GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 5, 1998, 415-424

# ON A CLASS OF COVARIANCE OPERATORS

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Abstract. This paper is the continuation of [1] in which complex symmetries of distributions and their covariance operators are investigated. Here we also study the most general quaternion symmetries of random vectors. Complete classification theorems on these symmetries are proved in terms of covariance operator spectra.

Let $\xi = (\xi', \xi'')$ be a random vector with components from the separable real Hilbert space $H_{\mathbb{R}}$ with a scalar product $(\cdot|\cdot)_{\mathbb{R}}$. Thus $\xi$ takes values from the direct sum $H_0 = H_{\mathbb{R}} \oplus H_{\mathbb{R}}$ whose scalar product has the form

$$(h|g)_0 = (h'|g')_{\mathbb{R}} + (h''|g'')_{\mathbb{R}} \ \ \forall\, h = (h'|h''),\ g = (g', g'') \in H_0.$$

Each element of the space $L_0 = L(H_0)$ of linear bounded operators on $H_0$ can be represented as a block matrix

$$A = (a_{ij})_{i,j=1}^2 = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix}, \quad a_{ij} \in L(H_{\mathbb{R}}), \quad i, j = 1, 2. \tag{1}$$

Let us consider the operator $U(x', x'') = (-x'', x')\ \forall (x', x'') \in H_0$ whose representation (1) can be written as

$$U = \begin{pmatrix} 0, & -I \\ I, & 0 \end{pmatrix}, \tag{2}$$

where $I$ is the unit operator from $L(H_{\mathbb{R}})$. The operator $U$ possesses the important property $U^*U = UU^* = I$, implying that $U$ is orthogonal and

$$U^2 = -I. \tag{3}$$

**Definition 1.** A random vector $\xi$ is called symmetric if $(\xi', \xi'')$ and $(-\xi'', \xi')$ are equally distributed, i.e., if $\xi$ and $U\xi$ are such.

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1991 *Mathematics Subject Classification.* 60B11, 60B05, 60A10.

*Key words and phrases.* Random vectors with values from a Hilbert space, covariance operator, proper covariances and distributions, Gaussian distributions, complex and quaternion Hilbert spaces.



1072-947X/98/0900-0415$15.00/0 © 1998 Plenum Publishing Corporation

Clearly, if $\xi$ is symmetric, then $\xi$, $-\xi$, $U\xi$, $-U\xi$ are equally distributed.

**Proposition 1.** *Let $\xi_1$, $\xi_2$, $\xi_2$, $\xi_4$ be the independent copies of a random vector $\xi$. Then a random vector*

$$\xi_a = \frac{1}{2}[\xi_1 - \xi_2 + U\xi_3 - U\xi_4]$$

*is symmetric.*

The proof immediately follows from (3).

**Definition 2.** $\xi_a$ is called an additive symmetrization of a random vector $\xi$.

**Proposition 2.** *Let $\mathfrak{M}$ be a random operator taking the values $I$, $-I$, $U$, $-U$, each with probability $\frac{1}{4}$. Then a random vector*

$$\xi_m = \mathfrak{M}\xi$$

*is symmetric if $\xi$ and $\mathfrak{M}$ are independent.*

*Proof.* It follows from (3) that the operator $U\mathfrak{M}$ has the same distribution as $\mathfrak{M}$, and does not depend on $\xi$, so that $\xi_m$ and $U\xi_m$ are equally distributed. $\square$

**Definition 3.** $\xi_m$ is called a multiplicative symmetrization.

**Proposition 3.** *Let a symmetric random vector $\xi$ have a strong moment of second order. Then it is centered and its covariance $K_0$ satisfies the relation*

$$UK_0 = K_0U. \tag{4}$$

*Proof.* Since $\xi$ and $-\xi$ are equally distributed, $\xi$ is centered. Since $\xi$ and $U\xi$ are equally distributed, we have $K_0 = U^*K_0U$. But in that case (3) implies (4).

Denote by $L_1$ the commutant of the operator $U$, i.e., $L_1 = \{A|A \in L_0, AU = UA\}$. Then (4) implies that $K_0 \in L_1$. $\square$

**Proposition 4.** *The operator $A$ belongs to $L_1$ if and only if its matrix representation has the form*

$$A = \begin{pmatrix} a, & -b \\ b, & a \end{pmatrix}, \quad a, b \in L(H_{\mathbb{R}}). \tag{5}$$

The proof follows immediately.

In [1], operators of form (5) are called proper and, accordingly, the centered random vector $\xi$ possessing the covariance $K_0$ is called proper for $K_0 \in L_1$, i.e., for

$$K_0 = \begin{pmatrix} R & -T \\ T & R \end{pmatrix}. \tag{6}$$

Generally speaking, if $\xi = (\xi', \xi'')$ is the centered random vector with covariance $K_0$, then $K_0$ has the form

$$K_0 = \begin{pmatrix} R_1 & T^* \\ T & R_2 \end{pmatrix}, \tag{7}$$

where $R_1$ and $R_2$ are the covariances of $\xi'$ and $\xi''$, respectively, and $T$ is their mutual covariance. Thus the properity of $\xi$ implies that

$$R_1 = R_2 = R, \quad T^* = -T. \tag{8}$$

**Definition 4.** For $A \in L_0$ we call the expression

$$\overline{A} = \frac{1}{2}[A + U^* A U]$$

the averaging of the operator $A$.

It is easy to verify that the block representation of the averaging $\overline{K_0}$ of the operator $K_0$ from (7) is written as

$$\overline{K_0} = \frac{1}{2}\begin{pmatrix} R_1 + R_2 & T^* - T \\ T - T^* & R_1 + R_2 \end{pmatrix}.$$

**Proposition 5.** *The operator $A \in L_0$ is proper, i.e., belongs to $L_1$ if and only if $\overline{A} = A$.*

The proof is obvious.

Let $\xi$ be a centered random vector with covariance $K_0$. Denote by $K_a$ and $K_m$ the covariances of additive and multiplicative symmetrizations of $\xi$.

**Proposition 6.** *The equalities*

$$K_a = K_m = \overline{K_0}$$

*are valid.*

*Proof.* The equality $K_a = \overline{K_0}$ immediately follows from Proposition 1. Indeed, since $\xi_a = \frac{1}{2}(\xi_1 - \xi_2 + U\xi_3 - U\xi_4)$, we have

$$K_a = \frac{1}{4}(K_0 + K_0 + U^*K_0U + U^*K_0U) = \frac{1}{2}(K_0 + U^*KU) = \overline{K_0}.$$

Let us prove the equality $K_m = \overline{K_0}$. We have

$$\begin{aligned}(K_m h|h)_0 &= E(\xi_m|h)_0^2 = E(\mathfrak{M}\xi|h)_0^2 = E(\xi|\mathfrak{M}^*h)_0^2 = E_{\mathfrak{M}}E_\xi(\xi|\mathfrak{M}^*h)_0^2 = \\ &= E_m(K_0\mathfrak{M}^*h|\mathfrak{M}^*h)_0 = \frac{1}{4}((K_0h|h)_0 + (K_0(-h)|(-h))_0 + \\ &+ (K_0U^*h|U^*h)_0 + (K_0(-U^*h)|(-U^*h))_0) = \\ &= \frac{1}{2}((K_0h|h)_0 + (UK_0U^*h|h)) = \frac{1}{2}((K_0 + U^*K_0U)h|h)_0.\end{aligned}$$

The latter equality follows from (3), which gives $U^* = -U$. □

As shown above (Proposition 4), a symmetric random vector $\xi$ with covariance is proper. The converse statement, generally speaking, does not hold, but we have

**Proposition 7.** *A random proper Gaussian vector is symmetric.*

*Proof.* For the characteristic functional of a proper Gaussian vector we have

$$\begin{aligned}\varphi(Ut) &= e^{-\frac{1}{2}(K_0Ut|Ut)_0} = e^{-\frac{1}{2}(U^*K_0Ut|t)_0} = \\ &= e^{-\frac{1}{2}(U^*UK_0t|t)_0} = e^{-\frac{1}{2}(K_0t|t)_0} = \varphi(t),\end{aligned}$$

from which it follows that $\xi$ and $U\xi$ are equally distributed, i.e., $\xi$ is symmetric. □

Thus if $\xi$ is a proper Gaussian vector, then $\xi$ and $\xi_a$ are equally distributed. Moreover, we have

**Theorem 1.** *The distributions of $\xi$ and $\xi_a$ coincide if and only if $\xi$ is a random Gaussian vector.*

This theorem is a corollary of a more general theorem proved by N. Vakhania in [2].

The averaging $\overline{K_0}$ of the covariance operator $K_0$ naturally appears as the covariance operator of $\xi$ from some complex Hilbert space $H_1$. Indeed, let us define the multiplication of vectors $x \in H_0$ by scalars $\lambda \in \mathbb{C}$ as

$$\lambda x = (\lambda' + i\lambda'')x = (\lambda' I + \lambda'' U)x \quad \forall\, x \in H_0, \lambda \in \mathbb{C} \tag{9}$$

and its Hermitian form as

$$(x|y)_1 = (x|y)_0 + i(x|Uy)_0 \quad \forall\, x, y \in H_0. \tag{10}$$

As a result, the real space $H_0$ becomes the complex Hilbert space $H_1$ with the Hermitian form $(\cdot|\cdot)_1$ and algebra of $\mathbb{C}$-linear operators $L_1$. Hence our random vector $\xi$ becomes the complex random vector $\xi = \xi' + i\xi''$, which by virtue of (9) implies that $\xi = (\xi', 0) + U(\xi'', 0)$, and its covariance $K_1$ with respect to the Hermitian form $(\cdot|\cdot)_1$ is defined by the relation

$$(K_1 h|h)_1 = \frac{1}{2} E|(\xi|h)_1|^2. \tag{11}$$

**Proposition 8.** *The equality*

$$K_1 = \overline{K_0}$$

*is valid.*

*Proof.*

$$(K_1 h|h)_1 = \frac{1}{2} E|(\xi|h)_1|^2 = \frac{1}{2} E\big((\xi|h)_0^2 + (\xi|Uh)_0^2\big) = \frac{1}{2}(K_0 h|h)_0 + \\ + \frac{1}{2}(U^* K_0 U h|h)_0 = \Big(\frac{1}{2}(K_0 + U^* K_0 U)h|h\Big)_0 = (\overline{K_0} h|h)_0.$$

Since by (11) $(K_1 h|h)_1$ is real, by (10) $(K_1 h|h)_1 = (K_1 h|h)_0$ and therefore $K_1 = \overline{K_0}$. $\square$

This important equality is thoroughly investigated in [1].

The above reasoning can be summarized as

**Theorem 2.** *The following conditions are equivalent:* (*a*) $\xi$ *is a proper random vector* (*see* (6) *and* (8))*;* (*b*) $K_0 = \overline{K_0}$*;* (*c*) $K_0 = K_a$*;* (*d*) $K_0 = K_m$*;* (*e*) $K_0 = K_1$*.*

Let $\xi$ be centered and have the covariance $K_0$. Recall that $K_0$ is a completely continuous, positive-definite operator. Denote by $M_i$ $(i = 1, 2, \dots)$ the eigensubspaces of $K_0$ which correspond to the eigenvalues $\mu_i$, $(i = 1, 2, \dots)$, $\mu_1 > \mu_2 > \cdots$; $\mu_i \to 0$. If the operator $K_0$ is degenerate, denote $M_0 = \operatorname{Ker} K_0$. Let us now describe proper covariance operators in terms of their eigenvalues.

**Definition 5.** A random vector $\xi$ is called orthogonally proper if there exists an orthogonal operator $O$ such that $O\xi$ is a proper random vector.

**Theorem 3.** *A random vector* $\xi$ *is orthogonally proper if and only if* $\dim M_k$ *is even for all* $k \geq 0$*.*

*Proof. Sufficiency.* Let $\dim M_k$ be even for all $k \geq 0$. Then there exists a basis of the space $H_0$ consisting of eigenvectors of the operator $K_0$ : $f_1, f_2, \dots$ such that for $K_0 f_{2p-1} = \mu f_{2p-1}$ we have $K_0 f_{2p} = \mu f_{2p}$, $p = 1, 2, \dots$. Let further $g_p$ $(p = 1, 2, \dots)$ be a basis of $H_{\mathbb{R}}$. Then a sequence of vectors $e_1, e_2, \dots$ from $H_0$, for which $e_{2p-1} = (g_p, 0)$, $e_{2p} = (0, g_p)$, $p = 1, 2, \dots$, is the orthonormal basis $H_0$. By (2) $Ue_{2p-1} = e_{2p}$, $Ue_{2p} = -e_{2p-1}$ $(p = 1, 2, \dots)$. Let us now define the orthogonal operator $O$ as $Oe_k = f_k$ $(k = 1, 2, \dots)$. The covariance $O\xi$ has the form $\Gamma = O^{-1}K_0O$. Next we shall prove that $\Gamma U = U\Gamma$. Indeed, we have

$$\begin{aligned} U\Gamma e_{2p-1} &= UO^{-1}K_0Oe_{2p-1} = UO^{-1}K_0f_{2p-1} = UO^{-1}\mu_k f_{2p-1} = \\ &= \mu_k UO^{-1}f_{2p-1} = \mu_k e_{2p-1} = \mu_k e_{2p}; \\ \Gamma U e_{2p-1} &= O^{-1}K_0e_{2p} = O^{-1}K_0f_{2p} = \mu_k O^{-1}f_{2p} = \mu_k e_{2p}, \end{aligned}$$

$p = 1, 2, \dots$. The same reasoning is true for $e_{2p}$ $(p = 1, 2, \dots)$.

*Necessity.* Let $O$ be an orthogonal operator such that $O\xi$ is proper. Then $M'_k = O^{-1}M_k$ $(k \geq 0)$ are the eigensubspaces of $\Gamma$ and $\mu_k$ are the corresponding eigenvalues. Indeed, if $x \in M_k$, then $\Gamma x = O^{-1}K_0Ox$. Since $Ox \in M_k$, we have $\Gamma x = O^{-1}\mu_k Ox = \mu_k x$. Conversely, assume that $\Gamma x = \mu_k x$. Then, on the one hand, $\mu_k x = \Gamma x = O^{-1}(K_0Ox)$, and, on the other, $O^{-1}(\mu_k Ox) = \mu_k x$. Hence it follows that $K_0Ox = \mu_k Ox$ and therefore $Ox \in M_k$, i.e., $x \in M'_k$. Since $\Gamma$ is a proper operator, we have $\Gamma U = U\Gamma$. Thus we conclude that $M'_k$ are invariant for $U$ as well. Indeed, assuming that $x \in M'_k$, we obtain $\Gamma Ux = U\Gamma x = U\mu_k x = \mu_k Ux$. But this implies that, together with $x$, $Ux$ also belongs to $M'_k$. But this is possible only when $\dim M'_k$ is even. Therefore $\dim M_k$ will be even too. (Throughout the paper it is assumed that "infinity" may have any multiplicity). $\square$

*Remark* 1. The orthogonal property is understood as a property with respect to $U_1 = OUO^{-1}$ in the sense that $U_1$ is orthogonal, $U_1^2 = -I$, and $U_1K_0 = K_0U_1$.

Since $K_0$ is the self-conjugate operator, $H_0$ can be represented as an orthogonal sum $H_0 = \overline{\operatorname{im} K_0} + \operatorname{Ker} K_0$. $e_1, e_2, \dots$ is the orthonormal basis of the subspace $\overline{\operatorname{im} K_0}$ consisting of the eigenvectors of $K_0$ corresponding to the values $\lambda_1 \geq \lambda_2 \geq \cdots$.

**Theorem 4.** *A self-conjugate invertible operator $S$, permutable with $K_0$ and such that a random vector $\eta = S\xi$ is orthogonally proper, exists if and only if* $\inf_{p\geq 1} \frac{\lambda_{2p}}{\lambda_{2p-1}} = c_0 > 0$ *and* $\operatorname{Ker} K_0$ *is even.*

*Proof. Sufficiency.* We write $S$ as $S_{e_k} = \sigma_k e_k$, $S_x = x$ $\forall x \in \operatorname{Ker} K_0$, where $\sigma_k$ are defined by the equalities $\sigma_{2p} = 1$, $\sigma_{2p-1} = (\lambda_{2p}/\lambda_{2p-1})^{1/2}$ $p = 1, 2, \dots$. It is obvious that $S$ is permutable with $K_0$ and self-conjugate. We shall prove that it is invertible. Note that $\sqrt{c_0} \leq \sigma_k \leq 1$ for all $x$.

The right-hand side of the inequality follows from the definition of $\sigma_k$ and from $\lambda_{2p} \leq \lambda_{2p-1}$ $(p = 1, 2, \dots)$, while the left-hand side is provided by the condition of the theorem $\inf_{p\geq 1} \frac{\lambda_{2p}}{\lambda 2p-1} = c_0 > 0$. Therefore $S$ is invertible. Finally, let us prove that the random vector $\eta = S\xi$ is orthogonally proper. Since $S$ is self-conjugate and permutable with $K_0$, the covariance $\eta$ is $S^*K_0S$, we obtain $S^*K_0S = S^*SK_0 = S^2K_0$. Thus the covariance $\eta$ in the basis $e_1, e_2, \dots$ can be represented in the diagonal form $\sigma_1^2\lambda_1, \sigma_2^2\lambda_2, \dots$. But $\sigma_{2p}^2\lambda_{2p} = \lambda_{2p}$, $\sigma_{2p-1}^2\lambda_{2p-1} = (\lambda_{2p}/\lambda_{2p-1})\lambda_{2p-1} = \lambda_{2p}$, $p = 1, 2, \dots$. Therefore all eigensubspaces $S^*K_0S$ (including $\operatorname{Ker} K_0$) are even, i.e., $\eta = S\xi$ is orthogonally proper.

*Necessity.* Let $S$ satisfy the conditions of the theorem: $S$ is invertible, self-conjugate, permutable with $K_0$, and the dimensions of the eigensubspaces $S^*K_0S$ are even. Hence it immediately follows that $\operatorname{Ker} K_0 = \operatorname{Ker} S^*K_0S$ and $\operatorname{im} S^*K_0S$. Clearly, $S_{e_k} = \sigma_k e_k$, $k = 1, 2, \dots$, $\alpha_0 \leq \sigma_k \leq \alpha_1$, for all $k$ and $\sigma_k^2\lambda_k = \beta_k$ are such that $\beta_{2p-1} = \beta_{2p}$ $p = 1, 2, \dots$. Further, $\lambda_{2p}/\lambda_{2p-1} = (\sigma_{2p-1}/\sigma_{2p})^2\beta_{2p}/\beta_{2p-1} \geq (\frac{\alpha_0}{\alpha_1})^2 > 0$. $\square$

We shall now give an example of a random vector which does not belong to the above-mentioned class. Let $\xi$ be a random vector with covariance $K_0$ whose eigenvalues $\lambda_k = e^{-k^\alpha}$ $(\alpha > 1)$ $(k = 1, 2, \dots)$. It is assumed that the conditions of Theorem 4 are fulfilled for $\alpha = 1$ and for all orders less than $e^{-k}$.

**Proposition 9.** *The operators from $L_0 = L(H_0)$*

$$U = \begin{pmatrix} 0, & -I \\ I, & 0 \end{pmatrix}, \quad V = \begin{pmatrix} v, & 0 \\ 0, & -v \end{pmatrix}, \tag{12}$$

*where $v$ is an orthogonal operator in $H_{\mathbb{R}}$ such that $v^2 = -I$ (for the existence of such an operator see* [3], [4]*), give, in $H_0$, an additional structure (see* [5]*, Ch. 11, §2), i.e., $U$ and $V$ are orthogonal in $H_0$ and*

$$U^2 = V^2 = -I, \quad UV = -VU. \tag{13}$$

*Proof.* Follows immediately. $\square$

**Theorem 5.** *The covariance $K_0$ of a proper random vector (see* (6), (8)*) is permutable with $V$ if and only if*

$$Rv = vR, \quad T_v = -vT.$$

*Proof.* Follows immediately. $\square$

**Definition 6.** A proper random vector $\xi$ is called twice proper if its covariance operator $K_0$ is permutable with $V$.

The operators defined by formulas (12), (13) allow us to introduce, in the space $H_0$, the so-called Hermitian form defining one more covariance $K$ for $\xi$. We shall see below that for twice proper random vectors $K$ and $K_0$ coincide.

Consider the set

$$\mathbb{H} = \big\{A|A \in L_0 A = aI + bU + cV + dW,\ a,b,c,d \in \mathbb{R}\big\},$$

where $W = UV$. Note that $W$ is an orthogonal operator such that $WU = -UW$, $WV = -VW$. It is easy to verify that $\mathbb{H}$ is a four-dimensional $\mathbb{R}$-algebra with involution with respect to standard operations $A+B$, $AB$, $\lambda A$, $A^*$. Moreover, $\mathbb{H}$ is a body whose every nonzero element is invertible. Indeed, for any $A = aI + bU + cV + dW$ we have $AA^* = (a^2+b^2+c^2+d^2)I$, which means that if $a \neq 0$, then $A^{-1}$ exists and $A^{-1} = (a^2+b^2+c^2+d^2)^{-1}A^*$. The set $\mathbb{H}$ acts on $H_0$ in a natural way: if $A \in \mathbb{H}$, $x \in H_0$, then its action is simply $Ax$. Finally, the mapping $(\cdot|\cdot) : H_0 \times H_0 \to \mathbb{H}$ defined by the formula

$$\begin{aligned}\forall (x,y) \in H_0 \times H_0\ (x|y) = (x|y)_0 I + (x|Uy)_0 U + \\ +(x|Vy)_0 V + (x|Wy)_0 W\end{aligned} \tag{14}$$

gives the Hermitian form. Indeed, it is easy to verify that

$$(x+y|z) = (x|z) + (y|z)\ \forall\ \ x,y \in H_0,$$
$$(Ax|y) = A(x|y)\ \forall A \in \mathbb{H},\ x,y \in H_0;\ (x|y)^* = (y|x)\ \forall x,y \in H_0,$$

and, finally, $(x|x) \geq 0\ \forall x \in \mathbb{H}\ (x|x) = 0 \Leftrightarrow x = 0$, where $A \geq 0$ means that the operator $A$ is nonnegative. But since $|(|x)| = (x|x)_0$, the topologies generated by the Hermitian forms $\cdot|\cdot)$ and $\cdot|\cdot)_0$ in $H_0$ coincide.

For the centered random vector $\xi$ with covariance $K_0$ we define the covariance $K$ with respect to the Hermitian form (14) by the formula $(Kh|h) = \frac{1}{4}E|(\xi|h)|^\alpha$.

**Proposition 10.** *The covariance $K$ has the form*

$$K = \frac{1}{4}(K_0 + U^*K_0U + V^*K_0V + W^*K_0W). \tag{15}$$

*Proof.* (14) implies

$$\begin{aligned}K &= \frac{1}{4}\big[E(\xi|h)_0^2 + E(\xi|Uh)_0^2 + E(\xi|Vh)_0^2 + E(\xi|Wh)_0^2\big] = \\ &= \frac{1}{4}\big[(K_0h|h)_0 + (K_0Uh|Uh)_0 + (K_0Vh|Vh)_0 + (K_0Wh|Wh)_0\big] = \\ &= \frac{1}{4}\big((K_0 + U^*K_0U + V^*K_0Vh + W^*K_0W)h|h\big)_0.\end{aligned}$$

Since, by definition, $(Kh|h)$ coincides with $(Kh|h_0)_0$, we obtain (15). $\square$

**Proposition 11.** *If a random vector is twice proper, then*

$$K_0 = K.$$

So far we have been concerned with introducing new structures in the space $H_0$. A similar situation occurs in the case of the narrowing of a scalar field when a rich structure is given for the initial space and we are interested in the behavior of a random vector in different structures. The case $\mathbb{R} \subset \mathbb{C}$ is treated in [1]. Here we shall consider the restriction of scalars in the quaternion case. Recall that the quaternion body $\mathbb{H}$ is a four-dimensional $\mathbb{R}$-algebra with the basis $1, i, j, k$ and multiplication table

| | i | j | k |
|---|---|---|---|
| i | -1 | k | -j |
| j | -k | -1 | i |
| k | j | -i | -1 |

The conjugate quaternion $\overline{\mu}$ to $\mu = \mu' + i\mu'' + j\mu''' + k\mu^{IV}$ is defined as $\overline{\mu} = \mu' - i\mu'' - j\mu''' - k\mu^{IV}$. If $\lambda = \lambda' + i\lambda'' + j\lambda''' + k\lambda^{IV}$, then the expression $(\lambda|\mu) = \lambda\overline{\mu}$ defines the quaternion Hermitian form, while $(\lambda|\mu)_0 = \operatorname{Re}\lambda\overline{\mu}$ gives a Euclidean scalar product $\mathbb{H}$ over $\mathbb{R}^0 : (\lambda|\mu)_0 = \lambda'\mu' + \lambda''\mu'' + \lambda'''\mu''' + \lambda^{IV}\mu^{IV}$ with the norm $|\lambda\mu|_0 = |\lambda\mu| = |\lambda|_0|\mu|_0$, where, as usual, it is assumed that $|\cdot|_0 = (\cdot|\cdot)_0^{1/2}$. Note that $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. The quaternion Hilbert space $H$ is an Abelian group with respect to the sum $x + y$ of the vectors from $H$, where the product $\lambda x$ of scalars from $\mathbb{H}$ and of scalars from $H$ is defined. Moreover, this product is complete and separable with respect to the following Hermitian form with values in $H$ given on $\mathbb{H} : \forall x, y \in H$:

$$(x|y) = (x|y)_0 + (x|iy)_0 i + (x|jy)_0 j + (x|ky)_0 k.$$

Here

$$(\cdot|\cdot)_0 = \operatorname{Re}(\cdot|\cdot)$$

is the usual real scalar product. Thus, in the case of the restriction of the scalars from $\mathbb{H}$ to $\mathbb{R}$, the space $H$ transforms to the real Hilbert space $H_0$, while in the case of the restriction to $\mathbb{C}$, it transforms to the complex Hilbert space $H_1$ with the Hermitian form

$$\big(\,\cdot\,|\,\cdot\,\big) = \big(\,\cdot\,|\,\cdot\,\big)_0 + \big(\,\cdot\,|i(\cdot)\big)_0 i.$$

Since $|\cdot| = |\cdot|_1 = |\cdot|_0$, we find that $H^0$, $H_1$, $H_0$, being topological Abelian groups, completely coincide.

Denote by $L$, $L_1$, $L_0$ the spaces of bounded linear operators $\mathbb{H}$, $\mathbb{C}$, $\mathbb{R}$, respectively. The restriction of the scalars gives the natural embeddings

$L \subset L_1 \subset L_0$. Now we introduce in $H_0$ the operators $U$ and $V$

$$U = iI, \quad V = jI.$$

which, as is easy to verify, satisfy (13). Therefore all the above arguments hold for the random vector $\xi$ with values from $H_0$. Thus Theorem 3 will read as follows:

**Theorem 6.** *A random vector $\xi$ is twice orthogonally proper if and only if* $\dim M_k$ *is a multiple of four for all $k \geq 0$.*

## Acknowledgment

This study was financially supported by the Soros Foundation.

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(Received 01.08.1996)

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