0, 0]$:

(i) The matrix $B(\mu)$ of the linearized system (1.1_μ) has two complex eigenvalues

$$\lambda_{1,2}(\mu) = \theta(\mu) \pm i\omega(\mu)$$

in some neighbourhood of $\tilde{\mu}$ and for $\mu = \tilde{\mu}$ these eigenvalues are purely imaginary, that is,

$$\theta(\tilde{\mu}) = 0,$$

(ii) for the above-mentioned complex eigenvalues

$$\frac{d\,\theta(\mu)}{d\,\mu}\,|_{\mu=\tilde{\mu}} \neq 0$$

holds in some neighbourhood of $\tilde{\mu}$.

Notice first that $f(0) \notin I$ if and only if

$$(1 - \operatorname{tr} A) \det A = (1 - d - f(0))(f(0)d - cb) > 0$$

since the convex quadratic function $(1 - d - f(0))(f(0)d - cb)$ of the variable $f(0)$ has two roots σ_1 and σ_2 and it is strictly positive outside the interval I.

This fact together with the relations (3.3) yields that the assumption (4.1) is equivalent to a proposition that the coefficients p_0 and p_2 of the characteristic polynomial are non-zero and of the same sign.

The condition (i) can be fulfilled for $\mu = \tilde{\mu}$ such that

$$p(\lambda) = (\lambda - i\omega)(\lambda + i\omega)(\lambda - \lambda_3) = \lambda^3 - \lambda_3\lambda^2 + \omega^2\lambda - \lambda_3\omega^2, \tag{4.3}$$

where $\lambda_3 = \lambda_3(\tilde{\mu})$ and $\omega = \omega(\tilde{\mu})$.

Using the equalities (3.2), (3.3) and (4.3), we get

$$p_0 = -\omega^2(\tilde{\mu})\lambda_3(\tilde{\mu}) = \det A, \tag{4.4}$$

$$p_1(\tilde{\mu}) = \omega^2(\tilde{\mu}) = \det A - \operatorname{tr} A - \tilde{\mu}bd, \tag{4.5}$$

$$p_2 = -\lambda_3(\tilde{\mu}) = 1 - \operatorname{tr} A. \tag{4.6}$$

Multiplication of the equalities (4.5) and (4.6) and subtraction from the equality (4.4) yield

$$0 = \det A - (\det A - \operatorname{tr} A - \tilde{\mu}bd)(1 - \operatorname{tr} A),$$

which is a necessary condition for the existence of the Hopf bifurcation. Since the condition (4.1) excludes the case where $1 - \operatorname{tr} A = 0$, the Hopf bifurcation may occur for $\mu = \tilde{\mu}$, where

$$\tilde{\mu} = -\frac{1}{bd}\frac{\operatorname{tr} A(1 - \operatorname{tr} A + \det A)}{1 - \operatorname{tr} A}\,,$$

which is the expression (4.2).

On the other hand, if $\mu = \tilde{\mu}$, the coefficient p_1 of the characteristic polynomial satisfies

$$p_1 = \det A - \operatorname{tr} A + \frac{\operatorname{tr} A(1 - \operatorname{tr} A + \det A)}{1 - \operatorname{tr} A} = \frac{\det A}{1 - \operatorname{tr} A} = \frac{p_0}{p_2}$$

according to (3.3). Consequently the characteristic polynomial $p(\lambda)$ has the form (3.2):

$$p(\lambda) = \lambda^3 + p_2\lambda^2 + \frac{p_0}{p_2}\lambda + p_0 = \Big(\lambda + p_2\Big)\Big(\lambda^2 + \frac{p_0}{p_2}\Big).$$

Since the condition (4.1) guaranties that p_0 and p_2 are of the same sign, the characteristic polynomial has two purely imaginary roots $\pm i\omega(\tilde{\mu})$, where

$$\omega(\tilde{\mu}) = \sqrt{\frac{\det A}{1 - \operatorname{tr} A}} \neq 0. \tag{4.7}$$

The condition (ii) may be proved using the implicit function theorem. For each $\mu \in \mathbb{R}$ and the corresponding system (1.1_μ), we define

$$F(\mu, \lambda) = p(\lambda)$$

as a function of two variables μ and λ, where $p(\lambda)$ is the characteristic polynomial of the system (1.1_μ) defined by (3.1) or (3.2) and (3.3), respectively.

The complex eigenvalues $\lambda(\mu) = \theta(\mu) \pm i\omega(\mu)$ are the roots of the characteristic polynomial. Hence, for these eigenvalues, we have

$$F(\mu, \lambda(\mu)) = 0.$$

The equation $F(\mu, \lambda) = 0$ represents an implicit function of two variables μ and λ. Computing the derivative of the function $F(\mu, \lambda(\mu))$ at the point $\tilde{\mu}$, we find

$$\frac{\partial F}{\partial \lambda}(\mu, \lambda) \mid_{(\mu,\lambda)=(\tilde{\mu},\pm i\omega(\tilde{\mu}))} = 3\lambda^2 + 2p_2\lambda + p_1 \mid_{\mu=\tilde{\mu}} = 3(\pm i\omega)^2 + 2p_2(\pm i\omega) + p_1 \mid_{\mu=\tilde{\mu}}$$

and using the equalities (4.5) and (4.6), we have

$$\frac{\partial F}{\partial \lambda}(\mu, \lambda) \mid_{(\mu,\lambda)=(\tilde{\mu},\pm i\omega(\tilde{\mu}))} = -3\omega^2(\tilde{\mu}) \pm i2(1 - \operatorname{tr} A)\omega(\tilde{\mu}) + \omega^2(\tilde{\mu}).$$

Using the relation (4.7), we get

$$\frac{\partial F}{\partial \lambda}(\mu, \lambda) \mid_{(\mu,\lambda)=(\tilde{\mu},\pm i\omega(\tilde{\mu}))} = -2\omega^2(\tilde{\mu}) \pm i2(1 - \operatorname{tr} A)\omega(\tilde{\mu}) \neq 0. \tag{4.8}$$

According to the implicit function theorem the condition (4.8) makes it possible to write

$$\frac{d\,\lambda}{d\,\mu}(\tilde{\mu}) = -\frac{\frac{\partial F}{\partial \mu}}{\frac{\partial F}{\partial \lambda}} \mid_{\mu=\tilde{\mu}}\,. \tag{4.9}$$

Since the characteristic polynomial can be written in the form (3.1), we have

$$\frac{\partial F}{\partial \mu}(\mu,\lambda)\mid_{(\mu,\lambda)=(\tilde{\mu},\pm i\omega(\tilde{\mu}))}=-bd(\pm i\omega(\tilde{\mu})). \tag{4.10}$$

Hence the equalities (4.8) and (4.10) together with the relation (4.9) yield

$$\frac{d\,\lambda}{d\,\mu}(\tilde{\mu})=\frac{1}{2}\frac{\pm ibd}{-\omega(\tilde{\mu})\pm i(1-\operatorname{tr}A)}.$$

The following arrangements

$$\begin{aligned}\frac{d\,\lambda}{d\,\mu}(\tilde{\mu})&=\frac{1}{2}\frac{\pm ibd}{-\omega(\tilde{\mu})\pm i(1-\operatorname{tr}A)}\cdot\frac{-\omega(\tilde{\mu})\mp i(1-\operatorname{tr}A)}{-\omega(\tilde{\mu})\mp i(1-\operatorname{tr}A)}=\\&=\frac{1}{2}\frac{(1-\operatorname{tr}A)bd}{\omega^2(\tilde{\mu})+(1-\operatorname{tr}A)^2}+i\frac{1}{2}\frac{\mp bd\omega(\tilde{\mu})}{\omega^2(\tilde{\mu})+(1-\operatorname{tr}A)^2}\end{aligned}$$

together with the assumptions (4.1) and (1.2) give

$$\frac{d\,\theta(\mu)}{d\,\mu}\mid_{\mu=\tilde{\mu}}=\operatorname{Re}\frac{d\,\lambda}{d\,\mu}(\tilde{\mu})=\frac{1}{2}\frac{(1-\operatorname{tr}A)bd}{\omega^2(\tilde{\mu})+(1-\operatorname{tr}A)^2}\neq 0. \qquad \square$$

Remark 4.2. If the system (1.1_μ) has exactly one zero critical point P_0, the values of the function f do not reach $\frac{bc}{d}$, that is,

$$f(x)<\frac{bc}{d}$$

since the condition (2.3) cannot be satisfied at any point. Thus, for the parameters b, c, d satisfying

$$\frac{bc}{d}\leq 1-d, \tag{4.11}$$

the condition (4.1) may be omitted.

Corollary 4.3. *Let* (1.5_μ) *be a three-dimensional cubic* μ*-parametric system satisfying the assumption* (1.6) *and let*

$$a_1\notin I \tag{4.12}$$

hold for the coefficients. Then

$$\tilde{\mu}=\frac{(a_1+d)(1-a_1-d+a_1d-cb)}{bd(a_1+d-1)}$$

is a critical value of the parameter μ *and there exists a unique one-parametric system of periodic trajectories in the neighbourhood of the trivial critical point* P_0 *for some values of the parameter* μ *close to the critical value* $\tilde{\mu}$.

Proof. The proof follows from Theorem 4.1 since $f(x_1)=a_1+a_2x_1+a_3x_1^2$ and $f(0)=a_1$. $\square$

Remark 4.4. Notice that the parameter $\tilde{\mu}$ does not depend on the parameters a_2 and a_3, which implies that the Hopf bifurcation of the nonlinear cubic system (1.5_μ) depends strictly on the linear terms.

Remark 4.5. If the system (1.5_μ) has exactly one zero critical point P_0, the upper bound of the function f

$$\max_{x\in\mathbb{R}} f(x) = \max_{x\in\mathbb{R}}\{a_1 + a_2x + a_3x^2\} = a_1 - \frac{a_2^2}{4a_3}$$

does not reach $\frac{bc}{d}$, that is,

$$a_1 - \frac{a_2^2}{4a_3} < \frac{bc}{d}\,. \tag{4.13}$$

Corollary 4.6. *Let* (1.5_μ) *be a three-dimensional cubic* μ*-parametric system satisfying the assumptions* (1.6), (4.11) *and* (4.13)*. Then*

$$\tilde{\mu} = \frac{(a_1+d)(1-a_1-d+a_1d-cb)}{bd(a_1+d-1)}$$

is a critical value of the parameter μ *and there exists a unique one-parametric system of periodic trajectories in the neighbourhood of the trivial critical point* P_0 *for some values of the parameter* μ *close to the critical value* $\tilde{\mu}$*.*

Proof. The conditions (4.11) and (4.13) imply that

$$a_1 < \frac{bc}{d} \le 1-d,$$

that is, $a_1 \notin I$. □

5. Existence of a Globally Attractive Set

In this section, the system (1.1_μ) with assumptions (1.2) and (1.3), where $\mu \in \mathbb{R}$ is a given constant, is studied and a positively invariant set, which is globally attractive is found using a suitable Lyapunov-like function. A sufficient condition for a global asymptotic stability of the trivial solution is derived. This result is useful for the characterization of the global behaviour of solutions of (1.1_μ).

Denote

$$m = \max_{x\in\mathbb{R}}\left\{cx^2\left(f(x)+\frac{1}{2}+\frac{\mu^2bd}{2c}\right)\right\}$$

for further computations.

Theorem 5.1. *Let* (1.1_μ) *be a system satisfying the assumptions* (1.2) *and* (1.3)*. Then*

$$\left\{[x_1,x_2,x_3]\in\mathbb{R}^3 \mid \frac{c}{2}x_1^2 - \frac{b}{2}x_2^2 + \frac{bd}{2}(x_3-\mu x_1)^2 \le R\right\}$$

is a positively invariant set of the system (1.1_μ) *for each* $R \ge M$*, where*

$$M = \begin{cases} -\frac{m}{d} & \text{for } d\in(-1,0),\\ m & \text{for } d\in(-\infty,-1\rangle. \end{cases} \tag{5.1}$$

Proof. Denote

$$g(x) = cx^2 \left(f(x) + \frac{1}{2} + \frac{bd\mu^2}{2c} \right).$$

Notice that the function $g(x)$ is bounded from above on $\mathbb{R}$, since g is continuous and the assumptions (1.2) and (1.3) give that

$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} cx^2 \cdot \lim_{x\to\pm\infty} (f(x) + L) = -\infty.$$

This together with $g(0) = 0$ yields

$$m = \max_{x\in\mathbb{R}} g(x) \geq 0.$$

Put $\mathbf{x} = [x_1, x_2, x_3]$ and consider a function

$$w(\mathbf{x}) = w(x_1, x_2, x_3) = \frac{c}{2}x_1^2 - \frac{b}{2}x_2^2 + \frac{bd}{2}(x_3 - \mu x_1)^2.$$

The function w is positively definite since, obviously, $w \geq 0$ and $w = 0$ if and only if $x_1 = 0$, $x_2 = 0$ and $x_3 - \mu x_1 = 0$, that is, for $[x_1, x_2, x_3] = [0, 0, 0]$.

Consider $d \in (-1, 0)$ first. Let $R \geq M \geq 0$ be arbitrary.

If $R = 0$, then the set given by the inequality $w(\mathbf{x}) \leq R$ is the origin which is a critical point of (1.1_μ) and the statement is proved in this case.

If $R > 0$, then $w(\mathbf{x}) = R^*$ is an ellipsoid for any $R^* \geq R$ and the set $w(\mathbf{x}) \leq R^*$ contains the origin and the ellipsoid $w(\mathbf{x}) = R$. Let $\mathbf{x} = \mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]$ be any non-trivial solution of (1.1_μ). If $w(\mathbf{x}(t)) = R^*$, the solution $\mathbf{x} = \mathbf{x}(t)$ intersects the ellipsoid $w(\mathbf{x}) = R^*$ at the time t. Then the following relation holds for any t such that $w(\mathbf{x}(t)) = R^*$:

$$bd\mu x_1 x_3 = \left(\frac{c}{2} + \frac{bd\mu^2}{2} \right) x_1^2 - \frac{b}{2}x_2^2 + \frac{bd}{2}x_3^2 - R^*. \tag{5.2}$$

The derivative of the function w along this solution $\mathbf{x}(t)$ satisfies

$$\begin{aligned} w'(\mathbf{x}(t)) &= \frac{\partial w}{\partial x_1}\dot{x}_1 + \frac{\partial w}{\partial x_2}\dot{x}_2 + \frac{\partial w}{\partial x_3}\dot{x}_3 \\ &= (cx_1 - \mu bd(x_3 - \mu x_1))(x_1 f(x_1) + bx_2) - bx_2(cx_1 + d(x_2 + x_3)) \\ &\quad + bd(x_3 - \mu x_1)[\mu(x_1 f(x_1) + bx_2) - x_3] \\ &= -bdx_2x_3 + bd\mu x_1 x_3 - bdx_2^2 - bdx_3^2 + cx_1^2 f(x_1). \end{aligned}$$

Substituting the equality (5.2), we get

$$w'(x(t)) = -bdx_2x_3 + g(x_1) - \frac{b}{2}(2d+1)x_2^2 - \frac{bd}{2}x_3^2 - R^* \tag{5.3}$$

for the time t of the intersection. Since

$$0 \leq \frac{bd}{2}(x_2 + x_3)^2 = \frac{bd}{2}x_2^2 + bdx_2x_3 + \frac{bd}{2}x_3^2,$$

we can estimate

$$-bdx_2x_3 \leq \frac{bd}{2}(x_2^2 + x_3^2). \tag{5.4}$$

Using this estimation (5.4) in the relation (5.3), it follows that

$$w'(\mathbf{x}(t)) \le \frac{bd}{2}(x_2^2 + x_3^2) + g(x_1) - \frac{b}{2}(2d+1)x_2^2 - \frac{bd}{2}x_3^2 - R^*,$$

that is,

$$w'(\mathbf{x}(t)) \le g(x_1) - R^* - \frac{b}{2}(d+1)x_2^2. \tag{5.5}$$

When $-1 < d < 0$, the coefficient $-\frac{b}{2}(d+1)$ is positive. Since we compute the derivative $w(\mathbf{x}(t))$ at the time t of the intersection of the solution $\mathbf{x}(t)$ with the ellipsoid $w(\mathbf{x}) = R^*$, we can rewrite the relation (5.5) as

$$w'(\mathbf{x}(t)) = g(x_1) - R^* + (d+1)R^* - (d+1)\frac{c}{2}x_1^2 - (d+1)\frac{bd}{2}(x_3 - \mu x_1)^2.$$

The assumptions (1.2) and $-1 < d < 0$ imply that the coefficients $-(d+1)\frac{c}{2}$ and $-(d+1)\frac{bd}{2}$ are negative. The function $g(x_1)$ can be bounded by m from above, which gives an inequality

$$w'(\mathbf{x}(t)) \le m + dR^* \le m + dR \le 0. \tag{5.6}$$

Clearly, the inequality $m + dR^* \le 0$ is strict for $R^* > R$. This implies that the trajectories of (1.1_μ) intersect the ellipsoids $w(\mathbf{x}) = R^*$, $R^* > R$, in the direction from their exteriors to their interiors. Consequently, if $\mathbf{x}(t^*)$ is an element of $w(\mathbf{x}) \le R$ for some t^*, i.e. if $w(\mathbf{x}(t^*)) \le R$, then the trajectory corresponding to the solution $\mathbf{x}(t)$ cannot leave the set $w(\mathbf{x}) \le R$ for $t \ge t^*$. Hence $w(\mathbf{x}) \le R$ is a positively invariant set.

In the second case, where $d \in (-\infty, -1\rangle$, the proof is identical until the equation (5.5), where the coefficient $-\frac{b}{2}(d+1)$ is non-positive and the last member can be estimated by zero. Consequently, we get the estimation

$$w'(\mathbf{x}(t)) \le m - R^* \le m - R \le 0 \tag{5.7}$$

instead of the relation (5.6). The above considerations are the same then and the proof is completed. □

Theorem 5.2. *Let* (1.1_μ) *be a system satisfying the assumptions* (1.2) *and* (1.3). *Let* M *be a constant defined by* (5.1). *Then the ellipsoid*

$$E = \left\{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid \frac{c}{2}x_1^2 - \frac{b}{2}x_2^2 + \frac{bd}{2}(x_3 - \mu x_1)^2 \le M \right\}$$

is globally attractive.

Proof. In view of Theorem 5.1, any set

$$\left\{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid \frac{c}{2}x_1^2 - \frac{b}{2}x_2^2 + \frac{bd}{2}(x_3 - \mu x_1)^2 \le R \right\}$$

where $R \ge M$, is positively invariant. Thus every solution $\mathbf{x}(t)$ of (1.1_μ) is bounded and therefore defined for all $t \ge t_0$, where t_0 is an initial value of t for the solution $\mathbf{x}(t)$. From the proof of Theorem 5.1, we can easily see that

$w'(\mathbf{x}(t)) < 0$ for any $t \geq t_0$ for which the inequality $w(\mathbf{x}(t)) > M$ is valid. This implies that $w(\mathbf{x}(t))$ is decreasing for all $t \geq t_0$ with the mentioned property.

Choose $R^* > M$ arbitrary. We need to prove that there exists $t^* \geq t_0$ such that $w(\mathbf{x}(t)) < R^*$ for $t \geq t^*$. Since any set $w(\mathbf{x}) \leq R$, where $R \geq M$, is positively invariant, it is sufficient to show that there exists $t^* \geq t_0$ such that $w(\mathbf{x}(t^*)) < R^*$.

Suppose on the contrary that there is not such t^*. Then, in view of the monotonicity of $w(\mathbf{x}(t))$, the limit

$$\lim_{t\to\infty} w(\mathbf{x}(t)) =: \xi$$

exists and

$$w(\mathbf{x}(t)) \geq \xi \geq R^*$$

for all $t \geq t_0$. Calculating the derivative $w'(\mathbf{x}(t))$ similarly to deriving the estimations (5.6) or (5.7) in the proof of Theorem 5.1, respectively, we obtain the estimations

$$w'(\mathbf{x}(t)) \leq m + d\xi =: \kappa < 0 \quad \text{for } d \in (-1, 0)$$

or

$$w'(\mathbf{x}(t)) \leq m - \xi =: \kappa < 0 \quad \text{for } d \in (-\infty, -1\rangle,$$

respectively, for $t \geq t_0$. The integration over $[t_0, t]$ yields

$$w(\mathbf{x}(t)) - w(\mathbf{x}(t_0)) \leq \kappa(t - t_0)$$

for all $t \geq t_0$. Consequently $w(\mathbf{x}(t)) \to -\infty$ for $t \to \infty$, which is a contradiction to the non-negativity of $w(\mathbf{x}(t))$. □

Theorem 5.3. *Let* (1.1_μ) *be a system satisfying the assumptions* (1.2) *and* (1.3). *If the function* f *satisfies the boundary condition*

$$\max_{x\in\mathbb{R}} f(x) \leq -L = -\frac{1}{2} - \frac{\mu^2 bd}{2c}, \tag{5.8}$$

then the trivial solution of (1.1_μ) *is globally asymptotically stable.*

Proof. If (5.8) holds, then we get $m = 0$. Choose $R \geq 0$ fixed, but arbitrary. Then according to the proof of Theorem 5.1, the inequality

$$w'(\mathbf{x}(t)) \leq m + dR = dR \quad \text{for } d \in (-1, 0)$$

or

$$w'(\mathbf{x}(t)) \leq m - R = -R \quad \text{for } d \in (-\infty, -1\rangle$$

holds for any t such that $w(\mathbf{x}(t)) = R$. Consequently the orbital derivative of w is negatively definite and this guarantees the global asymptotical stability of the trivial solution. □

Corollary 5.4. *Let* (1.5_μ) *be a cubic system satisfying the assumptions* (1.6) *and*

$$a_2^2 \leq 2a_3 \left(2a_1 + 1 + \frac{\mu^2 bd}{c}\right) \tag{5.9}$$

Then the trivial solution of (1.1_μ) *is globally asymptotically stable.*

Proof. Since $f(x_1) = a_1 + a_2x_1 + a_3x_1^2$, the boundary condition (5.8) from Theorem 5.3 can be rewritten as

$$\max_{x\in\mathbb{R}} \left\{ a_3x_1^2 + a_2x_1 + a_1 + \frac{1}{2} + \frac{\mu^2 bd}{2c} \right\} \leq 0.$$

Since $a_3 < 0$, the condition is equivalent to the statement that the discriminant

$$a_2^2 - 2a_3 \left(2a_1 + 1 + \frac{\mu^2 bd}{c} \right)$$

is not positive, which is the condition (5.9). □

Consider now the planar system (1.4) together with the assumptions (1.2) and (1.3) (and also its cubic version). As it was mentioned, this system is a restriction of the system (1.1_0) to its invariant set $x_3 = 0$. All the statements presented in this section can be concretized and restricted to this planar system (1.4). The ellipsoid is restricted to an ellipse, since $\mu = 0$ and $x_3 = 0$. As obvious consequences, they will be now presented without proofs.

Theorem 5.5. *Let* (1.4) *be a system satisfying the assumptions* (1.2) *and* (1.3)*, then*

$$\left\{ [x_1, x_2] \in \mathbb{R}^2 \mid \frac{c}{2}x_1^2 - \frac{b}{2}x_2^2 \leq R \right\}$$

is a positively invariant set of the system (1.4) *for each* $R \geq \bar{M}$*, where*

$$\bar{M} = \begin{cases} -\frac{\bar{m}}{d} & \text{for } d \in (-1, 0), \\ \bar{m} & \text{for } d \in (-\infty, -1\rangle \end{cases} \tag{5.10}$$

and

$$\bar{m} = \max_{x\in\mathbb{R}} \left\{ cx^2 \left(f(x) + \frac{1}{2} \right) \right\}.$$

Theorem 5.6. *Let* (1.4) *be a system satisfying the assumptions* (1.2) *and* (1.3)*. Let* $\bar{M}$ *be a constant defined by* (5.10)*. Then the ellipse*

$$\bar{E} = \left\{ [x_1, x_2] \in \mathbb{R}^2 \mid \frac{c}{2}x_1^2 - \frac{b}{2}x_2^2 \leq M \right\}$$

is globally attractive.

Theorem 5.7. *Let* (1.4) *be a system satisfying the assumptions* (1.2) *and* (1.3)*. If the function* f *satisfies the boundary condition*

$$\max_{x\in\mathbb{R}} f(x) \leq -\frac{1}{2},$$

then the trivial solution is globally asymptotically stable.

These results correspond to the previously published results [4] for the cubic planar system (1.4), where

$$f(x_1) = a_1 + a_2x_1 + a_3x_1^2$$

and assumptions (1.6) are satisfied. A similar (but not the same) estimation obtained to (5.10) of the radius R in Theorem 5.5 (and 5.6) can be obtained by the method used in [4]:

$$\tilde{M} = \max_{x\in\mathbb{R}} \left\{ cx^2 \left(-\frac{f(x)}{d} + \frac{1}{2} \right) \right\},$$

while we assume that $f(x)$ is such that $\tilde{M}$ is finite. It can be seen, that this estimation is better for $d \in (-\infty, -1)$, but it is better to use Theorems 5.5, 5.6 and 5.7 for $d \in (-1, 0)$.

6. Examples

Here, we will show some interesting examples of the phase portraits and globally attractive sets related to the system (1.1_μ) and also some planar restrictions of the system (1.1_0).

Example 6.1. Consider the system (1.1_μ) with given coefficients

$$b = -12, \quad c = 3, \quad d = -1$$

and the function f of the form

$$f(x) = \frac{30x + 18}{1 + e^x} - 9.5.$$

The only critical point of the system (1.1_μ) is zero since

$$f(x) < \frac{bc}{d} = 36.$$

Since $f(0) = -0.5$ and $I = \langle 2, 36\rangle$, $f(0) \notin I$, the assumptions of Theorem 4.1 are satisfied and the Hopf bifurcation occurs for the parameter

$$\tilde{\mu} = 1.95$$

according to (4.2). Let $\mu = 2$. Then

$$L = \frac{bd\mu^2}{2c} + \frac{1}{2} = 8.5$$

and

$$\lim_{x\to-\infty} f(x) = -\infty < -L, \quad \lim_{x\to\infty} f(x) = -9.5 < -L,$$

which is the condition (1.3). The critical point is unstable, with one negative real eigenvalue and two complex eigenvalues with positive real parts

$$\lambda_{1,2} \doteq 0.036 \pm 3.766i, \quad \lambda_3 \doteq -2.573.$$

According to Theorem 5.2 there exists a globally attractive ellipsoid

$$E = \left\{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid \frac{3}{2}x_1^2 + 6x_2^2 + 6(x_3 - 2x_1)^2 \le M \right\},$$

where $M \doteq 99.574$. The figure shows the trajectories converging to their ω-limit set, the limit cycle caused by the Hopf bifurcation, and a part of the ellipsoid E, which surrounds this set.

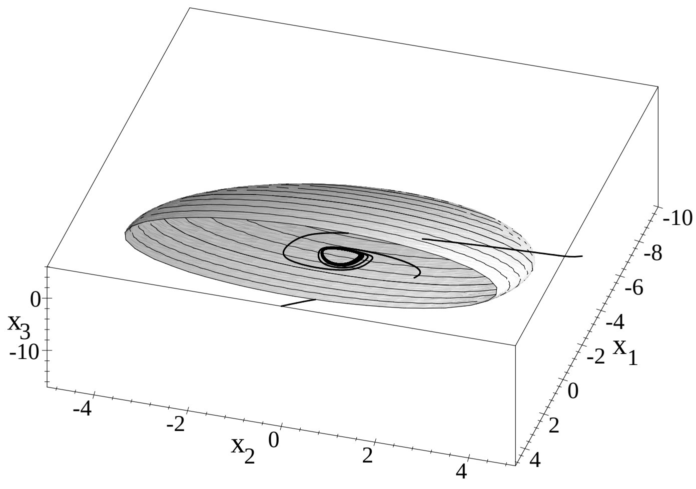

Fig.1.

Example 6.2. Consider the system (1.1_μ) with given coefficients

$$b = -0.1, \quad c = 1.2, \quad d = -0.6, \quad \mu = 1.2.$$

and the function f of the form

$$f(x) = 200\frac{x}{1 + e^{x+4}} - 2.$$

The only critical point of the system (1.1_μ) is zero, since

$$f(x) \leq \max_{x\in\mathbb{R}} f(x) \doteq -0.6614 < \frac{bc}{d} = 0.2.$$

The constant L of the condition (1.3) is

$$L = \frac{bd\mu^2}{2c} + \frac{1}{2} = 0.536$$

and

$$\lim_{x\to-\infty} f(x) = -\infty < -L, \quad \lim_{x\to\infty} f(x) = -2 < -L,$$

hence the condition (1.3) is satisfied. The zero critical point is a stable focus, with one negative real eigenvalue and two complex eigenvalues with negative real parts

$$\lambda_{1,2} \doteq -0.792 \pm 0.166i, \quad \lambda_3 \doteq -2.016.$$

Since

$$\max_{x\in\mathbb{R}} f(x) \doteq -0.6614 < -L = -0.536,$$

the condition (5.8) is fulfilled. According to Theorem 5.3, the trivial solution is even globally asymptotically stable. Each set

$$E_M = \left\{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid \frac{3}{5}x_1^2 + \frac{1}{20}x_2^2 + \frac{3}{100}\Big(x_3 - \frac{6}{5}x_1\Big)^2 \leq M\right\}$$

for arbitrary $M \geq 0$ is a positively invariant set.

The first figure shows the trajectories converging to their ω-limit set - the zero critical point. The second figure shows one of the positively invariant ellipsoids.

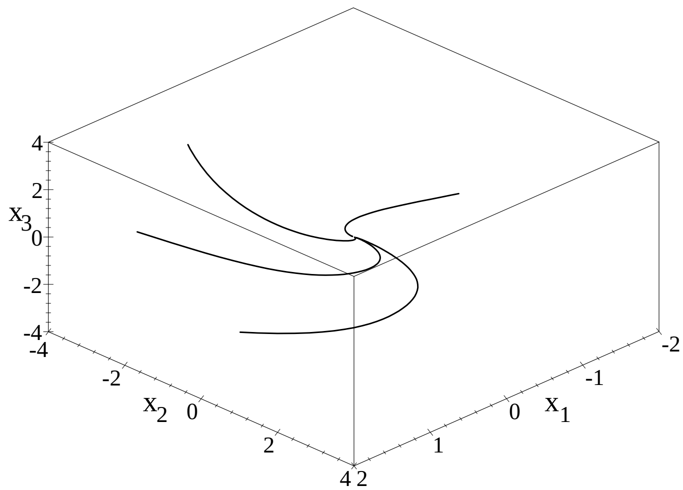

Fig.2.

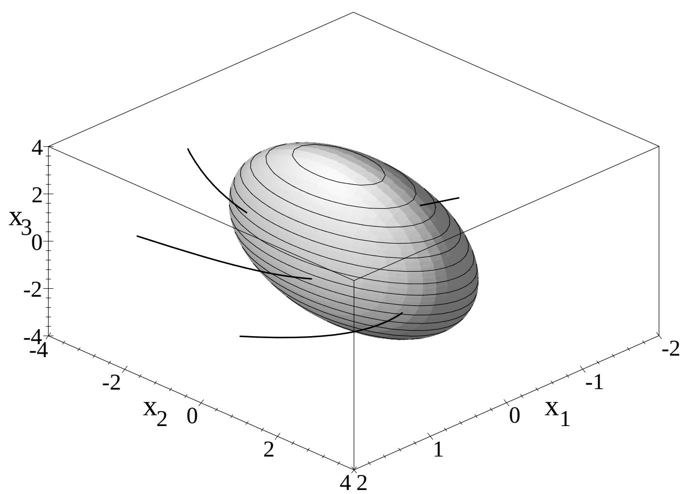

Fig.3.

Example 6.3. Consider the system (1.1_0) with given coefficients

$$b = -0.05555555556, \quad c = 26, \quad d = -1$$

and the function f in the form

$$f(x) = -0.05555555556(x^2 - 30)$$

It is easy to see that the system has three critical points. Its restriction to the plane $x_3 = 0$ is the two-dimensional system (1.4) satisfying (1.2) and (1.3). Points [2,52] and [-2,-52] are both stable nodes and it can be shown that they are surrounded by unstable limit cycles (see [4]). The point [0,0] is a saddle. According to Theorem 5.5 there exists a positively invariant set. The Poincaré–Bendixson theorem then guarantees existence of a stable limit cycle which contains unstable cycles and a saddle. The first figure shows the phase portrait of the system (1.4) with one trajectory converging to its ω-limit set and α-limit set. The second figure shows one trajectory intersecting the globally attractive ellipsoid

$$E = \left\{[x_1, x_2, x_3] \in \mathbb{R}^3 : 13x_1^2 + 0.02777777778x_2^2 + 0.02777777778x_3^2 \le M\right\},$$

where $M = \max_{x\in\mathbb{R}}\{26x^2(-0.05555555556(x^2-30)+\frac{1}{2})\} = 549.25$. Its ω-limit set is the stable limit cycle guaranteed by the Poincaré–Bendixson theorem.

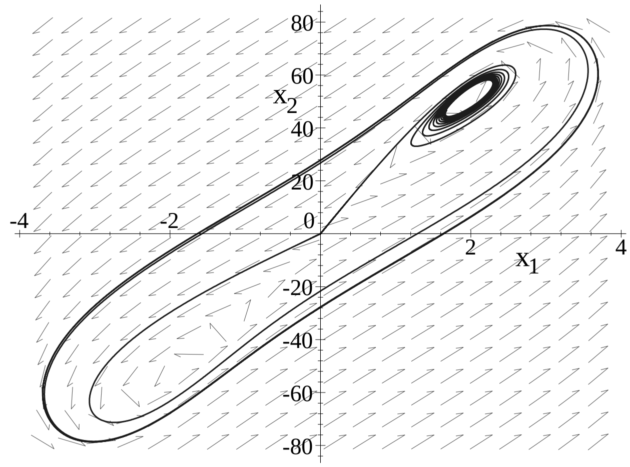

Fig.4.

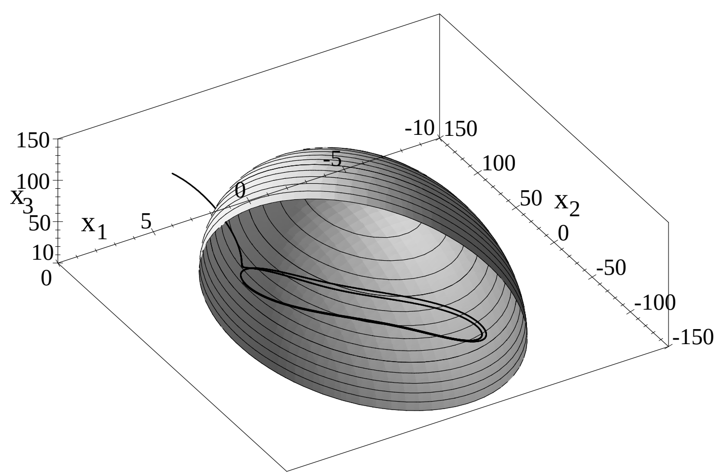

Fig.5.

Here we come to an open problem. For $\mu = 0$, this is a kind of a singular situation, but similar behaviour probably remains for values of μ close to zero. The system does not seem to behave chaotically inside the globally attractive set. It is probable that there exists an invariant surface (for $\mu = 0$ it is the plane $x_3 = 0$), which contains all critical points and possible limit cycles. In this case this asymptotic behaviour could be proved using the Poincaré–Bendixson theorem.

7. Applications in Economics

In this section, a possible application of the previous statements to the dynamical macroeconomic model of inflation (see [5]) is shown. This model is an extension of the neo-keynesian macroeconomic model IS-LM.

Neo-keynesian macroeconomic model IS-LM can be formulated as a planar dynamical system given by equations

$$\begin{aligned} \dot{y} &= \alpha[i(y,r) - s(y,r)], \\ \dot{r} &= \beta[l(y,R) - m], \end{aligned} \tag{7.1}$$

where $\alpha, \beta > 0$ are parameters and the following notation is used:

Y ... real net product, $Y \in \mathbb{R}_+$,
y ... $y = \ln Y$, $y \in \mathbb{R}$,
r ... real interest rate, $r \in \mathbb{R}$,
R ... nominal interest rate, $R \in \mathbb{R}_+$,
$I(Y,r)$ ... real investments,
$i(y,r)$ ... propensity to invest or real investment-real net product ratio, i.e., $\frac{I}{Y}$,
$S(Y,r)$ ... real savings,
$s(y,r)$ ... propensity to save or real saving-real income ratio i.e. $\frac{S}{Y}$,
$L(Y,R)$ ... demand for money,
$l(y,R)$ ... $l(y,R) = L(e^y, R)$,
m ... real supply of money.

i, s

s

i

y

M N

Fig. 6.

The continuity of all functions and their first derivatives in both variables is assumed. In general, i and s are non-linear functions. The derivatives of functions i, s, l are assumed to have the usual properties

$$i_y > 0, \quad i_r < 0, \quad s_y > 0, \quad s_r > 0, \quad l_y > 0, \quad l_R < 0. \tag{7.2}$$

Economic observations yields that the functions i, s are of sigmoid form (S-shaped) for any fixed r (for more economic details see [6], pp. 442–443, or [9], pp. 144–145). Furthermore, it is assumed that for some fixed r_0 curves i, s intersects at three points. That corresponds to existence of two points M, N, where $-\infty < M < N < \infty$ such that

$$\frac{\partial(i(y,r)-s(y,r))}{\partial y} < 0 \qquad \text{for} \quad y \in (-\infty, M) \cup (N, \infty),$$
$$\frac{\partial(i(y,r)-s(y,r))}{\partial y} > 0 \qquad \text{for} \quad y \in (M, N).$$

These assumptions on the shape and position of the functions $i(y,r)$, $s(y,r)$ are called the Caldor assumptions in economics.

The nominal interest rate R is commonly estimated by the sum of the real interest rate r and the inflation π

$$R \simeq r + \pi.$$

The model of inflation can be written in the following form:

$$\begin{aligned} \dot{y} &= \alpha[i(y,r) - s(y,r)], \\ \dot{r} &= \beta[l(y, r+\pi) - m], \\ \dot{\pi} &= \gamma[\mu\dot{y} - \pi], \end{aligned} \tag{7.3}$$

where $\alpha, \beta, \gamma, \mu > 0$ are parameters. The parameters α, β and γ can be excluded by suitable scale changes.

The first equation describes the goods market. Excess investments cause a product increase, the excess savings cause a decrease. The second equation describes the money market. The interest rate, as the price of money, is given by the equilibrium on the money market. An excess demand for money causes its increase, excess supply of money cause its decrease. The third equation can be explained by investigation of the labour market, for additional economic explanation see [5]. The economy model abstracts from the international trade (we consider a closed economy) and the supply of money is considered to be a constant at the concrete monitored term (the central bank determines the supply of money).

The model (7.1) is a restriction of the model (7.3), when we abstract from inflation, that is for $\pi \equiv 0$. Then the third equation is satisfied and the nominal interest rate is equal to the real interest rate.

For the IS-LM model (7.1) some general results have been published, especially the existence of a limit cycle was proved by Torre (see [8]). Torre's results are direct applications of Andronov's theorems about bifurcation (see [2] and [3]). The results presented in Section 4 are also based on the bifurcation theory (its extension for the three dimensional system respectively), but the parameter of the bifurcation is not the one used by Torre (Example 6.3 shows three limit cycles depending on these two different parameters).

Many economists and mathematicians also tried to apply the Poincaré–Bendixson theorem to the general system (7.1), but they never succeeded. In order to obtain the needed results, they had to choose one of two alternatives: either they changed some of the economic assumptions, or they had to specialize the situation by adding some other conditions. The first way was used by Chang and Smyth, who assumed $s_r < 0$ in (7.2) and that was why their work was not accepted by economists. To the second group of works belongs, for example, [7] (moreover, this model is more realistic since tax collections and government expenditure are considered). The applications of Theorems 5.5, 5.6 and 5.7 are also of this type.

To the best of my knowledge, the inflation model (7.3) presented in [5] has never been analyzed yet.

Consider the system (1.1_μ) together with the assumption (1.2). This system may represent the model of inflation with the following approximations:

$$i(y,r) - s(y,r) = x_1 f(x_1) + bx_2,$$

$$l(y,R) - m = cx_1 + d(x_2 + x_3),$$

where $x_1 = y - y^*$, $x_2 = r - r^*$ and $x_3 = \pi$. The economy equilibrium $[y^*, r^*, 0]$ is translated to zero. These approximations may be suitable in a lot of concrete economic situations: the function f is considered to be a general nonlinear function with a continuous first derivative, the demand for money l is approximated by a linear function, which is commonly used.

It is seen that the assumption (7.2) and the Caldor assumption imply the assumption (1.2) and also some properties of the function f. The cubic version (1.5_μ) with the assumption (1.6) is a special type of a model of inflation, where the function $i - s$ is approximated by a cubic polynomial. The Caldor assumption is satisfied, when $a_1x_1 + 2a_2x_1^2 + 3a_3x_1^3 = 0$ has two distinct roots. Although the cubic approximation is simple, it is much more realistic than the strictly linear approximation used in a lot of economic textbooks. It seems to be the simplest function which satisfies the Caldor assumption, the linear function, of course, does not satisfy it.

Results from the previous sections lead to the following consequences.

Sections 2 and 3 may help to specify types of critical points. A critical point of (7.3) corresponds to an aggregate equilibrium in economy. The type of this equilibrium does not say anything more than how the economy will behave "near" the equilibrium. That is usually not sufficient since we do not know whether we are sufficiently close to the equilibrium or not beforehand.

More useful consequences are given by the results of Section 4. We proved the existence of a limit cycle in the model of inflation, that is, under the assumptions given in Theorem 4.1 we found the critical value of the parameter μ, the limit cycle occurs nearby. A limit cycle (especially a stable limit cycle) of (7.1) represents a business cycle in the economy. In economic terminology, there is no difference between oscillations with a fixed or mildly variable amplitude or period, tending to some constant amplitude or period. All these situations are called business cycles. The assumptions of Theorem 4.1 are sufficient conditions for the existence of a business cycle, which is not evoked by external influences, but which is entirely determined by the internal structure of the system and which affects all: the goods market, the money market and the labour market too.

Consider the system (1.1_μ) to satisfy the assumption (1.3) too. According to Theorem 5.1, we can bound the level of product y, the interest rate r and the inflation π. If the initial values of y, r and π lie inside the positively invariant set (ellipsoid translated to the aggregate economy equilibrium), they will remain there in the future. Corollary 5.3 gives a condition for the global stability of the aggregate equilibrium. Many economists presume that the economic equilibrium is globally stable always, i.e., they assume there exists some mechanism of adaptation in economy. This is true for a linear model of inflation. If the economy satisfies the Caldor assumption such mechanism need not exist. Theorems 5.2 and 5.3 give us a very useful and effective economic tool. If the assumptions of Theorem 5.3 are fulfilled, the economy behaves like for the linear model. There exists some mechanism of adaptation and the macroeconomic quantities tend to the aggregate equilibrium (this may be the "invisible hand" of the liberalists or the Walras general equilibrium, i.e., the "nice behaviour" of the economy). Theorem 5.2 has to be explained in another way. The set E computed for a concrete economy situation may help to predict the reaction of economy, although we do not presume the existence of any mechanism of adaptation. Even when the aggregate equilibrium is unstable, the economy is stable as a complex. The macroeconomic quantities tend to some range of values. This may, for example, exclude the "bad" reactions of the economy like hyperinflation and so on.

Notice that for the cubic approximation, the condition (1.3) is fulfilled.

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References

1. H. Amann, Ordinary differential equations: an introduction to nonlinear analysis. *de Gruyter Studies in Mathematics* 13, *Walter de Gruyter, Berlin, New York,* 1990.
2. A. A. Andronov, E. A. Leontovicz, I. I. Gordon, and A. G. Maier, Theory of bifurcations od dynamical systems in the plane. (Russian) *Nauka, Moscow,* 1967.
3. A. A. Andronov and C. E. Chaikin, Theory of oscillations. (Translation from Russian) *Princeton University Press, Princeton, New Jersey,* 1949.
4. L. Baráková, Asymptotic behaviour and existence of a limit cycle in cubic autonomous systems. *Demonstratio Math.* **34**(2001), No. 3, 559–576.
5. J. Kodera, A dynamic model of inflation. *Bulletin of The Czech Econometric Society* **5**(1996), 81–87.
6. G. Gandolfo, Economic dynamics. Third, completely edition. *Springer, Berlin, Heidelberg, New York,* 1997.
7. G. I. Schinasi, Fluctuations in a dynamic, intermediate-run IS-LM model: applications of the Poincaré–Bendixson theorem. *J. Econom. Theory* **28**(1982), 369–375.
8. V. Torre, Existence of limit cycles and control in complete Keynesian kystem by theory of bifurcations. *Econometrica* **45**(1977), No. 6, 1457–1466.
9. P. N. S. Tu, Dynamical systems. *An Introduction with Applications in Economics and Biology, Springer, Berlin, Heidelberg, New York,* 1992.

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