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CONFORMAL AND QUASICONFORMAL MAPPINGS OF CLOSE MULTIPLY-CONNECTED DOMAINS

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Abstract. Doubly-connected and triply-connected domains close to each other in a certain sense are considered. Some questions connected with conformal and quasiconformal mappings of such domains are studied using integral equations.

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1. Conformal Mapping of Close Triply-Connected Domains

Let G be a finite triply-connected domain of a complex plane Z bounded by the simple Lyapunov curves Γ_0, Γ_1, Γ_2, one of which Γ_0 envelops the other two and $z = 0 \in \operatorname{int} \Gamma_1$.

Assume first that the boundary $\Gamma = \bigcup\limits_{i=0}^{2} \Gamma_i$ belongs to the class C'_α ($\frac{1}{2} < \alpha \leq 1$), while singly-connected domains with boundaries Γ_1 and Γ_0 are star-like with respect to $z = 0$. Let the equations of these curves be given in terms of polar coordinates

$$t = g_1(\varphi) = \rho_1(\varphi) \cdot e^{i\varphi}, \quad t = g_0(\varphi) = \rho_0(\varphi) \cdot e^{i\varphi} \quad (0 \leq \varphi \leq 2\pi)$$

and the finite domain with boundary Γ_2 be star-like with respect to $z_0 \in \operatorname{int} \Gamma_2$. If we assume that the polar axis with a pole in z_0 is parallel to the abscissa axis, then the parametric equation for Γ_2 can be written in the form

$$t = g_2(\varphi) = z_0 + \rho_2(\varphi) e^{i\varphi} \quad (0 \leq \varphi \leq 2\pi).$$

Let us consider the second triply-connected domain $\widetilde{G}$ of type G bounded by the curves $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$ (with the same properties) whose parametric equations are

$$t = \widetilde{\rho_1}(\varphi) e^{i\varphi}, \quad t = z_0 + \widetilde{\rho_2}(\varphi) e^{i\varphi}, \quad t = \widetilde{\rho_0}(\varphi) e^{i\varphi} \quad (0 \leq \varphi \leq 2\pi).$$

We introduce the following notation:

$$d_1 = \rho(\Gamma_1; \Gamma_0), \quad d_2 = \rho(\Gamma_1; \Gamma_2), \quad d_3 = \rho(\Gamma_2; \Gamma_0), d_0 = \min\{d_1; d_2; d_3\}.$$

It is assumed that $\varepsilon \in (0; d_0/2)$.

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Definition 1. The domains G and $\widetilde{G}$ are called ε-close to each other if the conditions

$$|\rho_i(\varphi) - \widetilde{\rho}_i(\varphi)| \leq \varepsilon; \quad \|\rho_i'(\varphi) - \widetilde{\rho}_i(\varphi)\|_{C_\alpha} \leq \varepsilon \quad (i = 0, 1, 2) \tag{1}$$

are fulfilled.

An infinite set of domains ε-close to G are formed for any $\varepsilon \in (0; d/2)$. We denote it by G_ε.

Let us conformally map the domains G and $\widetilde{G}$ onto the canonical domains $K(\rho; r; 1)$ and $\widetilde{K}(\widetilde{\rho}; \widetilde{r}; 1)$, respectively, using the assumptions of [1], where $K(\rho; r; 1)$ and $\widetilde{K}(\widetilde{\rho}; \widetilde{r}; 1)$ are annuli with concentric cuts along the arc of the circumferences $|W| = r$ and $|W| = \widetilde{r}$, $(\rho < r < 1,\ \widetilde{\rho} < \widetilde{r} < r < 1)$, respectively. Then for the definition of radii we have [1]

$$\begin{aligned}
\ln \rho &= \frac{1}{\pi} \int_0^{2\pi} \nu_1(\varphi) \sqrt{\rho_1^2(\varphi) + [\rho_1'(\varphi)]^2}\, d\varphi\,, \\
\ln \widetilde{\rho} &= \frac{1}{\pi} \int_0^{2\pi} \widetilde{\nu}_1(\varphi) \sqrt{\widetilde{\rho}_1^2(\varphi) + [\widetilde{\rho}_1'(\varphi)]^2}\, d\varphi\,, \\
\ln r &= \frac{1}{\pi} \int_0^{2\pi} \nu_2(\varphi) \sqrt{\rho_2^2(\varphi) + [\rho_2'(\varphi)]^2} d\varphi\,, \\
\ln \widetilde{r} &= \frac{1}{\pi} \int_0^{2\pi} \widetilde{\nu}_2(\varphi) \sqrt{\widetilde{\rho}_2^2(\varphi) + [\widetilde{\rho}_2'(\varphi)]^2}\, d\varphi\,.
\end{aligned} \tag{2}$$

It is assumed here that

$$\nu(t(\varphi)) = \begin{cases} \nu_1(\varphi), & \text{when } \ t \in \Gamma_1, \\ \nu_2(\varphi), & \text{when } \ t \in \Gamma_2, \\ \nu_0(\varphi), & \text{when } \ t \in \Gamma_0. \end{cases}$$

The reasoning for $\widetilde{\nu}(\widetilde{t}(\varphi))$ is analogous. It is assumed that $\nu(t)$ and $\widetilde{\nu}(t)$ are unique solutions of the integral equations

$$\nu(t_0) + \frac{1}{\pi} \int_\Gamma K_0(t; t_0) \nu(t) dt = -\ln |t_0|, \quad t_0 \in \Gamma, \tag{3}$$

$$\widetilde{\nu}(t_0) + \frac{1}{\pi} \int_{\widetilde{\Gamma}} \widetilde{K}_0(t; t_0) \widetilde{\nu}(t) dt = -\ln |t_0|, \quad t_0 \in \widetilde{\Gamma}, \tag{4}$$

where

$$K_0(t; t_0) = \begin{cases} \operatorname{Im}\left(\dfrac{1}{t - t_0} \cdot \dfrac{dt}{ds}\right) - 1, & \text{when } \ t, t_0 \in \Gamma_j \ \ (j = 1, 2), \\ \operatorname{Im}\left(\dfrac{1}{t - t_0} \cdot \dfrac{dt}{ds}\right), & \text{in all other cases.} \end{cases}$$

$\widetilde{K}_0(t;t_0)$ is defined analogously.

Let us pose the problem: derive an estimate through ε for a difference of the solutions of equations (3) and (4) (in an appropriate norm), and also for the expressions $|\rho-\widetilde{\rho}|$, $|r-\widetilde{r}|$.

We can obtain such estimates by using the statements proved below. For this, the integral equations (3) and (4) are represented in the complex form:

$$\nu(t_0)+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{1}{t'}K_*(t;t_0)\nu(t)dt=-\ln|t_0|, \tag{3_1}$$

$$\widetilde{\nu}(t_0)+\frac{1}{2\pi i}\int\limits_{\widetilde{\Gamma}}\frac{1}{t'}\widetilde{K}_*(t;t_0)\widetilde{\nu}(t)dt=-\ln|t_0|, \tag{4_1}$$

where

$$K_0(t;t_0)=\begin{cases}(t'-t_0')+(\bar{t}_0'-\bar{t}')\dfrac{t-t_0}{\bar{t}-\bar{t}_0}+\left[t_0'\dfrac{\bar{t}-\bar{t}_0}{t-t_0}-\bar{t}_0'\right]\dfrac{t-t_0}{\bar{t}-\bar{t}_0}\dfrac{1}{t-t_0}-2i, \\ \qquad\text{when } t,t_0\in\Gamma_i\ \ (i=1,2), \\ (t'-t_0')+(\bar{t}_0'-\bar{t}')\dfrac{t-t_0}{\bar{t}-\bar{t}_0}+\left[t_0'\dfrac{\bar{t}-\bar{t}_0}{t-t_0}-\bar{t}_0'\right]\dfrac{t-t_0}{\bar{t}-\bar{t}_0}\dfrac{1}{t-t_0} \\ \qquad\text{in all other cases.}\end{cases} \tag{5}$$

Here $t'=g_j'(\varphi)$, $t_0'=g_j'(\varphi_0)$ and j take values $0,1,2$ depending on the fact to which contour Γ_j $(j=0,1,2)$ the point t or t_0 belongs. $\widetilde{K}^*(t;t_0)$, too, is constructed analogously to (5). Clearly, in that case $t=\widetilde{g}_j(\varphi)$, $t_0=\widetilde{g}_j(\varphi_0)$, $t'=\widetilde{g}_j'(\varphi)$, $t_0'=\widetilde{g}_j'(\varphi_0)$ $(j=0,1,2)$.

Let us represent the integral equations (3_1), (4_1) in the operator form:

$$A\nu=(I+H)\nu=f_0, \tag{$3_1'$}$$

$$\widetilde{A}\widetilde{\nu}=(I+\widetilde{H})\widetilde{\nu}=\widetilde{f}_0. \tag{$4_1'$}$$

It is assumed that (analogously to $\nu[t(\varphi)]$), $\widetilde{\nu}(t)$, $f_0(t)$, $\widetilde{f}_0(t)$ are column-matrices, I is the uni t matrix of third order, and

$$H\nu=\begin{pmatrix}H_{11}\nu_1+H_{12}\nu_2+H_{10}\nu_0\\ H_{21}\nu_1+H_{22}\nu_2+H_{20}\nu_0\\ H_{01}\nu_1+H_{02}\nu_2+H_{00}\nu_0\end{pmatrix},$$

$$\widetilde{H}\widetilde{\nu}=\begin{pmatrix}\widetilde{H}_{11}\widetilde{\nu}_1+\widetilde{H}_{12}\widetilde{\nu}_2+\widetilde{H}_{10}\widetilde{\nu}_0\\ \widetilde{H}_{21}\widetilde{\nu}_1+\widetilde{H}_{22}\widetilde{\nu}_2+\widetilde{H}_{20}\widetilde{\nu}_0\\ \widetilde{H}_{01}\widetilde{\nu}_1+\widetilde{H}_{02}\widetilde{\nu}_2+\widetilde{H}_{00}\widetilde{\nu}_0\end{pmatrix}.$$

Here H_{ij}, $\widetilde{H}_{ij}$ $(i,j=0,1,2)$ are the concrete integral operators. Before representing them explicitly, let us make some additional observations.

Observe that

$$\frac{dt}{t-t_0}\bigg|_{\substack{t\in\Gamma_j\\ t_0\in\Gamma_k}}=\frac{g_j'(\varphi)d\varphi}{g_j(\varphi)-g_k(\varphi_0)}=M_{jk}(\varphi;\varphi_0)\Big(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\Big)d\varphi,$$

where the function

$$M_{jk}(\varphi;\varphi_0)=\frac{g_j'(\varphi)}{2ie^{i\varphi}}\cdot\frac{e^{i\varphi}-e^{i\varphi_0}}{g_j(\varphi)-g_k(\varphi_0)} \tag{6}$$

satisfies the Hölder condition with index α with respect to its arguments, i.e., belongs to the class $C_\alpha(\Gamma_j\times\Gamma_k)$.

The function

$$R_{jk}(t_0;t)=\frac{1}{t'}\left\{(t'-t_0')+(\bar{t}_0'-\bar{t}')\frac{t-t_0}{\bar{t}-\bar{t}_0}+\left[t_0'\frac{\bar{t}-\bar{t}_0}{t-t_0}-\bar{t}_0'\right]\frac{t-t_0}{\bar{t}-\bar{t}_0}\right\}$$

($t\in\Gamma_j$, $t_0\in\Gamma_k$, $j,k=0,1,2$) figuring in (5) also belongs to the class $C_\alpha(\Gamma_j\times\Gamma_k)$ and $R_{kk}(t_0;t_0)=0$.

We have in terms of the parameter φ

$$\begin{aligned}S_{j\,k}(\varphi;\varphi_0)=R_{j\,k}[g_j(\varphi);g_k(\varphi_0)]=\frac{1}{g_j'(\varphi)}\bigg\{&g_j'(\varphi)-g_k'(\varphi_0)\\&+(\bar{g}_k'(\varphi_0)-\bar{g}_j'(\varphi))\,\frac{g_j(\varphi)-g_k(\varphi_0)}{\bar{g}_j(\varphi)-\bar{g}_k(\varphi_0)}\\&+\left[g_k'(\varphi_0)\,\frac{\bar{g}_j(\varphi)-\bar{g}_k(\varphi_0)}{g_j(\varphi)-g_k(\varphi_0)}-\bar{g}_k'(\varphi_0)\right]\frac{g_j(\varphi)-g_k(\varphi_0)}{\bar{g}_j(\tau)-\bar{g}_k(\tau_0)}\bigg\},\end{aligned} \tag{5$'$}$$

$S_{jk}(\varphi;\varphi_0)\in C_\alpha(\Gamma_j\times\Gamma_k)$, $S_{kk}(\varphi_0;\varphi_0)=0$ $(0\le\varphi\le 2\pi)$.

$\widetilde{M}_{jk}(\varphi,\varphi_0)$ and $\widetilde{S}_{jk}(\varphi;\varphi_0)$ are defined analogously.

Having introduced the notation, we can write that

$$\begin{aligned}&[H_{11}\nu_1+H_{12}\nu_2+H_{10}\nu_0](\varphi_0)\\&=\frac{1}{2\pi i}\int_0^{2\pi}\left\{\frac{S_{11}(\varphi_0;\varphi)M_{11}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\Big(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\Big)|e^{i\varphi}-e^{i\varphi_0}|^\beta-2i\right\}\nu_1(\varphi)d\varphi\\&+\frac{1}{2\pi i}\int_0^{2\pi}\frac{S_{12}(\varphi_0;\varphi)M_{12}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\Big(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\Big)|e^{i\varphi}-e^{i\varphi_0}|^\beta\nu_2(\varphi)d\varphi\\&+\frac{1}{2\pi i}\int_0^{2\pi}\frac{S_{10}(\varphi_0;\varphi)M_{10}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\Big(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\Big)|e^{i\varphi}-e^{i\varphi_0}|^\beta\nu_0(\varphi)d\varphi,\end{aligned}$$

$$\begin{aligned}&[H_{21}\nu_1+H_{22}\nu_2+H_{20}\nu_0](\varphi_0)\\&=\frac{1}{2\pi i}\int_0^{2\pi}\left\{\frac{S_{21}(\varphi_0;\varphi)M_{21}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\Big(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\Big)|e^{i\varphi}-e^{i\varphi_0}|^\beta\right\}\nu_1(\varphi)d\varphi\end{aligned}$$

$$+\frac{1}{2\pi i}\int_0^{2\pi}\left\{\frac{S_{22}(\varphi_0;\varphi)M_{22}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\left(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\right)|e^{i\varphi}-e^{i\varphi_0}|^\beta-2i\right\}\nu_2(\varphi)d\varphi$$

$$+\frac{1}{2\pi i}\int_0^{2\pi}\left\{\frac{S_{20}(\varphi_0;\varphi)M_{20}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\left(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\right)|e^{i\varphi}-e^{i\varphi_0}|^\beta\right\}\nu_0(\varphi)d\varphi,$$

$$[H_{01}\nu_1+H_{02}\nu_2+H_{00}\nu_0](\varphi_0)$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\left\{\frac{S_{01}(\varphi_0;\varphi)M_{01}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\left(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\right)|e^{i\varphi}-e^{i\varphi_0}|^\beta\right\}\nu_1(\varphi)d\varphi$$

$$+\frac{1}{2\pi i}\int_0^{2\pi}\left\{\frac{S_{02}(\varphi_0;\varphi)M_{02}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\left(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\right)|e^{i\varphi}-e^{i\varphi_0}|^\beta\right\}\nu_2(\varphi)d\varphi$$

$$+\frac{1}{2\pi i}\int_0^{2\pi}\left\{\frac{S_{00}(\varphi_0;\varphi)M_{00}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\cdot\left(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\right)|e^{i\varphi}-e^{i\varphi_0}|^\beta\right\}\nu_0(\varphi)d\varphi,$$

where β is any number satisfying the condition $\frac{1}{2}<\beta\le\alpha$.

From the above formulas we obtain

$$\begin{aligned}(H_{11}-\widetilde{H}_{11})\nu_1&=\frac{1}{\pi i}\int_0^{2\pi}\frac{S_{11}(\varphi_0;\varphi)M_{11}(\varphi_0;\varphi)-\widetilde{S}_{11}(\varphi_0;\varphi)\widetilde{M}_{11}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\\&\qquad\times\frac{1}{2}\left(\operatorname{ctg}\frac{\varphi-\varphi_0}{2}+i\right)|e^{i\varphi}-e^{i\varphi_0}|^\beta\nu_1(\varphi)d\varphi\\&=\frac{1}{\pi i}\int_0^{2\pi}\frac{S_{11}(\varphi_0;\varphi)M_{11}(\varphi_0;\varphi)-\widetilde{S}_{11}(\varphi_0;\varphi)\widetilde{M}_{11}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta}\\&\qquad\times\frac{ie^{i\varphi}}{e^{i\varphi}-e^{i\varphi_0}}\;|e^{i\varphi}-e^{i\varphi_0}|^\beta\nu_1(\varphi)d\varphi\\&=\frac{1}{\pi i}\int_0^{2\pi}[K_{11}(\varphi_0;\varphi)-\widetilde{K}_{11}(\varphi_0;\varphi)]\nu_1(\varphi)d\varphi.\end{aligned}\tag{7}$$

In (7)

$$\begin{aligned}K_{11}(\varphi_0;\varphi)&=K_{11}^{(1)}(\varphi_0;\varphi)K_{11}^{(2)}(\varphi_0;\varphi),\\\widetilde{K}_{11}(\varphi_0;\varphi)&=\widetilde{K}_{11}^{(1)}(\varphi_0;\varphi)\widetilde{K}_{11}^{(2)}(\varphi_0;\varphi),\end{aligned}\tag{7$'$}$$

where

$$\begin{aligned}K_{11}^{(1)}(\varphi_0;\varphi)&=\frac{M_{11}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta};\quad K_{11}^{(2)}(\varphi_0;\varphi)=\frac{iS_{11}(\varphi_0;\varphi)e^{i\varphi}}{|e^{i\varphi}-e^{i\varphi_0}|^{1-\beta}\cdot e^{i\arg(e^{i\varphi}-e^{i\varphi_0})}};\\\widetilde{K}_{11}^{(1)}(\varphi_0;\varphi)&=\frac{\widetilde{M}_{11}(\varphi_0;\varphi)}{|e^{i\varphi}-e^{i\varphi_0}|^\beta};\quad \widetilde{K}_{11}^{(2)}(\varphi_0;\varphi)=\frac{i\widetilde{S}_{11}(\varphi_0;\varphi)e^{i\varphi}}{|e^{i\varphi}-e^{i\varphi_0}|^{1-\beta}\cdot e^{i\arg(e^{i\varphi}-e^{i\varphi_0})}}.\end{aligned}\tag{8}$$

According to [2], if $\varphi(t) \in C\mu(\Gamma)$, $0 < \mu \leq 1$, then the function of two variables $(t, t_0 \in \Gamma)$

$$\psi(t; t_0) = \frac{\varphi(t) - \varphi(t_0)}{|t - t_0|^{\lambda}},$$

$0 \leq \lambda < \mu \leq 1$, satisfies, on Γ, the Hölder condition with index $\mu - \lambda$. Moreover, the estimate

$$\|\psi(t_0; t)\|_{C_{\mu-\lambda}} \leq A^*(1 + \lambda) \tag{9}$$

holds true, where $A^* \geq \frac{|\varphi(t) - \varphi(t_0)|}{|t - t_0|^{\lambda}}$ (see §§5, 6 of [2]). Taking into account the structure of the functions $M_{11}(\varphi; \varphi_0)$, $S_{11}(\varphi; \varphi_0)$, $\widetilde{M}_{11}(\varphi; \varphi_0)$, $\widetilde{S}_{11}(\varphi; \varphi_0)$, this result implies that the functions $K_{11}(\varphi; \varphi_0)$ and $\widetilde{K}_{11}(\varphi; \varphi_0)$ are continuous in the Hölder sense with respect to φ_0 and φ with index δ, $\delta = \min\{\alpha - \beta; \alpha + \beta - 1\}$ and

$$\begin{aligned}\|K_{11}(\varphi; \varphi_0) - \widetilde{K}_{11}(\varphi; \varphi_0)\|_{C_\delta} &= \|K_{11}^{(1)} K_{11}^{(2)} - \widetilde{K}_{11}^{(1)} \widetilde{K}_{11}^{(2)}\|_{C_\delta} \\ &\leq \|K_{11}^{(2)}\|_{C_\delta} \cdot \|K_{11}^{(1)} - \widetilde{K}_{11}^{(1)}\|_{C_\delta} + \|\widetilde{K}_{11}^{(1)}\|_{C_\delta} \cdot \|K_{11}^{(2)} - \widetilde{K}_{11}^{(2)}\|_{C_\delta}.\end{aligned} \tag{10}$$

Let us define the order of smallness with respect to ε in (10). Preliminarily, we will prove the validity of the following propositions.

For small values of ε the following inequalities are fulfilled:

I. $\|[g_1'(\varphi) - g'(\varphi_0)] - [\widetilde{g}_1'(\varphi) - \widetilde{g}'(\varphi_0)]\|_{C_\alpha} \leq 2\varepsilon$,
$\|[\overline{g}_1'(\varphi) - \overline{g}'(\varphi_0)] - [\overline{\widetilde{g}}_1'(\varphi) - \overline{\widetilde{g}}'(\varphi_0)]\|_{C_\alpha} \leq 2\varepsilon$,

II. $$\left\|\frac{e^{i\varphi} - e^{i\varphi_0}}{g_1(\varphi) - g_1(\varphi_0)} \cdot \frac{g_1'(\varphi)}{2ie^{i\varphi}} - \frac{e^{i\varphi} - e^{i\varphi_0}}{\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)} \cdot \frac{\widetilde{g}_1'(\varphi)}{2ie^{i\varphi}}\right\|_{C_\alpha} \leq A_1 \cdot \varepsilon,$$

III. $$\left\|\frac{g_1(\varphi) - g_1(\varphi_0)}{\overline{g}_1(\varphi) - \overline{g}_1(\varphi_0)} - \frac{\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)}{\overline{\widetilde{g}}_1(\varphi) - \overline{\widetilde{g}}_1(\varphi_0)}\right\|_{C_\alpha} \leq A_2 \cdot \varepsilon,$$

IV. $$|\widetilde{g}_1'(\varphi)| \geq \frac{\min\limits_{[0,2\pi]} |g_1'(\varphi)|}{2} \quad \text{for} \quad \varepsilon \leq \frac{\min\limits_{[0,2\pi]} |g_1'(\varphi)|}{2},$$

V. $$\left\|\left(g_1'(\varphi_0) \cdot \frac{\overline{g}_1(\varphi) - \overline{g}_1(\varphi_0)}{g_1(\varphi) - g_1(\varphi_0)} - g_1'(\varphi_0)\right)\right. \left.\left(-\widetilde{g}_1'(\varphi_0) \cdot \frac{\overline{\widetilde{g}}_1(\varphi) - \overline{\widetilde{g}}_1(\varphi_0)}{\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0} - \overline{\widetilde{g}}_1'(\varphi_0)\right)\right\|_{C_\alpha} \leq A_3\varepsilon$$

and all constants contained in the estimates do not depend on $\widetilde{G}$.

Proof. Inequality I immediately follows from the definition of domain closeness, i.e., from (1).

Further we have

$$|g_1'(\varphi)| - |\widetilde{g}_1'(\varphi)| \leq \|g_1'(\varphi) - \widetilde{g}_1'(\varphi)\|_C \leq \|g_1'(\varphi) - \widetilde{g}_1'(\varphi)\|_{C_\alpha} < \varepsilon$$

and if it is assumed that $\varepsilon \le \frac{1}{2} \min\limits_{[0,2\pi]} |g_1'(\varphi)|$, then the validity of inequality IV is proved.

Before proving inequality II, note the following: for $|s - s_0| \le \frac{l_1}{2}$, where l_1 is the length of Γ_1, and s and s_0 are the arc abscissas of the points $t = g_1(\varphi)$ and $t_0 = g_1(\varphi_0)$, we have

$$\frac{|t - t_0|^2}{|\varphi - \varphi_0|^2} = \frac{|g_1(\varphi) - g_1(\varphi_0)|^2}{|s - s_0|^2} \cdot \frac{|s - s_0|^2}{|\varphi - \varphi_0|^2} \ge k_1^2 \cdot \rho_1^2, \tag{11}$$

where $\rho_1 = \min\limits_{[0,2\pi]} |g_1'(\varphi)| = \rho(0; \Gamma_1) > 0$, and k_1 $(0 < k_1 < 1)$ is the constant defined by giving the contour Γ_1.

On the other hand,

$$\begin{aligned}[g_1(\varphi) - g_1(\varphi_0)] - [\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)] &= [g_1(\varphi) - \widetilde{g}_1(\varphi)] - [g_1(\varphi_0) - \widetilde{g}_1(\varphi_0)] \\ &= \mathrm{Re}[g_1(\varphi) - \widetilde{g}_1(\varphi)] - \mathrm{Re}[g_1(\varphi_0) - \widetilde{g}_1(\varphi_0)] \\ &\quad + i\,(\mathrm{Im}[g_1(\varphi) - \widetilde{g}_1(\varphi)] - \mathrm{Im}[g_1(\varphi_0) - \widetilde{g}_1(\varphi_0)]) \\ &= (\varphi - \varphi_0)\ \mathrm{Re}[g_1'(\xi) - \widetilde{g}_1'(\xi)] + i\,(\varphi - \varphi_0)\ \mathrm{Im}[g_1'(\eta) - \widetilde{g}_1'(\eta)],\end{aligned}$$

where the numbers ξ and η lie between φ and φ_0.

By virtue of (1) we can write that

$$|[\widetilde{g}(\varphi) - \widetilde{g}(\varphi_0)] - [g_1(\varphi) - g_1(\varphi_0)]| \le 4\varepsilon|\varphi - \varphi_0|.$$

Moreover, since

$$\begin{aligned}&|g_1(\varphi) - g_1(\varphi_0)| - |\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)| \\ &\le |[g_1(\varphi) - g_1(\varphi_0)] - [\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)]| \le 4\varepsilon|\varphi - \varphi_0|\end{aligned}$$

we have

$$|\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)| \ge |g_1(\varphi) - g_1(\varphi_0)| - 4\varepsilon|\varphi - \varphi_0|.$$

Hence, assuming that $\varepsilon \le \frac{k_1\rho_1}{8}$, by (11) we obtain

$$\frac{|\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)|}{|\varphi - \varphi_0|} > \frac{k_1\rho_1}{2}. \tag{12}$$

Now we return to proving inequality II. Since

$$\begin{aligned}&\frac{e^{i\varphi} - e^{i\varphi_0}}{g_1(\varphi) - g_1(\varphi_0)} \cdot \frac{g_1'(\varphi)}{2ie^{i\varphi}} - \frac{e^{i\varphi} - e^{i\varphi_0}}{\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)} \cdot \frac{\widetilde{g}_1'(\varphi)}{2ie^{i\varphi}} \\ &= \frac{e^{i\,\varphi} - e^{i\,\varphi_0}}{2\;i\;e^{i\,\varphi}} \cdot \frac{g_1'(\varphi)[\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)] - \widetilde{g}_1'(\varphi)[g_1(\varphi) - g_1(\varphi_0)]}{[g_1(\varphi) - g_1(\varphi_0)][\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)]} = \frac{e^{i\varphi} - e^{i\varphi_0}}{2ie^{i\varphi}} \\ &\times \frac{(\varphi - \varphi_0)\Big[g_1'(\varphi) \int\limits_0^1 \widetilde{g}_1'[\varphi_0 + u(\varphi - \varphi_0)]\,du - \widetilde{g}_1'(\varphi) \int\limits_0^1 g_1'[\varphi_0 + u(\varphi - \varphi_0)]du\Big]}{[g_1(\varphi) - g_1(\varphi_0)][\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)]},\end{aligned}$$

by virtue of I, IV, (11) and (12) this expression obviously implies that inequality II is valid.

Just in the same way, after carrying out analogous transformations, from

$$\frac{g_1(\varphi)-g_1(\varphi_0)}{\widetilde{g}_1(\varphi)-\widetilde{g}_1(\varphi_0)}-\frac{\widetilde{g}_1(\varphi)-\widetilde{g}_1(\varphi_0)}{\bar{\widetilde{g}}_1(\varphi)-\bar{\widetilde{g}}_1(\varphi_0)}=\frac{(\varphi-\varphi_0)^2}{[\bar{g}_1(\varphi)-\bar{g}_1(\varphi_0)][\bar{\widetilde{g}}_1(\varphi)-\bar{\widetilde{g}}_1(\varphi_0)]}$$
$$\times\left[\int_0^1 g_1'[\varphi_0+u(\varphi-\varphi_0)]du\int_0^1\bar{\widetilde{g}}_1[\varphi_0+u(\varphi-\varphi_0)]du\right.$$
$$\left.-\int_0^1\widetilde{g}_1'[\varphi_0+u(\varphi+\varphi_0)]du\int_0^1\bar{g}_1[\varphi_0+u(\varphi-\varphi_0)]du\right]$$

follows the validity of inequality III.

Inequality V immediately follows from inequalities I, II and IV.

The validity of I–V is proved. □

Now taking into account formulas (6) and (5′) for $M_{11}(\varphi,\varphi_0)$, $S_{11}(\varphi,\varphi_0)$ and analogous formula for $\widetilde{M}_{11}(\varphi,\varphi_0)$, $\widetilde{S}_{11}(\varphi,\varphi_0)$, we can state by virtue of the above inequalities and $S_{11}(\varphi_0,\varphi_0)=\widetilde{S}_{11}(\varphi_0,\varphi_0)$ that

$$\begin{gathered}\|S_{11}(\varphi,\varphi_0)\,e^{i\,arg(e^{i\,\varphi}-e^{i\,\varphi_0})}-\widetilde{S}_{11}(\varphi,\varphi_0)\,e^{i\arg(e^{i\,\varphi}-e^{i\,\varphi_0})}\|_{c_\alpha}\le B_1\cdot\varepsilon,\\ \|M_{11}(\varphi,\varphi_0)-\widetilde{M}_{11}(\varphi,\varphi_0)\|_{c_\alpha}\le B_2\cdot\varepsilon,\end{gathered}\tag{13}$$

where the constants B_1 and B_2 depend only on the domain G.

By virtue of (9) and (13), inequality (10) immediately gives rise to the following estimates of its individual terms:

$$\begin{gathered}\|K_{11}^{(1)}(\varphi,\varphi_0)-\widetilde{K}_{11}^{(1)}(\varphi,\varphi_0)\|_{c_\delta}\le N_1\cdot\varepsilon,\\ \|K_{11}^{(2)}(\varphi,\varphi_0)-\widetilde{K}_{11}^{(2)}(\varphi,\varphi_0)\|_{c_\delta}\le N_2\cdot\varepsilon,\\ \|\widetilde{K}_{11}^{(1)}(\varphi,\varphi_0)\|_{c_\delta}\le N_3,\quad \|\widetilde{K}_{11}^{(2)}(\varphi,\varphi_0)\|_{c_\delta}\le N_4,\end{gathered}\tag{14}$$

($\delta=\min\{\alpha-\beta;\alpha+\beta-1\}$)), where all constants are expressed in terms of the initial domain G, i.e.,for small ε the estimates are uniform with respect to domains $\widetilde{G}\in G_\varepsilon$.

Therefore for the kernels represented by formulas (7′) we have

$$\|K_{11}(\varphi;\varphi_0)-\widetilde{K}_{11}(\varphi;\varphi_0)\|_{c_\delta}\le N_1(\beta)\cdot\varepsilon.\tag{15}$$

The estimates

$$\begin{gathered}\|K_{22}(\varphi;\varphi_0)-\widetilde{K}_{22}(\varphi;\varphi_0)\|_{c_\delta}\le N_2(\beta)\cdot\varepsilon,\\ \|K_{00}(\varphi;\varphi_0)-\widetilde{K}_{00}(\varphi;\varphi_0)\|_{c_\delta}\le N_0(\beta)\cdot\varepsilon\end{gathered}\tag{16}$$

are established analogously.

If the points $t(\varphi)$, $t(\varphi_0)$ and the corresponding points $\widetilde{t}(\varphi)$, $\widetilde{t}(\varphi_0)$ lie on different curves Γ_i, Γ_j and $\widetilde{\Gamma}_i$, $\widetilde{\Gamma}_j$ $(i\ne j)$, respectively, then estimates for the values $\|K_{ij}-\widetilde{K}_{ij}\|_{c_\delta}$ $(i\ne j,\ i,j=0,1,2)$ are established immediately and have order

$O(\varepsilon)$. This can be verified at once if, for the kernels of equations (3_1), (4_1) representable by formulas of form (5), the difference

$$K_*[g_i(\varphi),\ g_j(\varphi_0)] - \widetilde{K}_*[\widetilde{g}_i(\varphi);\ \widetilde{g}_j(\varphi_0)]$$

is reduced to the common denominator and the latter is estimated from below by the number $d_0 = \min\{d_1; d_2; d_3\}$ assuming that $0 < \varepsilon < \frac{d_0}{2}$, while the difference of their numerators is estimated using inequality (1).

We have thus shown that the following theorem holds for small values of the parameter ε.

Theorem 1. *If the domains G and $\widetilde{G} \in G_\varepsilon$ $(0 < \varepsilon \leq \varepsilon_0)$ belong to the class G'_α $(\frac{1}{2} < \alpha \leq 1)$, then the inequality*

$$\|K_*(\varphi;\varphi_0) - \widetilde{K}_*(\varphi;\varphi_0)\|_{c_\delta} < A_0(G;\beta)\cdot\varepsilon \tag{17}$$

holds, where $K_*(\varphi;\varphi_0)$, $\widetilde{K}_*(\varphi;\varphi_0)$ are the kernels of the integral equations (3_1), (4_1), respectively, $\delta = \min\{\alpha-\beta; \alpha+\beta-1\}$ and $\frac{1}{2} < \beta \leq \alpha$. The constant $A_0(G;\beta)$ and small ε_0 are completely defined by giving the initial domain G

$$\varepsilon_0 = \frac{1}{2}\min\Big\{d_0;\ k\rho/4;\ g(\varphi)\Big\},$$

where k is the constant defined by giving the contour Γ, $\rho = \min\{\rho(0,\Gamma_j) : j = 0,1,2\}$, $g(\varphi) = \min\{|g_j'(\varphi)| : j = 0,1,2\}$.

The proven theorem makes it possible to obtain an estimate for the difference $\nu(\varphi) - \widetilde{\nu}(\varphi)$ in an adequate norm. It is of order $O(\varepsilon)$, but it can be obtained in a stronger form if we use the result from [3]. This technique implies estimating, through ε, the difference of the corresponding integral operators.

Let us estimate the difference $(H - \widetilde{H})\nu$. Having in mind the structure of this difference, it suffices to confine the investigation to the case $(H_{11} - \widetilde{H}_{11})\nu_1$.

From (7) we have

$$(H_{11} - \widetilde{H}_{11})\ \nu_1$$
$$= \frac{1}{\pi i}\int_0^{2\pi} \widetilde{K}^*_{11}(\varphi;\varphi_0) i e^{i\,\varphi}\cdot\exp(-i\arg(e^{i\varphi} - e^{i\varphi_0}))\ \frac{\nu_1(\varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^{1-\beta}}, \tag{18}$$

where

$$\widetilde{K}^*_{11}(\varphi,\varphi_0) = \frac{S_{11}(\varphi;\varphi_0)M_{11}(\varphi;\varphi_0) - \widetilde{S}_{11}(\varphi;\varphi_0)\widetilde{M}_{11}(\varphi;\varphi_0)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta}. \tag{19}$$

As has already been noted, the function $\widetilde{K}^*_{11}(\varphi;\varphi_0)$ is continuous in the Hölder sense with index $\alpha - \beta$. Taking into account (9) and (19) as well as inequalities I–V, we see that the inequality

$$\|\widetilde{K}^*_{11}(\varphi;\varphi)\|_{c_{\alpha-\beta}} \leq C_1(\beta)\ \varepsilon \tag{20}$$

is valid for small values of the parameter ε.

Let us use the following result from [2]. A function of form (18)

$$\omega(t(\varphi_0)) = \frac{1}{\pi} \int_0^{2\pi} \widetilde{K}_{11}^*(\varphi;\varphi_0)\, r(\varphi;\varphi_0)\, \frac{\nu_1(\varphi)\, d\varphi}{|e^{i\,\varphi} - e^{i\,\varphi_0}|^{1-\beta}},$$

where $r(\varphi;\varphi_0) = ie^{i\varphi}\exp(-i\arg(e^{i\varphi} - e^{i\varphi_0}))$ belongs to the class C_δ for any bounded function $\nu_1(\varphi)$ and $\delta = \alpha - \beta$ (see §51 of [2]).

Now, by virtue of this result, (18) and (20) give rise to the estimate

$$\|(H_{11} - \widetilde{H}_{11})\, \nu_1\|_{c_{\alpha-\beta}} \le C^{'}(\beta)\, \|\nu_1\|_{C_{\alpha-\beta}} \cdot \varepsilon, \tag{21}$$

where $C'(\beta)$ and $C_1(\beta)$ from (20) are the absolute constants, while $\nu(\varphi)$ is any function of the class $C_{\alpha-\beta}$.

The proof that the values $\|(H_{ii} - \widetilde{H}_{ii})\nu_i\|_{C_{\alpha-\beta}}$ for $i = 0, 2$ have order $O(\varepsilon)$ repeats the proof of inequality (21).

The estimate for the values $\|(H_{ij} - \widetilde{H}_{ij})\nu_j\|_{C_{\alpha-\beta}}$ when $i \ne j$, i.e., when the points $t(\varphi)$ and $t_0(\varphi_0)$ belong to different curves Γ_1 and Γ_2, while the corresponding points $\widetilde{t}(\varphi)$ and $\widetilde{t}_0(\varphi_0)$ belong to the contours $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, is obtained immediately if it is assumed, for instance, $\varepsilon < \frac{d_0}{2}$. The constants in all such estimates are expressed only in terms of the curves Γ_1 and Γ_2.

Hence the estimate

$$\|(H - \widetilde{H})\, \nu\|_{c_{\alpha-\beta}} \le C_0(\beta) \cdot \|\nu\|_{C_{\alpha-\beta}} \cdot \varepsilon, \tag{22}$$

where $C_0(\beta)$ is the absolute constant, is valid.

Assume now that $\nu(\varphi)$ and $\widetilde{\nu}(\varphi)$ are solutions of equations (3_1) and (4_1), respectively. From $(3_1')$ and $(4_1'')$ we have

$$\begin{aligned}\nu - \widetilde{\nu} &= \widetilde{A}^{-1}(\widetilde{A} - A)\, \nu - \widetilde{A}^{-1}\, (\bar{f}_0 - f_0) \\ &= \widetilde{A}^{-1}(\widetilde{H} - H)\, \nu - \widetilde{A}^{-1}(\widetilde{f}_0 - f_0).\end{aligned} \tag{23}$$

But by (22)

$$\begin{aligned}\|A - \widetilde{A}\|_{c_{\alpha-\beta}} &= \sup_{\|\nu\|_{C_{\alpha-\beta}}=1} \|(\widetilde{A} - A)\, \nu\|_{c_{\alpha-\beta}} \\ &\le \sup_{\|\nu\|_{C_{\alpha-\beta}}=1} C_0(\beta)\|\nu\|_{c_{\alpha-\beta}} \cdot \varepsilon = C_0(\beta) \cdot \varepsilon\end{aligned} \tag{24}$$

and

$$\|(\widetilde{A} - A)\, A^{-1}\|_{c_{\alpha-\beta}} \le \|A - \widetilde{A}\|_{c_{\alpha-\beta}} \cdot \|A^{-1}\|_{c_{\alpha-\beta}} \le C_0(\beta)\, \|A^{-1}\|_{c_{\alpha-\beta}} \cdot \varepsilon$$

and if it is assumed that $\varepsilon < 1/(C_0(\beta) \cdot \|A^{-1}\|_{C_{\alpha-\beta}})$, then the norm of the inverse operator $\widetilde{A}^{-1}$ [4] is uniformly (with respect to $\widetilde{\Gamma}$) bounded in the space $C_{\alpha-\beta}$,

$$\|\widetilde{A}^{-1}\|_{C_{\alpha-\beta}} \le \frac{\|A^{-1}\|_{C_{\alpha-\beta}}}{1 - \|(\widetilde{H} - H)A^{-1}\|_{C_{\alpha-\beta}}}.$$

Using further the obvious estimate $\|f_0 - \widetilde{f}_0\|$ and (24), from (23) it follows that the inequality

$$\|\nu(\varphi) - \widetilde{\nu}(\varphi)\|_{C_{\alpha-\beta}} < B_0(\beta) \cdot \|\nu\|_{C\alpha-\beta} \cdot \varepsilon$$

is valid for all $\varepsilon \leq \varepsilon_1$, where

$$\varepsilon_1 = \left\{ \frac{d_0}{2};\ \frac{g(\varphi)}{2};\ \ \frac{1}{C_0(\beta)\|A^{-1}\|_{C_{\alpha-\beta}}} \right\},$$

$B_0(\beta)$ is the constant depending only on G and β (β is any positive number smaller than α). Thus we have proved

Theorem 2. *If the boundaries of the domains G and $\widetilde{G} \in G_\varepsilon$, $0 < \varepsilon \leq \varepsilon_1$, belong to the class C'_α $(0 < \alpha < 1)$, then the inequality*

$$\|\nu(\varphi) - \widetilde{\nu}(\varphi)\|_{C_{\alpha-\beta}} < B_0(\beta) \cdot \|\nu\|_{C_{\alpha-\beta}} \cdot \varepsilon \tag{25}$$

is valid, where $\nu(\varphi)$ and $\widetilde{\nu}(\varphi)$ are unique solutions of the integral equations (3_1), (4_1), *respectively. The constant $B_0(\beta)$ and small ε_1 are defined by giving the initial domain G; β is any positive number smaller than α.*

The proven theorem solves the problem we have posed. As for estimating the differences $\|r - \widetilde{r}\|$ and $\|\rho - \widetilde{\rho}\}$, such an estimate is immediately implied by Theorem 2 and formulas (2).

Corollary 1. *If the triply-connected domains G and $\widetilde{G}$ are ε-close to each other* $(0 < \varepsilon \leq \varepsilon_1)$, *then the inequalities*

$$\|\rho - \widetilde{\rho}\| < Q_1 \cdot \varepsilon; \quad \|r - \widetilde{r}\| < Q_2 \cdot \varepsilon \tag{26}$$

are valid, where ρ, r, $\widetilde{\rho}$, $\widetilde{r}$ are the radii defining the canonical domains $K(\rho; r; 1)$ and $\widetilde{K}(\widetilde{\rho}; \widetilde{r}; 1)$, respectively, while the constant Q_1 and Q_2 depend only on the domain G.

2. Conformal Mapping of Close Doubly-Connected Domains

In the complex plane Z let us consider the finite doubly-connected domain G whose boundary Γ consists of the simple closed Lyapunov curves Γ_0 and Γ_1, one of which Γ_0 envelops the other, and $z = 0 \in \operatorname{int} \Gamma_1$.

Assume that the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of the given domain belongs to the class C'_α $(0 < \alpha < 1)$ and is given parametrically by the equations

$$t = g_1(\tau), \quad t = g_0(\tau) \quad (0 \leq \tau \leq 2\pi;\ \ g_i(0) = g_i(2\pi),\ \ i = 0, 1).$$

Let $d_0 = \rho(\Gamma_1; \Gamma_0)$ and assume that $0 < \varepsilon < d_0/2$.

Consider another doubly-connected domain $\widetilde{G}$ of type G.

Definition 2. The domains G and $\widetilde{G}$ whose boundary $\widetilde{\Gamma}$ consists of $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ whose parametric equations are

$$t=\widetilde{g}_1(\tau), \quad t=\widetilde{g}_0(\tau) \quad (0\leq\tau\leq 2\pi; \;\; \widetilde{g}_i(0)=\widetilde{g}_i(2\pi), \;\; i=0,1)$$

are called ε-close to each other if the conditions

$$|g_i(\tau)-\widetilde{g}_i(\tau)|\leq\varepsilon, \;\; \|g_i'(\tau)-\widetilde{g}_i'(\tau)\|_{C_\alpha}\leq\varepsilon \;\; (i=0,1) \tag{1'}$$

are fulfilled.

As has already been noted, G_ε denotes a set of domains ε-close to G for any $0<\varepsilon<d_0/2$.

Let us map conformally (under the assumptions of [5]) the close domains G and $\widetilde{G}$ onto the canonical domains $K(\rho;1)$ and $\widetilde{K}(\widetilde{\rho};1)$, where $K(\rho;1)$ and $\widetilde{K}(\widetilde{\rho};1)$ are respectively annuli with $\rho<|w|<1$ and $\widetilde{\rho}<|w|<1$, while the radii ρ and $\widetilde{\rho}$ are defined by the formulas

$$\ln\rho=\frac{1}{\pi}\int_0^{2\pi}\nu_1(\tau)\,|g_1'(\tau)|d\tau,$$

$$\ln\widetilde{\rho}=\frac{1}{\pi}\int_0^{2\pi}\widetilde{\nu}_1(\tau)\,|\widetilde{g}_1'(\tau)|d\tau,$$

where

$$\nu(t)=\begin{cases}\nu_1(t), & \text{when } \; t\in\Gamma_1,\\ \nu_0(t), & \text{when } \; t\in\Gamma_0.\end{cases}$$

Apply an analogous treatment to $\widetilde{\nu}(t)$ too. Note that $\nu(t)$ and $\widetilde{\nu}(t)$ are solutions of integral equations of form (3_1) and (4_1), respectively, derived for the doubly-connected domains G and $\widetilde{G}$.

Using the methods from Section 1 one can similarly obtain an estimate for the norm $\|\nu-\widetilde{\nu}\|_{C_{\alpha-\beta}}$ of difference of solutions of integral equations. Namely, we have

Theorem 3. *If the boundaries of doubly-connected domains G and $\widetilde{G}\in G_\varepsilon$, $0<\varepsilon\leq\varepsilon_0$, belong to the class C_α' $(\frac{1}{2}<\alpha<1)$, then the inequalities*

$$\|\nu(\tau)-\widetilde{\nu}(\tau)\|_{C\alpha-\beta}<B_0'(\beta)\cdot\|\nu\|_{C\alpha-\beta}\cdot\varepsilon,$$
$$|\ln\rho-\ln\widetilde{\rho}|<Q_0\cdot\varepsilon$$

are valid, where $0<\beta<\alpha$ and the constants $B_0'(\beta)$, Q_0 and ε_0 are defined by giving the initial domain G.

These estimates allow us to construct, with the aid of the function $\nu(t)$ defined by giving the initial domain G, an approximation to the function $w=\widetilde{f}_{\widetilde{\nu}}(z)$ $(\widetilde{f}_{\widetilde{\nu}}(\widetilde{z}_1)=1, \; \widetilde{z}_1>0, \; \widetilde{z}_1\in\widetilde{\Gamma}_0)$ which maps conformally an arbitrary doubly-connected domain $\widetilde{G}\in G_\varepsilon$ $(0<\varepsilon\leq\varepsilon_0)$ onto the canonical domain $K(\widetilde{\rho};1)$.

By virtue of inequalities (25) and (26), we can regard the function (see [3])

$$w = \widetilde{f}_\nu(z) = z \cdot \exp\left(\frac{1}{\pi i}\int\limits_{\widetilde{\Gamma}} \frac{\nu(t)dt}{t-z} + ic\right) \tag{27}$$

($\widetilde{f}_\nu(z_1) = 1,\ z_1 > 0,\ z_1 \in \widetilde{\Gamma}_0$), where $\nu(t)$ is a solution of equation (3_1), as an approximation to the function $w = \widetilde{f}_{\widetilde{\nu}}(z)$. By the proof of the Plemelj–Privalov theorem [2] (§18) the use of (25) and (26) leads to

Theorem 4. *The function* $w = \widetilde{f}_\nu(z)$ ($\widetilde{f}_\nu(z_1) = 1,\ z_1 > 0,\ z_1 \in \widetilde{\Gamma}_0$) *given in the doubly-connected domain* $\widetilde{G}$ *by formula* (27), *where* $\widetilde{G} \in G_\varepsilon$ ($0 < \varepsilon \le \varepsilon_1^*$), *admits, in* $\widetilde{G}$, *an estimate*

$$|\widetilde{f}_\nu(z) - \widetilde{f}_{\widetilde{\nu}}(z)| < P \cdot \varepsilon,$$

where P *depends only on the domain* G.

Note that it is assumed here that the given boundary points z_1 and $\widetilde{z}_1$ correspond to one and the same value of the parameter τ (say, $\tau = 0$).

3. To the Quasiconformal Mapping of Close Doubly-Connected Domains

Let us consider the problem of quasiconformal mapping of close domains. As a construction tool we take the method of integral equations [5], which stipulates the knowledge of the concrete global homeomorhism of the Beltrami equation

$$W_{\bar z} = q(z) \cdot W_z, \tag{28}$$
$$|q(z)| \le q_0 < 1,$$

constructed by I. N. Vekua's scheme [6]. This homeomorphism figures in the kernels of integral equations whose solutions are used to construct the wanted functions.

Assume that the coefficient $q(z)$ of the Beltrami equation is given in some doubly-connected domain G_0 containing the initial domain G and all domains $\widetilde{G} \in G_\varepsilon$ which are ε-close to G (in the sense of ($1'$)). As G_0 we can take, for instance, an annulus $\frac{\rho_0}{2} < |z| < R_0$, where $\rho_0 = \rho(0;\Gamma_1)$, $R_0 = \max\limits_{t_0 \in \Gamma_0} \rho(0;t_0)$.

Assume further that the boundary of the domain G belongs to the class C'_α ($0 < \alpha < 1$), and $q(z) \in C'_\gamma(\bar G_0)$, ($0 < \gamma < 1$), and the so-called Vekua basic homeomorphism $\widetilde{W}(z)$ of equation (28) is constructed with the coefficient

$$\widetilde{q}(z) = \begin{cases} q(z), & \text{when } z \in \bar G_0, \\ 0, & \text{when } z \text{ lies outside } G_0. \end{cases}$$

In that case $\widetilde{q}(z)$ belongs to any Lebesgue class $L_p(E)$ (where E is the entire complex plane) and, according to [7], $\widetilde{W}_{\bar z}$ and $\widetilde{W}_z$ satisfy the Hölder condition with index γ_0 and $0 < \gamma_0 < \min\{\alpha;\gamma\}$. In what follows this global homeomorphism of equation (28) is denoted by $\widetilde{W}_{G_0}(z)$.

We use the technique of exit in the plane of this homeomorphism. Then the integral equations (3_1) and (4_1) take the form

$$\mu(\xi_0)+\frac{1}{2\pi i}\int\limits_{\widetilde{W}_{G_0}(\Gamma)}\frac{1}{\xi'}K_*(\xi;\xi_0)\mu(\xi)d\xi$$
$$=-\ln|\xi_0-\widetilde{W}_{G_0}(0)|,\quad \xi_0\in\widetilde{W}_{G_0}(\Gamma), \tag{29}$$
$$\widetilde{\mu}(\xi_0)+\frac{1}{2\pi i}\int\limits_{\widetilde{W}_{G_0}(\widetilde{\Gamma})}\frac{1}{\xi'}\widetilde{K}_*(\xi;\xi_0)\widetilde{\mu}(\xi)d\xi$$
$$=-\ln|\xi_0-\widetilde{W}_{G_0}(0)|,\quad \xi_0\in\widetilde{W}_{G_0}(\widetilde{\Gamma}), \tag{30}$$

where $\xi=\widetilde{W}_{G_0}[g_k(\tau)]$, $\xi=\widetilde{W}_{G_0}[\widetilde{g}_k(\tau)]$ $(k=0,1)$ are the parametric equations of the curves $\widetilde{W}_{G_0}(\Gamma_k)$, $\widetilde{W}_{G_0}(\widetilde{\Gamma}_k)$, respectively. $K_*(\xi;\xi_0)=K_*[\widetilde{W}_{G_0}[g_k(\tau);g_l(\tau_0)]]$ $(k,l=0,1)$. In an analogous manner we define $\widetilde{K}_*(\xi;\xi_0)$. Moreover, $[\widetilde{W}_{G_0}(z)]_{\bar z}$ $[\widetilde{W}_{G_0}(z)]_z\in C_{\gamma_0}(\bar G_0)$ [7] and $\mu[\xi(\tau)]$, $\widetilde{\mu}[\xi(\tau)]\in C_{\gamma_0}[0;2\pi]$. We can also write that $\mu(\xi_0)=\nu[\widetilde{W}_{G_0}^{-1}(g(\tau_0))]$.

By virtue of ($1'$) we have

$$|\widetilde{W}_{G_0}[g_j(\tau)]-\widetilde{W}_{G_0}[\widetilde{g}_j(\tau)]|\le C_j(\widetilde{W}_{G_0};G_0)|g_j(\tau)-\widetilde{g}_j(\tau)|$$
$$<C_j(\widetilde{W}_{G_2};G_0)\cdot\varepsilon\ \ (j=0,1). \tag{31}$$

For convenience, denote $\widetilde{W}_{G_0}(z)=\widetilde{W}(z)$.

We have

$$(\widetilde{W}[g_j(\tau)])'_\tau=\widetilde{W}_t[g(\tau)](g'_j(\tau))_\tau+\widetilde{W}_{\bar t}[g_j(\tau)](\bar g_j(\tau))'_\tau,$$
$$(\widetilde{W}[\widetilde{g}_j(\tau)])'_\tau=\widetilde{W}_t[\widetilde{g}(\tau)](\widetilde{g}'_j(\tau))_\tau+\widetilde{W}_{\bar t}[\widetilde{g}_j(\tau)](\bar{\widetilde{g}}_j(\tau))'_\tau$$

and

$$(\widetilde{W}[g_j])'_\tau\in C_{\gamma_0}[0,2\pi],\ \ (\widetilde{W}[\widetilde{g}_j(\tau)])'_\tau\in C_{\gamma_0}[0;2\pi];\ \ (j=0,1).$$

Compose the difference

$$(\widetilde{W}[g_j(\tau)])'_\tau-(\widetilde{W}[\widetilde{g}_j(\tau)])'_\tau=\widetilde{W}_t(g_j)(g'_j)_\tau-\widetilde{W}_t(g_j)(\widetilde{g}'_j)_\tau-\widetilde{W}_t(\widetilde{g}_j)(\widetilde{g}_j)'_\tau$$
$$+\widetilde{W}_t(g_j)(\widetilde{g}'_j)_\tau+\widetilde{W}_{\bar t}(g_j)(\widetilde{g}_j)'_\tau-\widetilde{W}_{\bar t}(g_j)(\bar{\widetilde{g}}_j)'_\tau$$
$$-\widetilde{W}_{\bar t}(\widetilde{g}_j)(\bar{\widetilde{g}}_j)'_\tau+\widetilde{W}_{\bar t}(g_j)(\bar{\widetilde{g}}_j)'_\tau\quad(j=0,1).$$

Using ($1'$) in these expressions and taking into account the inequalities

$$\|g'_j(\tau)-\widetilde{g}'(\tau)\|_{C_{\gamma_0/2}}\le \mathrm{const}_{j1}\cdot\|g'_j(\tau)-\widetilde{g}_j{}'(\tau)\|_{C_\alpha},$$
$$\|\bar g'_j(\tau)-\bar{\widetilde{g}}_j{}'(\tau)\|_{C_{\gamma_0/2}}\le \mathrm{const}_{j2}\cdot\|\bar g'_j(\tau)-\bar{\widetilde{g}}_j{}'(\tau)\|_{C_\alpha},$$

we obtain

$$\|\widetilde{W}_t(g_j)-\widetilde{W}_t(\widetilde{g}_j)\|_{C_{\gamma_0/2}}\le C^*_{2j}(\widetilde{W}_{G_0};G_0)|g_j(\tau)-\widetilde{g}_j(\tau)|^{\gamma_0/2}$$
$$\le C^*_{3j}\,\varepsilon^{\gamma_0/2}\quad(j=0,1). \tag{32}$$

We see that the closeness conditions $(1')$ of the contours Γ_j and $\widetilde{\Gamma}_j$ in the plane of the homeomorphism $\widetilde{W}_{G_0}(z)$ for the corresponding curves $\widetilde{W}_{G_0}(\Gamma_j)$ and $\widetilde{W}_{G_0}(\widetilde{\Gamma}_j)$ are replaced by conditions (31) and (32).

Now, by virtue of the estimate established in Theorem 3, we come to the validity of the following proposition.

Theorem 5. *If the doubly-connected domains $G \subset G_0$ and $\widetilde{G} \subset G_0$ whose boundaries belong to the class C'_α $(0 < \alpha < 1)$ are ε-close to each other in the sense of $(1')$, then the estimate*

$$\|\mu(\xi) - \widetilde{\mu}(\xi)\|_{C_{\frac{\gamma_0}{2}-\beta}} < Q^*(\widetilde{W}_{G_0}; G; \beta) \cdot \|\mu\|_{C_{\frac{\gamma_0}{2}-\beta}} \cdot \varepsilon^{\frac{\gamma_0}{2}} \tag{33}$$

holds for all $\varepsilon \in [0; \varepsilon^*]$; here $\mu(\xi)$ and $\widetilde{\mu}(\xi)$ are unique solutions of the integral equations (29) *and* (30)*, respectively, β is any positive number smaller than $\gamma_0/2$, the constant Q^* and small ε^* are completely defined by giving the initial domain G and the homeomorphism $\widetilde{W}_{G_0}(z)$.*

Note that in Theorem 5 the order of smallness for ε can be obtained arbitrarily close to γ_0 [3], for instance, $O(\varepsilon^{\gamma_0-\eta})$, where $\eta < \gamma_0$ is any positive number, but in that case we can estimate only "small" norms of the value $\|\mu(\xi) - \widetilde{\mu}(\xi)\|_{C_\eta}$ (the Hölder index η decreases). In the considered situation the choice $\eta = \frac{\gamma_0}{2}$ seems optimal to us.

In conclusion, also note that, analogously to the conformal case, the estimate in terms of ε for the difference of modules $\ln \rho(\widetilde{q}) - \ln \widetilde{\rho}(\widetilde{q})$ calculated by [8] can be established with the aid of estimate (33).

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