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ON SINGULAR FUNCTIONAL DIFFERENTIAL INEQUALITIES

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ABSTRACT. Classical theorems on differential inequalities [1, 2, 3] are generalized for initial value problems of the kind

$$\frac{dx(t)}{dt} \le f(x)(t), \quad \lim_{t\to a} \Big\|[x(t)-c_0]_+\Big\|/h(t) = 0$$

and

$$\frac{dx(t)}{dt} \ge f(x)(t), \quad \lim_{t\to a} \Big\|[x(t)-c_0]_-\Big\|/h(t) = 0,$$

where $f : C([a,b];R^n) \to L_{loc}(]a,b];R^n)$ is a singular Volterra operator, $c_0 \in R^n$, $h : [a,b] \to [0,+\infty[$ is continuous and positive on $]a,b]$, $\|\cdot\|$ is a norm in R^n, and $[u]_+$ and $[u]_-$ are respectively the positive and the negative part of the vector $u \in R^n$.

§ 1. Statement of the Basic Results

1.1. Main Notation and Definitions. Throughout this paper use will be made of the following notation:

R is the set of real numbers; $R_+ = [0,+\infty[$;

R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ $(i = 1,\dots,n)$ and norm $\|x\| = \sum_{i=1}^n |x_i|$;

$R_+^n=\{x=(x_i)_{i=1}^n \in R^n : x_i \ge 0\,(i=1,\dots,n)\}$; $R_\rho^n=\{x\in R^n : \|x\|\le \rho\}$;

if $x = (x_i)_{i=1}^n$, then

$$\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^n; \quad [x]_+ = \Big(\frac{|x_i|+x_i}{2}\Big)_{i=1}^n, \quad [x]_- = \Big(\frac{|x_i|-x_i}{2}\Big)_{i=1}^n;$$

if x and $y \in R^n$, then $x \le y \Longleftrightarrow y - x \in R_+^n$;

$x \cdot y$ is the scalar product of vectors x and $y \in R^n$;

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$C([a,b];R^n)$ is the space of continuous vector functions $x:[a,b]\to R^n$ with the norm $\|x\|_C=\max\{\|x(t)\|:\ a\le t\le b\}$;

$$C_\rho([a,b];R^n)=\big\{x\in C([a,b];R^n):\ \|x\|_C\le\rho\big\};$$
$$C([a,b];R_+)=\big\{x\in C([a,b];R):\ x(t)\ge 0 \text{ for } a\le t\le b\big\};$$

if $x\in C([a,b];R^n)$ and $a\le s\le t\le b$, then

$$\nu(x)(s,t)=\max\big\{\|x(\xi)\|: s\le\xi\le t\big\};$$

$L_{loc}(]a,b];R^n)$ is the space of vector functions $x:]a,b]\to R^n$ summable on every segment contained in $]a,b]$ with the topology of mean convergence on every segment from $]a,b]$;

$$L_{loc}(]a,b];R_+)=\big\{x\in L_{loc}(]a,b];R):\ x(t)\ge 0 \text{ for almost all } t\in[a,b]\big\}.$$

Definition 1.1. An operator $f:C([a,b];R^n)\to L_{loc}(]a,b];R^n)$ is said to be *Volterra* if for every $t_0\in]a,b]$ and every x and $y\in C([a,b];R^n)$ satisfying $x(t)=y(t)$ for $a<t\le t_0$, the equality $f(x)(t)=f(y)(t)$ is fulfilled a.e. on $]a,t_0[$.

Definition 1.2. We say that an operator $f:C([a,b];R^n)\to L_{loc}(]a,b];R^n)$ satisfies the *local Carathéodory conditions* if it is continuous and there exists a function $\gamma:]a,b]\times R_+\to R_+$ nondecreasing in the second argument such that $\gamma(\cdot,\rho)\in L_{loc}(]a,b];R)$ for $\rho\in R_+$, and for any $x\in C([a,b];R^n)$ the inequality $\|f(x)(t)\|\le\gamma(t,\|x\|_C)$ is fulfilled a.e. on $]a,b[$.

If

$$\int_a^b\gamma(t,\rho)dt<+\infty\quad\text{for}\quad\rho\in R_+,$$

then the operator f is called *regular*. Otherwise f is called *singular*.

Definition 1.3. An operator $f:C([a,b];R^n)\to L_{loc}(]a,b];R^n)$ is said to be *nondecreasing* if for every x and $y\in C([a,b];R^n)$ satisfying $x(t)\le y(t)$ for $a\le t\le b$, the inequality $f(x)(t)\le f(y)(t)$ is fulfilled a.e. on $]a,b[$.

In the present paper we consider the following weighted initial value problems:

$$\frac{dx(t)}{dt}=f(x)(t), \tag{1.1}$$
$$\lim_{t\to a}\frac{\|x(t)-c_0\|}{h(t)}=0; \tag{1.2}$$
$$\frac{dz(t)}{dt}\le f(z)(t), \tag{1.3}$$

$$\lim_{t\to a}\frac{\|[z(t)-c_0]_+\|}{h(t)}=0; \tag{1.4}$$

$$\frac{dz(t)}{dt}\geq f(z)(t), \tag{1.5}$$

$$\lim_{t\to a}\frac{\|[z(t)-c_0]_-\|}{h(t)}=0, \tag{1.6}$$

where everywhere below $f : C([a,b];R^n)\to L_{loc}(]a,b];R^n)$ is assumed to be a Volterra, in general singular, operator satisfying the local Carathéodory conditions, $c_0\in R^n$, and $h : [a,b]\to[0,+\infty[$ is continuous, nondecreasing and positive on $]0,a]$.

We shall separately consider the case where $h(a)>0$, so that the conditions (1.2), (1.4) and (1.6) have respectively the form

$$x(a)=c_0, \tag{1.2$_1$}$$

$$z(a)\leq c_0, \tag{1.4$_1$}$$

$$z(a)\geq c_0. \tag{1.6$_1$}$$

The question of estimating solutions of functional differential inequalities (1.3) and (1.5) satisfying the initial conditions

$$z(a)<c_0, \tag{1.4$_2$}$$

$$z(a)>c_0 \tag{1.6$_2$}$$

will also be studied.

The vector differential equation and differential inequalities with delay:

$$\frac{dx(t)}{dt}=f_0\big(t,x(t),x(\tau_1(t)),\dots,x(\tau_m(t))\big), \tag{1.7}$$

$$\frac{dz(t)}{dt}\leq f_0\big(t,z(t),z(\tau_1(t)),\dots,z(\tau_m(t))\big), \tag{1.8}$$

$$\frac{dz(t)}{dt}\geq f_0\big(t,z(t),z(\tau_1(t)),\dots,z(\tau_m(t))\big) \tag{1.9}$$

are important particular cases of the vector functional differential equation (1.1) and of the vector functional differential inequalities (1.3) and (1.4).

In the sequel, when considering equation (1.7) or inequalities (1.8) and (1.9), it will be assumed that the vector function $f_0 :]a,b[\times R^{(m+1)n}\to R^n$ satisfies the local Caratéodory conditions, i.e., $f_0(t,\cdot,\dots,\cdot) : R^{(m+1)n}\to R^n$ is continuous for almost all $t\in]a,b[$, $f_0(\cdot,x_0,x_1,\dots,x_n) :]a,b[\times R^n$ is measurable for all $x_k\in R^n$ $(k=0,1,\dots,m)$, and on the set $]a,b[\times R^{(m+1)n}$ the following inequality is fulfilled:

$$\|f_0(t,x_0,x_1,\dots,x_m)\|\leq\gamma\Big(t,\sum_{k=0}^m\|x_k\|\Big),$$

where $\gamma :]a,b] \times R_+ \to R_+$ does not decrease in the second argument, and $\gamma(\cdot,\rho) \in L_{loc}(]a,b];R_+)$ for $\rho \in R_+$. As for $\tau_i : [a,b] \to [a,b]$ $(i=1,\dots,m)$, they are measurable, and

$$\tau_i(t) \le t \quad \text{for} \quad a \le t \le b \quad (i=1,\dots,m).$$

By $f^*(\cdot,c_0,\rho)$ and $f_0^*(\cdot,c_0,\rho)$ are meant the functions given by the equalities

$$f^*(t,c_0,\rho) = \sup\big\{f(t,c_0+y)(t)\cdot \operatorname{sgn}(y(t)) :\; y \in C_\rho([a,b];R^n)\big\},$$
$$f_0^*(t;c_0,\rho) = \sup\Big\{f_0(t,c_0+y_0,c_0+y_1,\dots,c_0+y_m)\cdot \operatorname{sgn}(y_0) : \\ y_0 \in R_\rho^m,\dots,y_m \in R_\rho^n\Big\}.$$

Definition 1.4. If $b_0 \in]a,b]$, then

(i) for every $x \in C([a,b_0];R^n)$ by $f(x)$ is understood the vector function given by the equality $f(x)(t) = f(\overline{x})(t)$ for $a \le t \le b_0$, where $\overline{x}(t) = x(t)$ for $a \le t \le b_0$ and $\overline{x}(t) = x(b_0)$ for $b_0 < t \le b$.

(ii) a continuous vector function $x : [a,b_0] \to R^n$ is said to be a *solution of equation* (1.1) (*of inequality* (1.3) *or* (1.5)) *on the segment* $[a,b_0]$ if x is absolutely continuous on every segment contained in $]a,b_0]$ and satisfies equation (1.1) (inequality (1.3) or (1.5)) a. e. on $]a,b_0[$;

(iii) a vector function $x : [a.b_0[\to R^n$ is said to be a *solution of equation* (1.1) (*of inequality* (1.3) *or* (1.5)) *on a half-open interval* $[a,b_0[$ if for every $b_1 \in]a,b_0[$ the restriction of x on $[a,b_1]$ is a solution of this equation (inequality) on the segment $[a,b_1]$;

(iv) a solution x of equation (1.1) (of inequality (1.3) or (1.5)) satisfying the initial condition (1.2) (the initial condition (1.4) or (1.6)) is said to be a *solution of problem* (1.1), (1.2) (*of problem* (1.1), (1.4) *or* (1.5), (1.6)).

Definition 1.5. A solution x of equation (1.1) defined on a segment $[a,b_0] \subset [a,b[$ (on a half-open interval $[a,b_0[\subset [a,b[$) is said to be *continuable* if for some $b_1 \in]b_0,b]$ ($b_1 \in [b_0,b]$) equation (1.1) has a solution y on the segment $[a,b_1]$ satisfying the condition $x(t) = y(t)$ for $a \le t \le b_0$. Otherwise x is said to be *noncontinuable*.

Definition 1.6. A solution x^* (solution x_*) of problem (1.1), (1.2) defined in the interval $I_0 \subset [a,b]$ is said to be *upper* (*lower*) if an arbitrary solution x of this problem defined in some interval $I \subset [a,b]$ satisfies

$$x(t) \le x^*(t) \quad (x(t) \ge x_*(t)) \quad \text{for} \quad t \in I_0 \cap I.$$

1.2. Existence of a Non-Continuable Solution of Problem (1.1), (1.2). In [4] the following propositions are proved.

Theorem 1.1. *Let there exist a positive number ρ and summable functions p and $q : [a, b] \to R_+$ such that*

$$\limsup_{t\to a}\left(\frac{1}{h(t)}\int\limits_a^t p(s)ds\right) < 1, \quad \lim_{t\to a}\left(\frac{1}{h(t)}\int\limits_a^t q(s)ds\right) = 0 \tag{1.10}$$

and for every $y \in C_\rho([a, b]; R^n)$ the inequality

$$f(c_0 + hy)(t) \cdot \operatorname{sgn}(y(t)) \le p(t)\nu(y)(a, t) + q(t) \tag{1.11}$$

is fulfilled a.e. on $[a, b[$. Then problem (1.1)*,* (1.2) *has at least one noncontinuable solution and every continuable solution of this problem is a restriction of some noncontinuable solution.*

Corollary 1.1. *Let for some $\rho > 0$, on the set $]a, b[\, \times R_\rho^{(m+1)n}$ the following inequality be fulfilled:*

$$f_0\big(t, c_0 + h(t)y_0, c_0 + h(\tau_1(t))y_1, \dots, c_0 + h(\tau_m(t))y_m\big) \cdot \operatorname{sgn}(y_0) \le$$
$$\le \sum_{k=0}^m p_k(t)\|y_k\| + q(t),$$

where $p_k : [a, b] \to R_+$ $(k = 0, \dots, m)$ and $q : [a, b] \to R_+$ are summable functions satisfying conditions (1.10)*, where $p(t) = \sum_{k=0}^m p_k(t)$. Then problem* (1.7)*,* (1.2) *has at least one noncontinuable solution, and every continuable solution of this problem is a restriction of some noncontinuable solution.*

1.3. Theorems on the Existence of Upper and Lower Solution of Problem (1.1), (1.2).

Theorem 1.2. *Let f be a nondecreasing operator satisfying the conditions of Theorem* 1.1*. Let, moreover, $b^* \in\,]a, b]$ be such that the interval of definition of an arbitrary noncontinuable solution of problem* (1.1)*,* (1.2) *contains the segment $[a, b^*]$. Then problem* (1.1)*,* (1.2) *has on the segment $[a, b^*]$ the upper and the lower solution.*

Corollary 1.2. *Let f be a nondecreasing operator and for some $\rho > 0$ and $b^* \in\,]a, b]$ let the inequality*

$$\int\limits_a^{b^*} f^*(t; c_0, \rho)dt < \rho \tag{1.12}$$

be fulfilled. Then the interval of definition of an arbitrary noncontinuable solution of problem (1.1), (1.2_1) *contains the segment* $[a, b^*]$ *and this problem has on* $[a, b^*]$ *the upper and the lower solution.*

Let $n = 1$ and $f(x)(t) = g(t, x(t))$, where $f_0 : [a, b] \times R \to R$ is a function from the Carathéodory class. Then we can exclude from Corollary 1.2 the requirement for f to be a non-decreasing operator (see [1], Ch. II, §1, Theorem 1.2). However, if f is an arbitrary operator, then this requirement is essential and it cannot be neglected. To verify the above-said, let us consider

Example 1.1. J. Heidel [5][1] proved the existence of a continuous function $p : [a, b] \to]-\infty, 0[$ such that the equation has on the segment $[a, b]$ a non-trivial solution $u''(t) = p(t)u^{1/3}(t)$ which in any neighborhood of the point b changes sign, and $u(b) = u'(b) = 0$. The function $x(t) = u(b+a-t)$ is obviously a non-zero solution of the problem

$$\frac{dx(t)}{dt} = \int_a^t p(a+b-s)x^{1/3}(s)ds, \tag{1.13}$$

$$x(a) = 0. \tag{1.14}$$

On the other hand, if for $\rho = 1$ we choose a number b^* such that

$$(b^* - a)\int_a^{b^*} |p(a+b-s)|ds < 1,$$

then the operator $f(x)(t) = \int_a^t p(a+b-s)x^{1/3}(s)ds$ will satisfy condition (1.12). However, since p is negative, the operator f is not nondecreasing. Suppose that problem (1.13), (1.14) has on the interval $[a, b^*]$ the upper solution x^*. Since this problem has as well the zero solution, $x^*(t) \geq 0$ for $a \leq t \leq b^*$, whence, owing to the fact that p is negative and because of (1.14), we obtain $\frac{dx^*(t)}{dt} \leq 0$ for $a \leq t \leq b^*$ and $x^*(t) = 0$ for $a \leq t \leq b^*$. But this contradicts the inequality $x^*(t) \geq x(t)$ for $a \leq t \leq b^*$, since the solution x takes positive values in any right-hand neighborhood of the point a. The above-obtained contradiction proves that problem (1.13), (1.14) has no upper solution. The fact that this problem has no lower solution either is proved analogously.

Corollary 1.3. *Let f be a nondecreasing operator satisfying the conditions of Theorem* 1.1. *Then problem* (1.1), (1.2) *has the noncontinuable upper and the noncontinuable lower solution.*

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[1] See also [6], Theorem 17.7.

Corollary 1.4. *Let the vector function f_0 be nondecreasing in the last $(m+1)n$ variables and let the conditions of Corollary* 1.1 *be fulfilled. Moreover, let $b^* \in]a,b]$ be such that the interval of definition of an arbitrary noncontinuable solution of problem* (1.7), (1.2) *contains the segment $[a,b^*]$. Then problem* (1.7), (1.2) *has on the segment $[a,b^*]$ the upper and the lower solution.*

Corollary 1.5. *Let the vector function f_0 be nondecreasing in the last $(m+1)n$ variables and let for some $\rho > 0$ and $b^* \in]a,b]$ the inequality*

$$\int_a^{b^*} f_0^*(t;c_0,\rho)dt < \rho$$

be fulfilled. Then problem (1.7), (1.2_1) *has on the segment $[a,b^*]$ the upper and the lower solution.*

Corollary 1.6. *Let the vector function f_0 be nondecreasing in the last $(m+1)n$ variables and let the conditions of Corollary* 1.2 *be fulfilled. Then problem* (1.7), (1.2) *has the noncontinuable upper and the noncontinuable lower solution.*

1.4. Theorems on Functional Differential Inequalities.

Theorem 1.3. *Let f be a nondecreasing operator satisfying the conditions of Theorem* 1.1. *Moreover, let x^* be the upper solution (x_* be the lower solution) of problem* (1.1), (1.2) *on the interval $I_0 \subset [a,b]$, and let z be a solution of problem* (1.3), (1.4) (*of problem* (1.5), (1.6)) *in the interval $I \subset [a,b]$. Then*

$$z(t) \le x^*(t) \quad (z(t) \ge x_*(t)) \quad \text{for} \quad t \in I \cap I_0. \tag{1.15}$$

Corollary 1.7. *Let f be a nondecreasing operator satisfying $f^*(\cdot;c_0,\rho) \in L([a,b];R)$ for some $\rho > 0$. Moreover, let x^* be the upper solution (x_* be the lower solution) of problem* (1.1), (1.2_1) *on the interval $I \subset [a,b]$, and let z be a solution of problem* (1.3), (1.4_1) (*of problem* (1.5), (1.6_1)) *on the interval $I \subset [a,b]$. Then estimate* (1.15) *is valid.*

Corollary 1.8. *Let the vector function f_0 be nondecreasing in the last $(m+1)n$ variables and let the conditions of Corollary* 1.1 *be fulfilled. Moreover, let x^* be the upper solution (x_* be the lower solution) of problem* (1.7), (1.2) *on the interval $I_0 \subset [a,b]$, and let z be a solution of problem* (1.8), (1.4) (*of problem* (1.9), (1.6)) *on the interval $I \subset [a,b]$. Then the estimate* (1.15) *is valid.*

Corollary 1.9. *Let the vector function f_0 be nondecreasing in the last $(m+1)n$ variables and let $f_0^*(\cdot; c_0, \rho) \in L([a,b]; R)$ for some $\rho > 0$. Moreover, let x^* be the upper solution (x_* be the lower solution) of problem* (1.7), (1.2$_1$), *and let z be a solution of problem* (1.8), (1.4$_1$) (*of problem* (1.9), (1.6$_1$)) *on the interval $I \subset [a,b]$. Then estimate* (1.15) *is valid.*

Theorem 1.4. *Let f be a nondecreasing operator satisfying the conditions of Theorem* 1.1. *Moreover, let x_* be the upper solution (x_* be the lower solution) of problem* (1.1), (1.2) *on the interval I_0, and let z be a solution of problem* (1.3), (1.4$_2$) (*of problem* (1.5), (1.6$_2$)) *on the interval I. Then*

$$z(t) < x_*(t) \quad (z(t) > x^*(t)) \quad \text{for} \quad t \in I \cap I_0. \tag{1.16}$$

Corollary 1.10. *Let the vector function f_0 be nondecreasing in the last $(m+1)n$ variables and let the conditions of Corollary* 1.1 *be fulfilled. Moreover, let x_* be the upper solution (x^* be the lower solution) of problem* (1.7), (1.2), *and let z be a solution of problem* (1.8), (1.4$_2$) (*of problem* (1.9), (1.6$_2$)). *Then the estimate* (1.16) *is valid.*

§ 2. Auxiliary Propositions

2.1. Lemma on an a priori estimate. In [4] the following Lemma is proved.

Lemma 2.1. *Let p and $q : [a,b] \to R_+$ be summable functions satisfying conditions* (1.10) *and let numbers $b_0 \in]a,b]$ and $\alpha \in]0,1[$ be such that*

$$\int_a^t p(s)ds \le \alpha h(t) \quad \text{for} \quad a \le t \le b_0.$$

Then any solution $x : [a, b_0] \to R^n$ of the differential inequality

$$x''(t) \cdot \operatorname{sgn}(x(t) - c_0) \le p(t)\nu\Big(\frac{1}{h}(x - c_0)\Big)(a,t) + q(t) \tag{2.1}$$

satisfying the initial condition (1.2) *admits the estimate*

$$\|x(t) - c_0\| \le h(t)\varepsilon(t) \quad \text{for} \quad a \le t \le b_0, \tag{2.2}$$

where $\varepsilon(a) = 0$ and

$$\varepsilon(t) = \frac{1}{1-\alpha} \sup\left\{\frac{1}{h(s)}\int_a^s q(\xi)d\xi : \ a < s \le t\right\} \quad \text{for} \quad a < t \le b_0. \tag{2.3}$$

2.2. Lemma on the Boundedness of a Set of Solutions of Problem (1.1), (1.2). Denote by $I^*(f;c_0,h)$ the set of those $b^* \in]a,b]$ for which the interval of definition of every noncontinuable solution of problem (1.1), (1.2) contains the segment $[a,b^*]$.

Lemma 2.2. *If the conditions of Theorem* 1.1 *are fulfilled, then* $I^*(f;c_0,h) \neq \varnothing$*. Moreover, for every* $b^* \in I^*(f;c_0,h)$ *the set* X *of restrictions of all noncontinuable solutions of problem* (1.1), (1.2) *on* $[a,b^*]$ *is a compact set of the space* $C([a,b^*];R^n)$*. Moreover, there exists a bounded function* $h^* : [a,b^*] \to R_+$ *such that*

$$\lim_{t\to a} \frac{h^*(t)}{h(t)} = 0 \tag{2.4}$$

and an arbitrary solution $x \in X$ *admitting the estimate*

$$\|x(t) - c_0\| \leq h^*(t) \quad \textit{for} \quad a \leq t \leq b^*. \tag{2.5}$$

Proof. Let ρ, b_0, α and ε be the numbers and the function appearing in Theorem 1.1 and Lemma 2.1. By (1.10) and (2.3), without loss of generality, we may assume that

$$\varepsilon(t) < \rho \quad \text{for} \quad a \leq t \leq b_0. \tag{2.6}$$

Let x be an arbitrary noncontinuable solution of problem (1.1), (1.2) which is defined on the interval I_0. Let us show that $[a,b_0] \subset I_0$ and

$$\|x(t) - c_0\| < \rho h(t) \quad \text{for} \quad a < t \leq b_0. \tag{2.7}$$

Assume the contrary. Then by Corollary 3.1 from [4], there exists $t_0 \in]a,b_0] \cap I_0$ such that

$$\|x(t) - c_0\| < \rho h(t) \quad \text{for} \quad a < t < t_0, \quad \|x(t_0) - c_0\| = \rho h(t_0). \tag{2.8}$$

Due to conditions (1.11) and (2.8), the vector function x is a solution of problem (2.1), (1.2) on the interval $[a,t_0]$. By Lemma 2.1, it admits the estimate $\|x(t) - c_0\| \leq \varepsilon(t)h(t)$ for $a \leq t \leq t_0$. From this, owing to (2.6), we find that $\|x(t_0) - c_0\| < \rho h(t_0)$, which contradicts condition (2.8). The obtained contradiction proves that $b_0 \in I^*(f;c_0,h)$, and an arbitrary noncontinuable solution x of problem (1.1), (1.2) admits estimate (2.7). Conditions (1.10), (2.7) and Lemma 2.1 result in estimate (2.2).

Let $b^* \in I^*(f,c_0,h)$, and let X be the set of restrictions of all noncontinuable solutions of problem (1.1), (1.2) on $[a,b^*]$. If $b^* = b_0$, then according to the above arguments, X is the bounded set. Consider the case where $b_0 < b^*$, and prove that X is bounded. For any $t \in]a,b^*]$ assume

$$r(t) = \sup\{\|x(t)\| : x \in X\}, \quad r^*(t) = \sup\{r(s) : a \leq s \leq t\}.$$

Because of (2.2), $r^*(b_0) < +\infty$. Denote by t^* the exact upper bound of the set of $t \in [a, b^*]$ for which $r^*(t) < +\infty$. Obviously, $t^* \in [b_0, b^*]$. Show that $r^*(t^*) < +\infty$. Assume the contrary. Then $t^* > b_0$, and there exist a sequence $t_k \in]b_0, t^*[$ $(k = 1, 2, \dots)$ and a sequence $(x_k)_{k=1}^{\infty}$ of non-continuable solutions of problem (1.1), (1.2) such that

$$\lim_{k\to\infty} t_k = t^*, \quad \lim_{k\to+\infty} \|x_k(t_k)\| = +\infty. \tag{2.9}$$

By virtue of (1.10) and (2.2), for every $\beta \in [b_0, t^*[$ and natural k we have $\|x_k(t) - c_0\| \le h_\beta(t)$ for $a \le t \le \beta$, where

$$h_\beta(t) = \begin{cases} \varepsilon(t)h(t) & \text{for} \quad a \le t \le b_0 \\ \|c_0\| + r^*(\beta) & \text{for} \quad b_0 < t \le \beta \end{cases}, \quad \lim_{t\to a} \frac{h_\beta(t)}{h(t)} = 0.$$

Applying now Lemma 2.1 from [4], it becomes clear that $(x_k)_{k=1}^{\infty}$ contains a subsequence $(x_{k_j})_{j=1}^{\infty}$ which converges uniformly on every segment contained in $[a, t^*[$, and

$$x_0(t) = \lim_{j\to+\infty} x_{k_j}(t) \quad \text{for} \quad a \le t < t^* \tag{2.10}$$

is a solution of problem (1.1), (1.2) on $[a, t^*[$. According to one of the conditions of the lemma, x_0 is a continuable solution, and thus

$$r_0 = \sup\big\{\|x_0(t)\| : \ a \le t < t^*\big\} < +\infty. \tag{2.11}$$

Let

$$\gamma(t;\delta) = \sup\big\{\|f(y)(t)\|; \ y \in C_\delta([a,b]; R^n)\big\}. \tag{2.12}$$

Then $\gamma(\cdot;\delta) \in L_{loc}(]a,b]; R_+)$ for $\delta \in R_+$, and for every $y \in C([a,b]; R^n)$ the inequality

$$f(y)(t) \cdot \operatorname{sgn}(y(t)) \le \gamma(t; \nu(y)(a,t)) \tag{2.13}$$

is fulfilled a.e. on $]a, b[$.

By (2.10), (2.11), and (2.13), there exist $t_* \in]b_0, t^*[$ and a natural number k_0 such that

$$\int_{t_*}^{t^*} \gamma(t, r_0 + 2)dt < 1, \quad \|x_k(t)\| < r_0 + 1 \ \text{ for } \ a \le t \le t_* \ (k = k_0, k_0 + 1, \dots),$$

$$x_k'(t) \cdot \operatorname{sgn}(x_k(t)) \le \gamma(t, \nu(x_k)(a,t)) \ \text{ for } \ a < t < t^* \ \ (k = k_0, k_0 + 1, \dots).$$

This, owing to Lemma 3.1 from [4], implies that

$$\|x_k(t)\| < r_0 + 2 \quad \text{for} \quad a < t < t^* \quad (k = k_0, k_0 + 1, \dots).$$

But this contradicts condition (2.9). The obtained contradiction proves that $r^*(t^*) < +\infty$.

If now we again apply Lemma 3.1 from [4], then from the condition $r^*(t^*) < +\infty$ and the definition of t^* we conclude that $t^* = b^*$. Thus the boundedness of X is proved. Moreover, we have proved that an arbitrary solution $x \in X$ admits estimate (2.5), where $h^*(t) = \varepsilon(t)h(t)$ for $a \le t \le b_0$, $h^*(t) = r^*(b^*)$ for $b_0 < t \le b^*$, the function h^* being bounded and satisfying condition (2.4). This, due to Lemma 2.1 from [4], implies that X is a compactum from $C([a, b^*]; R^n)$. $\square$

2.3. Lemmas on Differential Inequalities.

Lemma 2.3. *Let f be a nondecreasing operator satisfying the conditions of Theorem* 1.1*, and $b^* \in I^*(f; c_0, h)$. Then for an arbitrary solution z of problem* (1.3)*,* (1.4) (*of problem* (1.5)*,* (1.6)) *defined on some segment $[a, \overline{b}] \subset [a, b^*]$ there exists a solution $\overline{x}$* (*a solution $\underline{x}$*) *of problem* (1.1)*,* (1.2) *such that*

$$z(t) \le \overline{x}(t) \quad (z(t) \ge \underline{x}(t)) \quad \textit{for} \quad a \le t \le \overline{b}. \tag{2.14}$$

Proof. Let ρ be the number appearing in Theorem 1.1 and let $z : [a, \overline{b}] \to R^n$ be a solution of problem (1.3), (1.4).

By (1.4), there exists an absolutely continuous vector function $z_0 : [a, \overline{b}] \to R^n$ such that

$$z(t) < z_0(t) \quad \text{for} \quad a < t \le \overline{b} \tag{2.15}$$

and

$$\varepsilon_0(t) = \sup\left\{\frac{\|z_0(s) - c_0\|}{h(s)} \; : \; a < s \le t\right\} \to 0 \quad \text{for} \quad t \to a. \tag{2.16}$$

On the other hand, according to (1.10), we can choose the numbers $b_0 \in]a, \overline{b}[$ and $\alpha \in]0, 1[$ such that

$$\begin{gathered}\int_a^t p(s)ds \le \alpha h(t), \quad \varepsilon(t) = t - a + \frac{1+\alpha}{1-\alpha}\varepsilon_0(t) + \\ + \frac{1}{1-\alpha} \sup\left\{\frac{1}{h(s)} \int_a^s q(\xi)d\xi : \; a < s \le t\right\} < \rho \;\; \text{for} \;\; a < t \le b_0.\end{gathered} \tag{2.17}$$

It is also clear that

$$\lim_{t \to a} \varepsilon(t) = 0. \tag{2.18}$$

For any natural k suppose

$$t_k = a + \frac{b-a}{2k}, \quad f_k(x)(t) = \begin{cases} z_0'(t) & \text{for} \quad a < t \le t_k \\ f(x)(t) & \text{for} \quad t_k < t < b \end{cases} \tag{2.19}$$

and consider the differential equation

$$\frac{dx(t)}{dt} = f_k(x)(t). \tag{2.20}$$

By virtue of Theorem 1.1, for every natural k problem (2.20), (1.2_1) has a noncontinuable solution x_k defined on some interval I_k.

By (2.16) and (2.19), it is clear that $I_k \supset [a, t_k]$,

$$x_k(t) = z_0(t) \quad \text{for} \quad a \le t \le t_k \tag{2.21}$$

and on $]a, t_k]$ the following inequality is fulfilled:

$$\|x_k(t) - c_0\| < \varepsilon(t)h(t). \tag{2.22}$$

Let us show that $I_k \supset [a, b_0]$ and inequality (2.22) is fulfilled on the whole $]a, b_0]$. Assume the contrary. Then by Corollary 3.1 from [4], there exists $b_k \in I_k \cap]t_k, b_0[$ such that

$$\|x_k(t) - c_0\| < \varepsilon(t)h(t) \quad \text{for} \quad a < t < b_k, \tag{2.23}$$

$$\|x_k(b_k) - c_0\| = \varepsilon(b_k)h(b_k). \tag{2.24}$$

Owing to (1.11), (2.17), (2.19) and (2.23), the inequality

$$\|x_k(t) - c_0\|' = f(x_k)(t) \cdot \operatorname{sgn}(x_k(t) - c_0) \le$$
$$\le p(t)\nu\Big(\frac{1}{h}(x_k - c_0)\Big)(a, t) + q(t)$$

is fulfilled a.e. on $]t_k, b_k[$. On the other hand, $\|x_k(t_k) - c_0\| = \|z_0(t_k) - c_0\| \le \varepsilon_0(t_k)h(t_k) \le \varepsilon_0(t)h(t)$ for $t_k \le t \le b_k$ and

$$\nu\Big(\frac{1}{h}(x_k - c_0)\Big)(a, t) \le \nu\Big(\frac{1}{h}(x_k - c_0)\Big)(a, t_k) + \nu\Big(\frac{1}{h}(x_k - c_0)\Big)(t_k, t) \le$$
$$\le \nu\Big(\frac{1}{h}(z_0 - c_0)\Big)(a, t_k) + \nu\Big(\frac{1}{h}(x_k - c_0)\Big)(t_k, t) \le$$
$$\le \varepsilon_0(t) + \nu\Big(\frac{1}{h}(x_k - c_0)\Big)(t_k, t) \quad \text{for} \quad t_k \le t \le b_k.$$

Therefore

$$\|x_k(t) - c_0\| \le \|x_k(t_k) - c_0\| + \int_{t_k}^{t} p(s)\nu\Big(\frac{1}{h}(x_k - c_0)\Big)(a, s)ds + \int_{t_k}^{t} q(s)ds \le$$

$$\leq \varepsilon_0(t)\Big[h(t)+\int\limits_a^t p(s)ds\Big]+\nu\Big(\frac{1}{h}(x_k-c_0)\Big)(t_k,t)\int\limits_a^t p(s)ds+\int\limits_a^t q(s)ds,$$

$$\frac{\|x_k(t)-c_0\|}{h(t)}\leq(1+\alpha)\varepsilon_0(t)+\alpha\nu\Big(\frac{1}{h}(x_k-c_0)\Big)(t_k,t)+$$
$$+\frac{1}{h(t)}\int\limits_a^t q(s)ds \quad \text{for} \quad a<t\leq b_k$$

which by (2.17), implies

$$\nu\Big(\frac{1}{h}(x_k-c_0)\Big)(t_k,t)\leq(1-\alpha)[\varepsilon(t)-t+a]+$$
$$+\alpha\nu\Big(\frac{1}{h}(x_k-c_0)\Big)(t_k,t) \quad \text{for} \quad a<t\leq b_k.$$

Therefore $\nu(\frac{1}{h}(x_k-c_0))(t_k,b_k)\leq\varepsilon(b_k)-t_k+a<\varepsilon(b_k)$, which contradicts equality (2.24). Thus we have shown that $I_k\supset[a,b_0]$ and inequality (2.22) is fulfilled on $]a,b_0]$.

By (2.15) and (2.21), if $\widetilde{t}\in]t_k,b_0]$ and $\widetilde{t}-t_k$ is sufficiently small, then on $]a,\widetilde{t}]$ we have

$$z(t)<x_k(t). \tag{2.25}$$

Denote by t^* the exact upper bound of the set of $\widetilde{t}\in]t_k,b_0]$ for which inequality (2.25) is fulfilled on the interval $[a,\widetilde{t}]$. Then because of inequality (1.3) and the fact that the operator f is nondecreasing, we have

$$x_k(t^*)=z_0(t_k)+\int\limits_{t_k}^{t^*} f(x_k)(s)ds\geq z_0(t_k)+\int\limits_{t_k}^{t^*} f(z)(s)ds\geq$$
$$\geq z_0(t_k)+\int\limits_{t_k}^{t^*} z'(s)ds=z_0(t_k)-z(t_k)+z(t^*)>z(t^*).$$

This, by the definition of t^*, implies that $t^*=b_0$, and hence inequality (2.25) is fulfilled on $]a,b_0]$.

By Lemma 2.1 from [4], conditions (2.18), (2.19), and (2.22) ensure the existence of a subsequence $(x_{k_j})_{j=1}^{\infty}$ of the sequence $(x_k)_{k=1}^{\infty}$ such that $(x_{k_j})_{j=1}^{\infty}$ converges uniformly on $[a,b_0]$, and $x_0(t)=\lim_{j\to\infty}x_{k_j}(t)$ is a solution of problem (1.1), (1.2) in $[a,b_0]$. On the other hand, from (2.25) it is clear that

$$z(t)\leq x_0(t) \quad \text{for} \quad a\leq t\leq b_0. \tag{2.26}$$

For any $x \in C([b_0, \overline{b}]; R^n)$ assume

$$\chi(x)(t) = \begin{cases} x_0(t) + [x(b_0) - z(b_0)]_- & \text{for} \quad a \le t \le b_0 \\ z(t) + [x(t) - z(t)]_+ & \text{for} \quad b_0 < t \le \overline{b} \end{cases}, \tag{2.27}$$

$$\widetilde{f}(x)(t) = f(\chi(x))(t) \quad \text{for} \quad b_0 \le t \le \overline{b}. \tag{2.28}$$

Obviously, $\widetilde{f} : C([b_0, \overline{b}]; R^n) \to L([b_0, \overline{b}]; R^n)$ is a Volterra operator satisfying the local Carathéodory conditions. Moreover, for any $x \in C([b_0, \overline{b}]; R^n)$ the inequality $\|\widetilde{f}(x)(t)\| \le \widetilde{\gamma}(t, \|x\|_C)$ is fulfilled a.e. on $[b_0, b]$, where $\widetilde{\gamma} : [b_0, \overline{b}] \times R_+ \to R_+$ does not decrease in the second argument, and $\widetilde{\gamma}(\cdot; s) \in L([b_0, \overline{b}]; R)$ for $s \ge 0$.

By Theorem 1.1, the problem

$$\frac{dx(t)}{dt} = \widetilde{f}(x)(t), \tag{2.29}$$

$$x(b_0) = x_0(b_0) \tag{2.30}$$

has a noncontinuable solution $\widetilde{x}$ defined on some interval $\widetilde{I} \subset [b_0, \overline{b}]$ whose left end is equal to b_0.

Because of conditions (1.3), (2.26)–(2.28) and the fact that the operator f is nondecreasing, we have

$$\chi(\widetilde{x})(t) = \begin{cases} x_0(t) & \text{for} \quad a \le t \le b_0 \\ z(t) + [\widetilde{x}(t) - z(t)]_+ & \text{for} \quad t \in \widetilde{I} \end{cases}, \tag{2.31}$$

$$\chi(\widetilde{x})(t) \ge z(t) \quad \text{for} \quad t \in [a, b_0] \cup \widetilde{I}$$

and $\widetilde{x}'(t) = f(\chi(\widetilde{x}))(t) \ge f(z)(t) \ge z'(t)$ for almost all $t \in \widetilde{I}$. Taking into account (2.26), we arrive at

$$\widetilde{x}(t) \ge x_0(b_0) + z(t) - z(b_0) \ge z(t) \quad \text{for} \quad t \in \widetilde{I}. \tag{2.32}$$

Assume $\overline{I} = [a, b_0] \cup \widetilde{I}$ and $\overline{x}(t) = x_0(t)$ for $t \in [a, b_0]$, $\overline{x}(t) = \widetilde{x}(t)$ for $t \in \widetilde{I}$.

By virtue of (2.31) and (2.32), $\overline{x}(t) = \chi(\widetilde{x})(t)$ for $t \in \overline{I}$. This implies that $\overline{x}$ is a solution of problem (1.1), (1.2) on the interval $\overline{I}$. If now we take into consideration that $\widetilde{x}$ is a noncontinuable solution of problem (2.29), (2.30) and the interval of definition of an arbitrary noncontinuable solution of problem (1.1), (1.2) contains the segment $[a, \overline{b}]$, then it becomes clear that $\widetilde{I} = [b_0, \overline{b}]$ and $\overline{I} = [a, \overline{b}]$. Consequently, $\overline{x}$ is a solution of problem (1.1), (1.2) on the interval $[a, \overline{b}]$ and, as follows from (2.26), (2.31), it satisfies the inequality $z(t) \le \overline{x}(t)$ for $a \le t \le \overline{b}$. Analogously, we can prove that if $z : [a, \overline{b}] \to R^n$ is a solution of problem (1.5), (1.6), then there exists a solution $\underline{x}$ of problem (1.1), (1.2) satisfying the inequality $z(t) \geq \underline{x}(t)$ for $a \leq t \leq \overline{b}$. $\square$

Lemma 2.4. *Let f be a nondecreasing operator, $b_0 \in]a, b[$ and let x be a solution of equation* (1.1) *on the interval $[a, b_0]$. Let, moreover,*

$$\rho_0 = \max\big\{\|x(t)\| : \ a \leq t \leq b_0\big\}, \tag{2.33}$$

and $b_1 \in]b_0, b]$ be such that

$$\int_{b_0}^{b_1} \gamma(t, \rho_0 + 2)dt < 1, \tag{2.34}$$

where γ is a function given by equality (2.12)*. Then equation* (1.1) *has on the interval $[a, b_1]$ a solution $\overline{x}$* (*a solution $\underline{x}$*) *such that $\overline{x}(t) = x(t)$ $(x(t) = \underline{x}(t))$ for $a \leq t \leq b_0$, and for any $\overline{b} \in]b_0, b_1]$ an arbitrary solution $z : [a, \overline{b}] \to R^n$ of differential inequality* (1.3) (*of differential inequality* (1.5)) *satisfying on $z(t) \leq x(t)$ $(z(t) \geq x(t))$ for $a \leq t \leq b_0$ admits the estimate $z(t) \leq \overline{x}(t)$ $(z(t) \geq \underline{x}(t))$ for $a \leq t \leq \overline{b}$.*

Proof. We will prove only the existence of the solution $\overline{x}$, since the existence of the solution $\underline{x}$ can be proved analogously.

For any $y \in C([b_0, b]; R^n)$ we assume

$$\chi(y)(t) = \begin{cases} x(t) + y(b_0) - x(b_0) & \text{for} \quad a \leq t \leq b_0 \\ y(t) & \text{for} \quad b_0 < t \leq b \end{cases}, \tag{2.35}$$

$$f_1(y)(t) = f(\chi(y))(t). \tag{2.36}$$

Let $c_k \in R^n$ $(k = 1, 2, \dots)$ be an arbitrary sequence satisfying the inequalities

$$\|x(b_0) - c_k\| < \frac{1}{k} \quad (k = 1, 2, \dots), \tag{2.37}$$

$$c_k > x(b_0) \quad (k = 1, 2, \dots). \tag{2.38}$$

For every natural k consider the initial value problem

$$\frac{dy}{dt} = f_1(y)(t), \tag{2.39}$$

$$y(b_0) = c_k. \tag{2.40}$$

Because of (2.12) and (2.33)–(2.35),

$$f_1^*(t; c_k, 1) = \sup\big\{\|f_1(c_k + y)(t) \cdot \operatorname{sgn}(y(t))\| : \ y \in C_1([b_0, b]; R^n)\big\} \leq$$

$$\leq \gamma(t,\rho_0+2) \quad \text{and} \quad \int\limits_{b_0}^{b_1} f_1^*(t;c_k,1)dt \leq 1.$$

This, according to Corollary 2.1 from [4], implies that problem (2.39), (2.40) has on the interval $[b_0,\overline{b}]$ a solution y_k such that $\|y_k(t)\| \leq \|c_k\| + \|y_k(t) - c_k\| \leq \rho_0+2$, $\|y_k'(t)\| \leq \gamma(t;\rho_0+2)$ for $b_0 \leq t \leq b_1$.

Hence, the sequence $(y_k)_{k=1}^{\infty}$ is uniformly bounded and equicontinuous. By Arzela–Ascoli's lemma, it can be assumed to be uniformly convergent. Let

$$\overline{y}(t) = \lim_{k\to\infty} y_k(t) \quad \text{for} \quad b_0 \leq t \leq b_1 \tag{2.41}$$

and

$$\overline{x}(t) = x(t) \;\text{ for }\; a \leq t \leq b_0, \quad \overline{x}(t) = \overline{y}(t) \;\text{ for }\; b_0 < t \leq b_1. \tag{2.42}$$

Since the operator f_1 is continuous, by condition (2.37) $\overline{y}$ is a solution of equation (1.1) under the initial condition $\overline{y}(b_0) = x(b_0)$. Taking this fact into account, it follows from (2.35), (2.36) that $\chi(\overline{y})(t) \equiv \overline{x}(t)$, $f_1(y)(t) \equiv f(\overline{x})(t)$, and hence $\overline{x}$ is a solution of equation (1.1) on the interval $[a,b_1]$.

Let $\overline{b} \in]b_0,b_1]$ and let $z:[a,\overline{b}] \to R^n$ be a solution of differential inequality (1.3) satisfying the condition

$$z(t) \leq x(t) \quad \text{for} \quad a \leq t \leq b_0. \tag{2.43}$$

Our aim is to prove that

$$z(t) \leq \overline{x}(t) \quad \text{for} \quad a \leq t \leq \overline{b}. \tag{2.44}$$

Due to (2.38) and (2.43), for every natural k the inequality

$$z(t) < y_k(t) \tag{2.45}$$

is fulfilled in a right-hand neighborhood of the point b_0. Let us show that this inequality is fulfilled on the whole $[b_0,\overline{b}]$. Assume the contrary. Then there exists $\underline{b} \in]b_0,\overline{b}]$ such that on $[b_0,\underline{b}[$ inequality (2.45) is fulfilled but for $t=\underline{b}$ it does not hold. By (1.3), (2.35), (2.36) and the fact that the operator f is nondecreasing, $\chi(y_k)(t) > z(t)$ for $a \leq t < \underline{b}$ and the inequality $y_k'(t) \geq f(z)(t) \geq z'(t)$ is fulfilled a.e. on $]a,b_1[$. Therefore $y_k(\underline{b}) - z(\underline{b}) \geq y_k(b_0) - z(b_0) \geq c_k - x(b_0) > 0$, which contradicts our assumption that inequality (2.45) does not hold for $t=\underline{b}$. Thus we have proved that inequality (2.45) is fulfilled on the whole $[a,\overline{b}]$.

By (2.41) and (2.42), from (2.43) and (2.45) we obtain estimate (2.44). $\square$

§ 3. Proof of the Main Results

Proof of Theorem 1.2. We will prove only the existence of the upper solution, since that of the lower solution is proved analogously.

Let X be the set of restrictions of all noncontinuable solutions of problem (1.1), (1.2) on the segment $[a, b^*]$. By Lemma 2.2, X is a compactum in $C([a, b^*]; R^n)$. Moreover, there exists a bounded function $h^* : [a, b^*] \to R_+$ satisfying condition (2.4) such that every solution $x \in X$ admits estimate (2.5). On the other hand, there exists $\gamma^* \in L_{loc}(]a, b^*]; R_+)$ such that

$$\|x(t) - x(s)\| \le \int_s^t \gamma^*(\xi) d\xi \quad \text{for} \quad x \in X, \quad a < s < t \le b^*. \tag{3.1}$$

Assume $x_i^*(t) = \sup\{x_i(t) : x = (x_j)_{j=1}^n \in X\}$ and $x^*(t) = (x_i^*(t))_{i=1}^n$. Then from (2.5) and (3.1) we find

$$\|x^*(t) - c_0\| \le n h^*(t) \quad \text{for} \quad a \le t \le b^*, \tag{3.2}$$

$$\|x^*(t) - x^*(s)\| \le n \int_s^t \gamma^*(\xi) d\xi \quad \text{for} \quad a < s < t \le b^*.$$

Consequently, x^* is continuous on $[a, b^*]$ and absolutely continuous on every segment contained in $]a, b^*]$.

If we take into account that the operator f is nondecreasing, then from the equality

$$x(t) = x(s) + \int_s^t f(x)(\xi) d\xi \quad \text{for} \quad x \in X, \quad a < s < t \le b^*$$

we have

$$x^*(t) < x^*(s) + \int_s^t f(x^*)(\xi) d\xi \quad \text{for} \quad a < s < t \le b^*,$$

that is,

$$\frac{1}{t-s}(x^*(t) - x^*(s)) \le \frac{1}{t-s} \int_s^t f(x^*)(\xi) d\xi \quad \text{for} \quad a < s < t \le b^*.$$

From the above it is clear that the inequality $\frac{dx^*(t)}{dt} \le f(x^*)(t)$ is fulfilled a.e. on $]a, b^*[$. On the other hand, by (2.4) and (3.2),

$$\lim_{t \to a} \frac{\|x^*(t) - c_0\|}{h_0(t)} = 0.$$

Hence $z(t) = x^*(t)$ is a solution of problem (1.3), (1.4) in $[a, b^*]$. Therefore, as follows from Lemma 2.3, problem (1.1), (1.2) has on the interval $[a, b^*]$ a solution $\overline{x}$ such that $x^*(t) \le \overline{x}(t)$ for $a \le t \le b^*$. However, by the definition of x^*, $\overline{x}(t) \le x^*(t)$ for $a \le t \le b^*$. Clearly, $x^*(t) \equiv \overline{x}(t)$, and x^* is the upper solution of problem (1.1), (1.2). $\square$

Proof of Corollary 1.2. By Theorem 1.2, it suffices to establish that the interval of definition of every noncontinuable solution of problem (1.1), (1.2_1) contains the segment $[a, b^*]$.

Let x be an arbitrary noncontinuable solution of problem (1.1), (1.2_1) defined on the interval I. Suppose $y(t) = x(t) - c_0$. Then the inequality $y'(t)\operatorname{sgn}(y(t)) \le f^*(t; c_0, \nu(y)(a,t))$ is fulfilled a.e. on I, and $\nu(y)(a,a) = 0$. This, by Lemma 3.1 from [4] and inequality (1.12), implies that $\|y(t)\| < \rho$ for $t \in [a, b^*] \cap I$. From this estimate, by virtue of Corollary 3.1 from [4], we have $I \supset [a, b^*]$. $\square$

Proof of Corollary 1.3. By Lemma 1.2 and Theorem 1.2, the set $I^*(f; c_0, h)$ is non-empty and for arbitrary fixed $b_0 \in I^*(f; c_0, h)$ problem (1.1), (1.2) has in the interval $[a, b_0]$ the upper solution x_0^*. Assume $\rho_0 = \max\{\|x_0^*(t)\| : a \le t \le b_0\}$.

Let γ be the function given by equality (2.12) and let $(t_k)_{k=1}^\infty$ be a sequence satisfying $b_0 < t_k < t_{k+1} < b$ $(k = 1, 2, \dots)$, $\lim_{k\to\infty} t_k = b$.

If $\int_{b_0}^{t_1} \gamma(t, \rho_0 + 2)dt \le 1$, then we assume $b_1 = t_1$. If, however, $\int_{b_0}^{t_1} \gamma(t, \rho_0 + 2)dt > 1$, then we choose $b_1 \in]b_0, t_1[$ such that $\int_{b_0}^{b_1} \gamma(t, \rho_0 + 2)dt = 1$. By virtue of Lemma 2.4, equation (1.1) has in the interval $[a, b_1]$ a solution x_1^* such that $x_1^*(t) = x_0^*(t)$ for $a \le t \le b_0$ and for any $\overline{b} \in]b_0, b_1]$ an arbitrary solution $x : [a, \overline{b}] \to R^n$ of equation (1.1) satisfying the condition $x(t) \le x_0^*(t)$ admits for $a \le t \le b_0$ the estimate $x(t) \le x_1^*(t)$ for $a \le t \le \overline{b}$.

Since x_0^* is the upper solution of problem (1.1), (1.2) on the interval $[a, b_0]$, it is clear that x_1^* is the upper solution of this problem on the interval $[a, b_1]$.

Proceeding from the above-said, we prove by induction the existence of an increasing sequence $b_k \in]b_0, b[$ $(k = 1, 2, \dots)$ and of a sequence of vector functions $x_k^* : [a, b_k] \to R^n$ $(k = 1, 2, \dots)$ such that for every natural k:

(i) x_k^* is the upper solution of problem (1.1), (1.2) on the interval $[a, b_k]$;

(ii) either $b_k = t_k$ or $\int_{b_{k-1}}^{b_k} \gamma(t, \rho_{k-1} + 2)dt = 1$, where $\rho_{k-1} = \max\{\|x_{k-1}^*(t)\| : a \le t \le b_{k-1}\}$.

Assume $b^* = \lim_{k\to\infty} b_k$, $x^*(t) = x_0^*(t)$ for $a \le t < b_0$, $x^*(t) = x_k^*(t)$ for $b_{k-1} < t < b_k$ $(k = 1, 2, \dots)$. Clearly, x^* is the upper solution of problem (1.1), (1.2) on $[a, b^*[$. If $b^* < b$, then owing to (ii), x^* is a noncontinuable solution.[1] The solution x^* is noncontinuable as well when $b^* = b$ and $\sup\{\|x^*(t)\| : a \le t < b\} = +\infty$. If, however, $\sup\{\|x^*(t)\| : a \le t <$

[1]See the proof of Theorem 3.2 in [4].

$b\} < +\infty$, then x^* has at the point b the left-hand limit $x^*(b-)$. Assume $\overline{x}(t) = x^*(t)$ for $a \le t < b$, $\overline{x}(b) = x^*(b-)$ for $t = b$. Then $\overline{x}$ is the noncontinuable upper solution of problem (1.1), (1.2). □

When

$$f(x)(t) \equiv f_0\big(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))\big), \tag{3.3}$$

from Theorem 1.1 and Corollary 1.3 follow respectively Corollaries 1.4 and 1.6, while from Corollary 1.2 follows Corollary 1.5.

Proof of Theorem 1.3. Let z be a solution of problem (1.3), (1.4) on the interval I. Prove that in $I_0 \cap I$ the inequality

$$z(t) \le x^*(t) \tag{3.4}$$

is fulfilled. Assume the contrary. Then there exists $\overline{b} \in (I_0 \cap I) \cap]a, b]$ such that for $t = \overline{b}$ inequality (3.4) does not hold.

By Lemmas 2.2, 2.3 and Theorem 1.2, the set $I^*(f; c_0, h)$ is non-empty and for any $t_0 \in]a, \overline{b}[\cap I^*(f; c_0, h)$ inequality (3.4) is fulfilled on $[a, t_0]$. Denote by b_0 the exact upper bound of the set of $t_0 \in]a, \overline{b}[$ for which inequality (3.4) is fulfilled on the segment $[a, t_0]$. Then according to our assumption, $b_0 < \overline{b}$.

By Lemma 2.4, there exist $b_1 \in]b_0, \overline{b}]$ and a solution $\overline{x} : [a, b_1] \to R^n$ of equation (1.1) such that $\overline{x}(t) = x^*(t)$ for $a \le t \le b_0$ and $z(t) \le \overline{x}(t)$ for $a \le t \le b_1$. Obviously, $\overline{x}$ is a solution of problem (1.1), (1.2). Therefore, $\overline{x}(t) \le x^*(t)$ for $a \le t \le b_1$. Consequently, inequality (3.4) is also fulfilled on the segment $[a, b_1]$, which contradicts the definition of b_0. The obtained contradiction proves that inequality (3.4) is fulfilled on $I_0 \cap I$.

It can be proved analogously that if z is a solution of problem (1.5), (1.6) on the interval I, then inequality $z(t) \ge x_*(t)$ is fulfilled on $I_0 \cap I$. □

When $h(t) \equiv 1$, from the above proved theorem we obtain Corollary 1.7, while when identity (3.3) is fulfilled, from Theorem 1.3 and Corollary 1.7 follow Corollaries 1.8 and 1.9.

Proof of Theorem 1.4. Let z be a solution of problem (1.3), (1.4_2) on the interval I. Then in a right-hand neighborhood of the point a the inequality

$$z(t) < x_*(t) \tag{3.5}$$

is fulfilled.

Let us show that it is fulfilled on the whole interval $I \cap I_0$. Assume the contrary. Then there exists $b_0 \in I \cap I_0 \cap]a, b]$ such that inequality (3.5) is fulfilled on $]a, b_0[$, but for $t = b_0$ it does not hold. On the other hand, since the operator f is nondecreasing, we have $(x_*(t) - z(t))' \ge f(x_*)(t) - f(z)(t) \ge 0$ a.e. on $]a, b_0[$. Therefore $x_*(b_0) - z(b_0) \ge x_*(a) - z(a) > 0$, which contradicts our assumption that (3.5) does not hold for $t = b_0$. Thus we have established that (3.5) is fulfilled on the whole $I \cap I_0$. It can be proved analogously that an arbitrary solution $z : I \to R^n$ of problem (1.5), (1.6_2) admits the estimate $z(t) > x^*(t)$ for $t \in I \cap I_0$. $\square$

If identity (3.3) is fulfilled, then from Theorem 1.9 we obtain Corollary 1.10.

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