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THE WEIGHTED BMO CONDITION AND A CONSTRUCTIVE DESCRIPTION OF CLASSES OF ANALYTIC FUNCTIONS SATISFYING THIS CONDITION

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Abstract. The problem of local polynomial approximation of analytic functions prescribed in finite domains with a quasiconformal boundary is investigated in weighted plane integral metrics; a constructive description of the class of analytic functions satisfying a weak version of the known BMO condition is obtained.

The First results of the investigation of the problem (formulated by V. I. Belyi) dealing with a local polynomial approximation of analytic functions prescribed in finite domains with quasiconformal boundary have been described in [1, 2] for weighted plane integral metrics. This problem is investigated in [3] for the nonweighted case, where a constructive description of Hölder classes as well as of some other classes of analytic functions has been obtained. In the present paper we continue the investigation of the above-mentioned problem for the weighted case; moreover, a constructive description of one more class of analytic functions in weighted plane integral metrics is obtained.

1. Notation and Definitions. The Basic Results

Let G be the domain with a quasiconformal boundary $\partial G = \Gamma$, and let $y = y(\zeta)-$ be a quasiconformal reflection across the curve Γ [4]. We will be concerned only with the special, so-called canonical, quasiconformal reflection (see relations (2.1) and (2.2)). Let w be some weight function (i.e., nonnegative and measurable) defined in the domain Γ. Let us introduce the

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notation

$$H'(G) = \{f : f \text{ holomorphic in } G\},$$
$$L_p(G,w) = \{f : |f|^p w \in L_1(G)\}, \quad H'_p(G,w) = L_p(G,w) \cap H'(G) \quad (p \geq 1).$$

Furthermore, let σ be the plane Lebesgue measure, and let μ be the Borel measure defined by the equality

$$\mu(E) = \iint\limits_E w(z) d\sigma_z \quad (E \subset G). \tag{1.1}$$

The integral with respect to the measure μ of the function f will be denoted by the symbols

$$\iint\limits_E f(z) d\mu_z = \iint\limits_E f d\mu.$$

If $\mu = \sigma$, then (in cases where this does not cause misunderstanding) we shall use the brief notation

$$\iint\limits_E f(z) d\sigma_z = \iint\limits_E f.$$

Let $Q = Q(z,r)$ be an open square with center at the point z, whose sides are parallel to the coordinate axes and have the length r, and let

$$F(G) = \{Q = Q(z,r) : z \in \Gamma, \quad r > 0\}.$$

For a square Q denote $|Q| = \sigma(Q)$.

Let the function f and the weight function w be defined in the domain G, let the measure μ be defined by equality (1.1), and

$$f_{\mu,Q\cap G} = \frac{1}{\mu(Q \cap G)} \iint\limits_{Q\cap G} f d\mu. \tag{1.2}$$

We say that f satisfies the weighted BMO condition $\mathrm{BMO}_p(G,w)$ (briefly, $f \in \mathrm{BMO}_p(G,w)$), if

$$\sup_{Q\in F(G)} \left(\frac{1}{\mu(Q\cap G)} \iint\limits_{Q\cap G} |f - f_{\mu,Q\cap G}|^p d\mu \right)^{\frac{1}{p}} \overset{df}{=} \|f\|_{BMO_p(G,w)} < \infty. \tag{1.3}$$

When $p = 1$ and $w(z) = 1$ everywhere in G, we shall use the usual notation $f \in \mathrm{BMO}(G)$ and $\|f\|_{\mathrm{BMO}(G)}$ respectively.

The $\mathrm{BMO}(G)$ condition is a weaker analogue of the well-known BMO (bounded mean oscillation) condition (see, e.g., [5, Ch. VI]).

Next, we say that the weight function w given in the domain G satisfies the condition $A_p(F(G))$ $(1<p<\infty)$ (briefly $w \in A_p(F(G))$), if (assuming $0\cdot\infty=0$)

$$\sup_{Q\in F(G)}\Big(\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G} w\Big)\Big(\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G} w^{-\frac{1}{p-1}}\Big)^{p-1}<\infty.$$

The condition $A_p(F(G))$, introduced for the first time in [6] (with the unit circle as G), is a weaker analogue of the well-known Muckenhoupt condition (A_p) [7].

Let $z_0 \in \Gamma$, $\rho_n(z_0)$ $(n\in N)$ be the distance from the point z_0 to the external level line $\Gamma_{1+1/n}$ of G, $u(z_0,r)=\{z:|z-z_0|<r\}$, $c_0>0$. Set

$$G(z_0,c_0)=\big\{z\in G:\ |\zeta-z|\ge c_0|\zeta-z_0|\quad \forall\zeta\in CG\big\},$$
$$G_n(z_0,c_0)=G(z_0,c_0)\cup\big\{u(z_0,\rho_n(z_0))\cap G\big\}.$$

The set $G(z_0,c_0)$ is a kind of a "nontangential" subset of G with the vertex at the point z_0.

In the sequel, for brevity we shall write

$$\Big(\iint\limits_{G_n(z_0,c_0)}|f(z)-P_n(z)|^p w(z)d\sigma_z\Big)^{\frac{1}{p}}\overset{df}{=}\|f-P_n\|_{z_0,p,w}.$$

Let us now formulate the basic results in which G denotes a finite domain with a quasiconformal boundary Γ, the weight function $w\in A_p(F(G))$ $(1<p<\infty)$, and μ is the measure defined by equality (1.1).

Theorem 1. *For the function f to belong to the class* $\mathrm{BMO}_p(G,w)\cap H'(G)$ *(neglecting its values on the set of measure zero), it is necessary and sufficient that a sequence of algebraic polynomials P_n of order not higher than n exist such that for all $z_0\in\Gamma$ and $n\in N$ the relation*

$$\|f-P_n\|_{z_0,p,w}\le c(c_0)\Big(\mu\{u(z_0,\rho_n(z_0))\cap G\}\Big)^{\frac{1}{p}} \tag{1.4}$$

holda, where the constant $c(c_0)$ does not depend on z_0 and n.

Theorem 2. *Let $f\in H'(G)$. The following conditions are equivalent:*
(a) $f\in\mathrm{BMO}(G)$;
(b) $f\in\mathrm{BMO}_p(G,w)$.

This theorem is an analogue of the well-known John and Nirenberg theorem (see, e.g., [5]).

Obviously, Theorem 2 allows us to formulate Theorem 1 as follows:

Theorem 1^***.** *Theorem* 1 *remains valid when the class* $\mathrm{BMO}_p(G)$ *is replaced by the class* $\mathrm{BMO}(G)$.

Thus we have given the constructive description of the class of functions $\mathrm{BMO}(G)\cap H'(G)$ in the weighted plane integral metrics.

2. Auxiliary Results

Let G be the domain with a quasiconformal boundary Γ, and let $0\in G$, $y=y(\zeta)$ be a quasiconformal reflection across the curve Γ [4]. As follows from the Ahlfors theorem [4] (see also [8]), the reflection $y=y(\zeta)$ can always be chosen to be canonical in the sense that it is differentiable for $\zeta\notin\Gamma$, and for any fixed sufficiently small $\delta>0$ it will satisfy the relations

$$|y_{\bar\zeta}(\zeta)|\asymp M,\ \ |y_\zeta(\zeta)|\preccurlyeq M,\ \ \delta<|\zeta|<1/\delta,\ \ \zeta\neq\Gamma, \tag{2.1}$$

$$|y_{\bar\zeta}(\zeta)|\preccurlyeq M|\zeta|^{-2},\ \ |y(\zeta)\asymp|\ M|\zeta|^{-1},\ \ |\zeta|\le\delta,\ \ |\zeta|\ge\frac{1}{\delta}, \tag{2.2}$$

where $M=M(\delta,\Gamma)$ is a constant depending only on δ and Γ.

The symbol $A\preccurlyeq B$ for the numbers A and B depending on some parameters denotes that $A\le cB$, where $c=\mathrm{const}>0$ does not depend on those parameters; the symbol $A\succcurlyeq B$ means that $B\preccurlyeq A$; $A\asymp B$ if simultaneously $A\preccurlyeq B$ and $A\succcurlyeq B$.

Let $w\in A_p(F(G))$ $(1<p<\infty)$, and let $y=y(\zeta)$ be a canonical quasiconformal reflection across the curve $\Gamma=\partial G$. Let us introduce the notation

$$w^*(z)=\begin{cases} w(z) & \text{for } z\in G,\\ w(y(z)) & \text{for } z\notin G,\end{cases}\qquad \mu^*(E)=\iint\limits_E w^*\,d\sigma. \tag{2.3}$$

It is clear that if $E\subset G$, then $\mu(E)=\mu^*(E)$. Suppose further that

$$\rho(E_1,E_2)=\inf\big\{|z_1-z_2|:\ z_1\in E_1,\ z_2\in E_2\big\},$$
$$\operatorname{diam}E=\sup\big\{|z_1-z_2|:\ z_1,\ z_2\in E\big\},$$
$$F(G,k)=\big\{Q:\ \operatorname{diam}Q\ge k\rho(Q,\Gamma),\ Q\cap G\neq\varnothing\big\}\ \ (k>0). \tag{2.4}$$

Let $w\in A_p(F(G))$ $(1<p<\infty)$, and let $w^*(z)$ be the function defined by equality (2.3). Then owing to relations (2.1) and (2.2), we can conclude that for all $Q\in F(G,k)$, $\operatorname{diam}Q<k_0$ $(k_0,k>0$ are arbitrary fixed numbers) the inequality

$$\bigg(\frac{1}{|Q|}\iint\limits_Q w^*\bigg)\bigg(\frac{1}{|Q|}\iint\limits_Q w^{*-\frac{1}{p-1}}\bigg)^{p-1}\le c(k,k_0)<\infty \tag{2.5}$$

holds, where $c(k,k_0)$ is a constant independent of Q.

Lemma 1 ([9], [10]). *Let $w \in A_p(F(G))$, and let $w^*(z)$ be the function defined by equality* (2.3). *There exist numbers $0 < \delta = \delta(k, k_0) < 1$, $0 < \varepsilon = \varepsilon(k, k_0) < 1$ such that for every $e \subset Q$ the inequality $|e| < \delta|Q|$ implies*

$$\iint_e w^* < \varepsilon \iint_Q w^*$$

for all $Q \in F(G,k)$, $\operatorname{diam} Q < k_0$.

Next, by virtue of the Hölder inequality and relation (2.5), we find that for all $Q(z,r)$, $Q(z,R) \in F(G,k)$ $(0 < r \le R \le k_0 < \infty)$ the inequality

$$\left(\frac{r}{R}\right)^2 \le c(k,k_0)\frac{\mu^*(Q(z,r))}{\mu^*(Q(z,R))} \tag{2.6}$$

holds, where μ^* is defined by equality (2.3).

In particular, it follows from (2.6) that μ^* satisfies the known "doubling" condition

$$\mu^*(2Q) \preccurlyeq \mu^*(Q), \quad (Q \in F(G,k), \quad \operatorname{diam} Q < k_0). \tag{2.7}$$

Let us now prove that for all squares $Q \in F(G)$ $(\operatorname{diam} Q < k_0)$ the relation

$$\mu(Q \cap G) \succcurlyeq \mu^*(Q) \tag{2.8}$$

holds, where μ and μ^* are defined respectively by equalities (1.1) and (2.3).

Indeed, let $Q = Q(z_0, r)$ $(z_0 \in \Gamma, r > 0)$, $\operatorname{diam} Q < k_0$. Owing to relations (2.1) and (2.2), we get

$$\mu^*(Q(z_0,(1/M)r) \cap CG) \preccurlyeq c(M)\mu^*(Q(z_0,r) \cap G) = c(M)\mu(Q \cap G),$$

where $M > 1$ is the constant from (2.1) and (2.2), and CG is the complement to the domain G. But then, using the "doubling" condition (2.7), we get

$$\begin{aligned}\mu^*(Q) &\preccurlyeq \mu^*(Q(z_0,(1/M)r) \cap G) + \mu^*(Q(z_0,(1/M)r) \cap CG) \preccurlyeq \\ &\preccurlyeq c(M)\mu^*(Q(z_0,r) \cap G) \preccurlyeq \mu(Q \cap G). \quad \square\end{aligned}$$

Let w be a weight function, and let μ^* be the measure defined by equality (2.3). Let f be a function given in the domain G, and let Q be a square. Introduce the notation

$$f^*(z) = \begin{cases} f(z) & \text{for } z \in G, \\ f(y(z)) & \text{for } z \notin G, \end{cases} \quad f^*_{\mu^*,Q} = \frac{1}{\mu^*(Q)} \iint_Q f^* d\mu^*. \tag{2.9}$$

In the case of the Lebesgue measure σ, we shall use f^*_Q instead of $f^*_{\sigma^*,Q}$.

Lemma 2. *Let w be some weight function, $p > 1$, $f \in \mathrm{BMO}_p(G, w)$. Then for all squares $Q \in F(G, k)$, $\operatorname{diam} Q < k_0$ ($k, k_0 > 0$ are fixed numbers) the relation*

$$\left(\frac{1}{\mu^*(Q)} \iint\limits_Q |f^* - f^*_{\mu^*,Q}|^p d\mu^*\right)^{\frac{1}{p}} \le c^* \|f\|_{\mathrm{BMO}_p(G,w)} \tag{2.10}$$

holds, where c^* is a constant independent of Q, p, f, and w.

Proof. Assume first that $Q \in F(G)$, $\operatorname{diam} Q < k_0$. Let $M > 1$ be the number from relations (2.1) and (2.2), and let MQ be the square obtained by an M-fold increase of the square Q. It follows from the "doubling" condition (2.7) that

$$\mu^*(Q) \succcurlyeq c(M)\mu^*(MQ) \ge c(M)\mu(MQ \cap G). \tag{2.11}$$

On account of relations (2.1) and (2.2) we obtain

$$\left(\iint\limits_{Q\cap CG} |f^* - f_{\mu,MQ}|^p d\mu^*\right)^{\frac{1}{p}} \preccurlyeq M^2 \left(\iint\limits_{MQ\cap G} |f - f_{\mu,MQ}|^p d\mu\right)^{\frac{1}{p}}. \tag{2.12}$$

Hence, using the Minkowsky inequality and relations (2.11), (2.12), and (1.3), we obtain

$$\left(\frac{1}{\mu^*(Q)} \iint\limits_Q |f^* - f^*_{\mu^*,Q}|^p d\mu^*\right)^{\frac{1}{p}} \preccurlyeq \left(\frac{2}{\mu^*(Q)} \iint\limits_Q |f^* - f_{\mu,MQ\cap G}|^p d\mu^*\right)^{\frac{1}{p}} \preccurlyeq$$

$$\preccurlyeq \left(\frac{M^2+1}{c(M)} \frac{1}{\mu(MQ \cap G)} \iint\limits_{MQ\cap G} |f - f_{\mu,MQ\cap G}|^p d\mu\right)^{\frac{1}{p}} \preccurlyeq c^* \|f\|_{\mathrm{BMO}_p(G,w)}.$$

Thus we have proved that inequality (2.10) is true for all $Q \in F(G)$, $\operatorname{diam} Q < k_0$. Using the "doubling" condition (2.7), it is not difficult to show that (2.10) holds for all $Q \in F(G, k)$ as well. □

Lemma 3. *Let $f \in \mathrm{BMO}(G)$ be an analytic function in the domain G, $\zeta \in G$, $Q = Q(\zeta, a) \notin F(G, k)$ $(k \le 1)$, $Q \subset G$. Then*

$$|f(\zeta) - f(z)| \le 4\|f\|_{\mathrm{BMO}(G)} \quad \forall (z \in Q(\zeta, a)).$$

Proof. Since $Q = Q(\zeta, a) \notin F(G, k)$ $(k \le 1)$, it follows from the definition of the set $F(G, k)$ (see (2.4)) that $\operatorname{diam} Q < k\rho(Q, \Gamma)$ $(k \le 1)$. Hence

$$|\zeta - z| \le \frac{1}{2} \operatorname{diam} Q < \frac{1}{2} k\rho(Q, \Gamma) < \frac{1}{2} k\rho(\zeta, \Gamma) < \frac{1}{2}\rho(\zeta, \Gamma)$$

for all $z \in Q$. Then assuming for brevity that $\rho(\zeta,\Gamma)=\rho$, owing to the mean value theorem and the condition $f \in \mathrm{BMO}(G)$, we obtain

$$|f(z)-f(\zeta)| \le \frac{1}{|u(z,\rho/2)|}\iint\limits_{u(z,\rho/2)} |f(\xi)-f(\zeta)|d\sigma_\xi \le$$

$$\le \frac{4}{|u(\zeta,\rho)}| \iint\limits_{u(\zeta,\rho)} |f(\xi)-f_{u(\zeta,\rho)}|d\sigma_\xi \le 4\|f\|_{\mathrm{BMO}(G)}.$$

for all $z \in Q(\zeta,a)$. □

Lemma 4. *Let $f \in \mathrm{BMO}(G)$ be an analytic function in the domain G, and let f^* and f^*_Q be defined by equalities* (2.9) (*the case $\mu=\sigma$*), *$Q \in F(G,k)$ ($k \le 1$), $\alpha > c^*\|f\|_{\mathrm{BMO}(G)}$ (c^* is a constant from* (2.10)). *Then there exists at most a countable set of nonintersecting squares $A=\{Q_j\}$ such that $Q_j \in F(G,k)$, and*

(1) $|f^*(z)-f^*_Q| \le 12\alpha \quad \forall(z \in Q\backslash Q_j,\ Q_j \in A);$ (2.13)

(2) $\alpha \le \dfrac{1}{|Q_j|}\displaystyle\iint_{Q_j} |f^*(z)-f^*_Q|d\sigma_z < 4\alpha \quad \forall(Q_j \in A);$ (2.14)

(3) $\displaystyle\sum_{Q_j\in A} |Q_j| \le \frac{1}{\alpha}\|f\|_{\mathrm{BMO}(G)}|Q|.$ (2.15)

Proof. It is obvious that the conditions (2.10) and $\alpha > c^*\|f\|_{\mathrm{BMO}(G)}$ yield

$$\frac{1}{|Q|}\iint\limits_{Q} |f^*(z)-f^*_Q|d\sigma_z < \alpha.$$

Let Q^* be the square obtained by partitioning Q into four equal squares.

In the case $Q^* \in F(G,k)$ we insert Q^* in A if $\iint_{Q^*}|f(z)-f_Q|d\sigma_z \ge \alpha|Q^*|$, while when the opposite inequality holds we again partition Q^* in four equal squares and argue as above.

Let us show that the squares $Q_j \in A$ obtained in such a way satisfy all the requirements of Lemma 2.

Let $\zeta \in \{Q \cap G\}\backslash \cup \{Q_j : Q_j \in A\}$. Then, obviously, there exist the squares Q_1 and Q_2 from the above-mentioned partitioning such that $\zeta \in Q_1 \subset Q_2$, $Q_1 \notin F(G,k)$, $Q_2 \in F(G,k)$, and

$$\frac{1}{|Q_2|}\iint\limits_{Q_2} |f^*(z)-f^*_Q| < \alpha,$$

whence it follows that

$$\frac{1}{|Q_1|}\iint\limits_{Q_1}|f^*(z)-f^*_Q|d\sigma_z<\frac{4}{|Q_2|}\iint\limits_{Q_2}|f^*(z)-f^*_Q|d\sigma_z<4\alpha.$$

Then, denoting by z_1 the center of the square Q_1 and using Lemma 3 and the mean value theorem, we obtain

$$|f(\zeta)-f_{Q_1}|\leq|f(\zeta)-f(z_1)|+|f(z_1)-f_{Q_1}|\leq$$
$$\leq 4\|f\|_{\mathrm{BMO}(G)}+\frac{2}{|Q_1|}\iint\limits_{Q_1}|f(z)-f_{Q_1}|d\sigma_z\leq 12\alpha.$$

Thus the validity of the first requirement of Lemma 2 is proved.

Further, it is evident that the left-hand side of the "double" inequality (2.14) holds for all $Q_j\in A$. Let us show that the right-side of that inequality is also valid.

Let Q_j^* be a square whose partitioning into four equal squares gives the square $Q_j\in A$. Clearly, $Q_j^*\supset Q_j$, and

$$\frac{1}{|Q_j^*|}\iint\limits_{Q_j^*}|f^*(z)-f^*_Q|d\sigma_z<\alpha.$$

Taking into account the above inequality, we obtain

$$\frac{1}{|Q_j|}\iint\limits_{Q_j}|f^*(z)-f^*_Q|d\sigma_z<\frac{4}{|Q_j^*|}\iint\limits_{Q_j^*}|f^*(z)-f^*_Q|d\sigma_z<4\alpha.$$

Thus relation (2.14) is proved.

Finally, using the already proven relation (2.14) and inequality (2.10) (the case where $\mu=\sigma$ is the Lebesgue measure), we get

$$|\bigcup_A Q_j|=\sum_{Q_j\in A}|Q_j|\leq\frac{1}{\alpha}\sum_{Q_j\in A}\iint\limits_{Q_j}|f^*(z)-f^*_Q|d\sigma_z\leq$$
$$\leq\frac{1}{\alpha}\iint\limits_{Q}|f^*(z)-f^*_Q|d\sigma_z\leq\frac{1}{\alpha}|Q|\,\|f\|_{\mathrm{BMO}(G)}.\qquad\square$$

The proof of the following lemma can be found in [11]. Let us formulate it in a way convenient for us.

Lemma 5. *Let G be a finite domain with a quasiconformal boundary Γ, $z_0\in\Gamma$, $n,m\in N$, $n>m$. Then*

$$\Big(\frac{m}{n}\Big)^2\preccurlyeq\frac{\rho_n(z_0)}{\rho_m(z_0)}\preccurlyeq\Big(\frac{m}{n}\Big)^\beta,\tag{2.16}$$

where $\beta = \beta(G) > 0$ is a constant depending only on G.

In particular, from relation (2.16) we obtain the known inequality

$$\rho_n(z_0) \succcurlyeq \Big(\frac{1}{n}\Big)^2. \tag{2.17}$$

Lemma 6. *Let G be a finite domain with a quasi-conformal boundary Γ, $p > 1$, $w \in A_p(F(G))$, $z_0 \in \Gamma$, $u(z_0, r) = \{z : |z - z_0| < r\}$, let μ be a measure defined by equality (1.1), and let $\{\Pi_n(z)\}_{n=1}^{\infty}$ be a sequence of algebraic polynomials of order not higher than n such that*

$$\|\Pi_n\|_{z_0,p,w} \le c_1\big(\mu\{u(z_0, \rho_n(z_0))\}\big)^{\frac{1}{p}},$$

where c_1 is a constant not depending on z_0 and n.

Then for all $z \in u(z_0, \rho_n(z_0))$ we have the inequality

$$|\Pi_n'(z)| \le c_2|\rho_n(z_0)|^{-1},$$

where c_2 is a constant not depending on z_0 and n.

This lemma is the analogue of the well-known theorem on the derivatives of algebraic polynomials [12, p.420], [13] which can be proved analogously to the result of [3, p.14].

3. Proofs of the Basic Results

Proof of Theorem 1. For brevity we shall use the notation $\mu\{u(z_0, t) \cap G\} = \mu(z_0, t)$ $(t \ge 0)$.

Let us prove first the necessity. Assume that $f \in \mathrm{BMO}_p(G, w) \cap H'(G)$ and let us show that relation (1.4) holds.

Let $n \in N$, $z_0 \in \Gamma$, $Q = Q(z_0, \rho_n(z_0))$, and μ and μ^* be the measures defined by the equalities (1.1) and (2.3). Relations (2.8), (2.6), and (2.17) yield

$$\mu(Q \cap G) \succcurlyeq \mu(Q^*) \succcurlyeq [\rho_n(z_0)]^2 \succcurlyeq \big(\frac{1}{n}\big)^4,$$

but then, obviously, we shall have

$$f_{\mu,Q\cap G} \preccurlyeq \frac{1}{\mu(Q \cap G)} \iint\limits_G |f| d\mu \preccurlyeq c(f, \mu)\big(\frac{1}{n}\big)^{-4}. \tag{3.1}$$

Clearly, $f \in H_p'(G, w)$. But then, repeating the arguments (and taking into account (3.1)) cited in [2, pp. 174, 182], we can see that there exists a sequence of algebraic polynomials P_n of order not higher than n, such that

$$\|f-P_n\|_{z_0,p,w} \le c\rho_n(z_0)\Big(\mu(z_0,\rho_n(z_0))\Big)^{\frac{1}{p}} \times$$
$$\times \int\limits_{\rho_n(z_0)}^{\infty} \frac{\sigma_p(f-f_{\mu,Q\cap G},w,z_0,t)}{t^2\mu^{1/p}(z_0,t)}dt. \qquad (3.2)$$

Now let us estimate the value $\sigma_p(f-f_{\mu,Q\cap G},w,z_0,t)$ for $t\ge\rho_n(z_0)$.

Let $Q_m = Q(z_0,2^m\rho_n(z_0))$ $(m\in N)$, $Q_0=Q$. Using relations (2.8) and (2.3) for all $m\ge 1$, we get

$$\mu(Q_{m-1}\cap G)\succcurlyeq \mu^*(Q_{m-1})\succcurlyeq \mu^*(2Q_{m-1}) = \mu^*(Q_m)\ge \mu(Q_m\cap G).$$

Then, on account of (1.3), we have

$$|f_{\mu,Q_m\cap G}-f_{\mu,Q_{m-1}\cap G}|\preccurlyeq \frac{1}{\mu(Q_m\cap G)}\iint\limits_{Q_m\cap G}|f-f_{\mu,Q-m\cap G}|d\mu\le\|f\|_{\mathrm{BMO}_p(G,w)}.$$

Thus, using the Minkowsky inequality and relation (1.3) for all $k\ge 1$, we obtain

$$\sigma_p\Big(f-f_{\mu,Q\cap G},w,z_0,2^k\rho_n(z_0)\Big)\le\bigg(\iint\limits_{Q_k\cap G}|f-f_{\mu,Q_k\cap G}|^p d\mu\bigg)^{\frac{1}{p}}+$$
$$+\sum_{m=1}^{k}|f_{\mu,Q_m\cap G}-f_{\mu,Q_{m-1}\cap G}|\bigg(\iint\limits_{Q_k\cap G}d\mu\bigg)^{\frac{1}{p}}\le$$
$$\le\|f\|_{\mathrm{BMO}_p(G,w)}(1+k)\bigg(\iint\limits_{Q_k\cap G}d\mu\bigg)^{\frac{1}{p}}=$$
$$=\|f\|_{\mathrm{BMO}_p(G,w)}\left(1+\log_2\frac{2^k\rho_n(z_0)}{\rho_n(z_0)}\right)\cdot\mu^{\frac{1}{p}}(z_0,2^k\rho_n(z_0)),$$

whence, obviously,

$$\sigma_p(f-f_{\mu,Q\cap G},w,z_0,t)\le\|f\|_{\mathrm{BMO}_p(G,w)}\Big(1+\log_2\frac{t}{\rho_n(z_0)}\Big)\cdot\mu^{\frac{1}{p}}(z_0,t)$$

for all $t\ge\rho_n(z_0)$.

Consequently, owing to (3.2), we have

$$\|f-P_n\|_{z_0,p,\omega}\preccurlyeq$$
$$\preccurlyeq\|f\|_{\mathrm{BMO}_p(G,w)}\rho_n(z_0)\Big(\mu(z_0,\rho_n(z_0))\Big)^{\frac{1}{p}}\int\limits_{\rho_n(z_0)}^{\infty}\frac{\big(1+\log_2\frac{t}{\rho_n(z_0)}\big)}{t^2}dt=$$

$$= \|f\|_{\mathrm{BMO}_p(G,w)}\Big(\mu(z_0,\rho_n(z_0))\Big)^{\frac{1}{p}} \int\limits_1^{\infty} \frac{\big(1+\log_2\tau\big)}{\tau^2}d\tau \preccurlyeq$$

$$\preccurlyeq \|f\|_{\mathrm{BMO}_p(G,w)}\big(\mu\{u(z_0,\rho_n(z_0))\cap G\}\big)^{\frac{1}{p}}. \qquad \square$$

Assume now that relation (1.4) is fulfilled for some function f given in G. Then, obviously, $f \in L_p(G,w)$. Moreover, $f \in H'(G)$ if we neglect the values of the function f on the set of measure zero. Indeed, if $z \in G$ is an arbitrary point and $z^* \in \Gamma$ is a point such that $|z - z^*| = \rho(z,\Gamma)$, then it is easy to check that $u(z,\rho(z,\Gamma)) \subset G_n(z^*,c_0)$ $(c_0 \le \frac{1}{2})$.Taking into account relation (1.4) it is not difficult to prove that the polynomials P_n converge uniformly on $u(z,\frac{1}{2}\rho(z,\Gamma))$. Cleary, the limiting analytic function coincides with the functions f a.e.

Further, let $z_0 \in \Gamma$, $r > 0$, $Q = Q(z_0,r)$, and let $n \in N$ be a number such that

$$\rho_{2^{n+1}}(z_0) < r \le \rho_{2^n}(z_0). \tag{3.3}$$

Using the Minkowsky inequality, we can see that

$$\Big(\iint\limits_{Q\cap G}|f(z) - f_{\mu,Q\cap G}|^p d\mu_z\Big)^{\frac{1}{p}} \le \Big(\iint\limits_{Q\cap G}|f(z) - P_{2^n}(z)|^p d\mu_z\Big)^{\frac{1}{p}} +$$

$$\Big(\iint\limits_{Q\cap G}|P_{2^n}(z) - f_{\mu,Q\cap G}|^p d\mu_z\Big)^{\frac{1}{p}} \overset{df}{=} I_1 + I_2.$$

By virtue of (1.4),

$$I_1 \le \mathrm{const}\, \mu^{1/p}(z_0,\rho_n(z_0)). \tag{3.4}$$

It remains to estimate I_2. Evidently,

$$I_2 \preccurlyeq \mu^{1/p}(Q\cap G)\cdot \max_{z\in Q\cap G}|P_{2^n}(z) - f_{\mu,Q\cap G}|. \tag{3.5}$$

Then it is obvious that for all $z \in Q\cap G$

$$|P_{2^n}(z) - f_{\mu,Q\cap G}| \le \frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f(\zeta) - P_{2^n}(z)|d\mu_\zeta \le$$

$$\le \frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f(\zeta) - P_{2^n}(\zeta)|d\mu_\zeta +$$

$$+ \max_{\zeta\in Q\cap G}|P_{2^n}(\zeta) - P_{2^n}(z)| \overset{df}{=} I_2' + I_2''.$$

Using the Hölder inequality and relation (1.4), we obtain

$$I_2' \leq \frac{1}{\mu(Q\cap G)}\Big(\iint\limits_{Q\cap G}|f-P_{2^n}|^p d\mu\Big)^{\frac{1}{p}}\Big(\iint\limits_{Q\cap G}d\mu\Big)^{1-\frac{1}{p}} \preccurlyeq \text{const}\,. \tag{3.6}$$

To estimate I_1'', let us consider the polynomial

$$\Pi_{2^k}(z)=P_{2^k}(z)-P_{2^{k-1}}(z) \quad (k\geq 1).$$

By (2.15), the Minkowsky inequality and relation (1.4) imply

$$\|\Pi_{2^k}\|_{z_0,p,w}\leq \|f-P_{2^k}\|_{z_0,p,w}+\|f-P_{2^{k-1}}\|_{z_0,p,w}\preccurlyeq \mu^{1/p}(z_0,\rho_n(z_0)).$$

But then, according to Lemma 6, we have

$$|\Pi_{2^k}'(z)|\preccurlyeq|\rho_{2^k}(z_0)|^{-1} \quad \forall (z\in u(z_0,\rho_{2^k}(z_0)), \quad k\geq 1).$$

Hence, taking into account (2.15), we obtain

$$\Big|P_{2^n}(\zeta)-P_{2^n}(z)\Big|=\Big|\big(P_1(\zeta)-P_1(z)\big)+$$
$$+\sum_{k=1}^{n}\Big(\Pi_{2^k}(\zeta)-\Pi_{2^k}(z)\Big)\Big|\preccurlyeq|\zeta-z|+\sum_{k=1}^{n}\int\limits_{[\zeta,z]}\Big|\Pi_{2^k}'(\xi)\Big||d\xi|\preccurlyeq$$
$$\preccurlyeq|\zeta-z|\Big(1+\sum_{k=1}^{n}\big|\rho_{2^k}(z_0)\big|^{-1}\Big)\preccurlyeq\Big(\rho_{2^n}(z_0)+\sum_{k=1}^{n}\frac{\rho_{2^n}(z_0)}{\rho_{2^k}(z_0)}\Big)\preccurlyeq$$
$$\preccurlyeq\Big(\rho_{2^n}(z_0)+\sum_{k=1}^{n}\big(\frac{1}{2^{n-k}}\big)^{\beta}\Big)\leq\text{const}$$

for all $z,\zeta\in Q\cap G$.

This means that $I_2''\leq$ const.

But then, taking into account (3.6) and (3.5), we get

$$I_2\leq \text{const}\,\mu^{1/p}(z_0,\rho_n(z_0)),$$

which, with regard to (3.4), completes the proof of Theorem 1. □

Proof of Theorem 2. Let us prove first that (b)⇒(a). Let $\omega\in A_p(F(G))$ $(1<p<\infty)$, $f\in \mathrm{BMO}_p(G,\omega)$ and let us show that $f\in\mathrm{BMO}(G)$.

Indeed,

$$\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G}|f-f_{Q\cap G}|d\sigma\leq$$
$$\leq\frac{2}{|Q\cap G|}\Big(\iint\limits_{Q\cap G}|f-f_{\mu,Q\cap G}|^p w d\sigma\Big)^{\frac{1}{p}}\Big(\iint\limits_{Q\cap G}w^{-\frac{1}{p-1}}d\sigma\Big)^{\frac{p-1}{p}}\preccurlyeq$$

$$\preccurlyeq \Big(\frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f-f_{\mu,Q\cap G}|^p d\mu\Big)^{\frac{1}{p}}\times$$

$$\times\Big(\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G} w\Big)^{\frac{1}{p}}\Big(\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G} w^{-\frac{1}{p-1}}\Big)^{\frac{p-1}{p}} \le \text{const}\,\|f\|_{\mathrm{BMO}(G)}.$$

It remains to prove that (a)$\Rightarrow$(b).

Let $f\in \mathrm{BMO}(G)$ be an analytic function in the domain G, let f^* be a function defined by the equality (2.9), $\omega\in A_P(F(G))$ $(1<p<\infty)$, and let μ^* be the measure defined by (2.3). We prove first that for all $Q\in F(G)$ and $\lambda>0$ the relation

$$\frac{1}{\mu^*(Q)}\mu^*\big\{z\in Q: |f^*(z)-f^*_Q|>\lambda\big\}\le C\exp\big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\big) \tag{3.7}$$

holds, where C and c are the constants independent of f^*, Q, and λ.

Choose a square $Q\in F(G)$. Let δ be the number from Lemma 1, and let c^* be the constant from (2.10). Without loss of generality we can assume that $\delta<1/c^*$. In this case we can apply Lemma 4 to the function f^* and $\alpha=(1/\delta)\|f\|_{\mathrm{BMO}(G)}$. Hence we get a family of disjoint squares $A_1=\{Q_j^1 : Q_j^1\in F(G,k)\}$ such that

$$|f^*(z)-f^*_Q|\le 12\alpha$$

for all $z\in Q\setminus \underset{A_1}{\cup} Q_j^1$,

$$|f^*_{Q_j^1}-f^*_Q|<4\alpha \tag{3.8}$$

according to (2.14), and by (2.15) we have

$$|\underset{A_1}{\cup} Q_j^1| = \sum_{A_1}|Q_j^1|\le \frac{1}{\alpha}\|f\|_{\mathrm{BMO}(G)}\cdot|Q|.$$

Since $(1/\alpha)\|f\|_{\mathrm{BMO}(G)}=\delta$, by virtue of Lemma 1 we obtain

$$\mu^*\{\underset{A_1}{\cup} Q_j^1\}\le \varepsilon\mu^*(Q). \tag{3.9}$$

Applying again Lemma 4 to the function f^* and $\alpha=(1/\delta)\|f\|_{\mathrm{BMO}(G)}$, for every Q_j^1 we obtain a family of nonintersecting squares $A_2=\{Q_j^2 : Q_j^2\in F(G,k)\}$ such that each of these squares is contained in one of the Q_j^1. Thus, by (3.8) and (2.13) the relation

$$|f^*-f^*_Q|\le|f^*-f^*_{Q_j^1}|+|f^*_{Q_j^1}-f^*_Q|<12\alpha+4\alpha<2\cdot 12\alpha$$

is fulfilled on $Q\setminus \underset{A_2}{\cup} Q_j^2$, while owing to (2.14) and (3.8) we have that

$$|f^*_{Q_j^2} - f^*_Q| \leq |f^*_{Q_j^2} - f^*_{Q_j^1}| + |f^*_{Q_j^1} - f^*_Q| < 4\alpha + 4\alpha < 2 \cdot 12\alpha.$$

Finally, according to (2.15), we have

$$\Big| \underset{Q_j^2 \subset Q_j^1}{\cup} Q_j^2 \Big| = \sum_{Q_j^2 \subset Q_j^1} |Q_j^2| \leq \frac{1}{\alpha} \|f\|_{\mathrm{BMO}(G)} \cdot |Q_j^1|$$

for every Q_j^1.

Then again, by virtue of Lemma 1 and (3.9), we obtain

$$\mu^*\{\underset{A_2}{\cup} Q_j^2\} = \sum_{Q_j^1 \in A_1} \mu^*\{\underset{Q_j^2 \subset Q_j^1}{\cup} Q_j^2\} \leq \sum_{Q_j^1 \in A_1} \varepsilon \mu^*(Q_j^1) \leq \varepsilon^2 \mu^*(Q).$$

Continuing this process ad infinitum, we obtain at the step n a family of intersecting squares $A_= \{Q_j^n\}$ such that

$$|f^* - f^*_Q| \leq 12\alpha \cdot n \quad \text{a.e. in} \quad Q\setminus \underset{A_n}{\cup} Q_j^n \quad \text{and} \quad \mu^*\{\underset{A_n}{\cup} Q_j^n\} \leq \varepsilon^n \mu^*(Q).$$

Assume now that $\lambda > 12\alpha$. Let $n \geq 1$ be a natural number such that $12\alpha n < \lambda \leq 12\alpha n + 12\alpha$. Then, obviously, we shall have

$$\mu^*\big\{z \in Q : |f^*(z) - f^*_Q| > \lambda\big\} \leq \mu^*\big\{z \in Q : |f^*(z) - f^*_Q| > 12\alpha n\big\} \leq$$
$$\leq \mu^*\big\{ \underset{A_n}{\cup} Q_j^n\big\} \leq \varepsilon^n \mu^*(Q) \leq \varepsilon^{\frac{\lambda}{12\alpha} - 1} \mu^*(Q) = \frac{1}{\varepsilon} \exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big) \mu^*(Q)$$

for $c = (1/12)\delta \cdot \ln(1/\varepsilon)$.

Hence, estimate (3.7) is valid for all $\lambda > 12\alpha$. But for all $0 < \lambda \leq 12\alpha$ we, obviously, have

$$\mu^*\{z \in Q : |f^*(z) - f^*_Q| > \lambda\} \leq \mu^*(Q) = \exp\Big(\frac{c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big) \times$$
$$\times \exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big) \mu^*(Q) \leq \exp\Big(\frac{12c}{\delta}\Big) \exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big) \mu^*(Q).$$

Consequently, assuming $C = \max\Big\{\frac{1}{\varepsilon}, \exp\Big(\frac{12c}{\delta}\Big)\Big\}$, we get estimate (3.7) for all $\lambda > 0$.

Relation (3.7) with regard to (2.8) implies that

$$\frac{1}{\mu(Q \cap G)} \mu\{z \in Q \cap G : |f(z) - f^*_Q| > \lambda\} \preccurlyeq \exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big). \tag{3.10}$$

The latter relation allows us to complete the proof of Theorem 2. Indeed, using first the Minkowsky inequality and then writing the corresponding integral in terms of a distribution function, applying estimate (3.10), we obtain

$$\left(\frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f-f_{\mu,Q\cap G}|^p d\mu\right)\le\left(\frac{2}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f-f_Q^*|^p d\mu\right)^{\frac{1}{p}}=$$

$$=\left(2p\int\limits_0^\infty\lambda^{p-1}\frac{1}{\mu(Q\cap G)}\mu\{z\in Q\cap G:|f(z)-f_Q^*|>\lambda\}d\lambda\right)^{\frac{1}{p}}\preccurlyeq$$

$$\preccurlyeq\left(2p\int\limits_0^\infty\lambda^{p-1}\exp\left(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\right)d\lambda\right)^{\frac{1}{p}}\preccurlyeq c(p)\big(\|f\|_{\mathrm{BMO}(G)}\big)^{\frac{1}{p}},$$

which implies that $f\in \mathrm{BMO}_p(G,w)$. □

References

1. L. N. Chikvinidze, Approximate functions in domains with quasi-conformal boundary in weighted plane integral metrics. (Russian) *Sibirskii Mat. J.* **XXX**(1989), No. 2, 172–184; English translation: *Siberian Math. J.* **30**(1989), No. 2, 306–316.

2. L. N. Chikvinidze, On approximation of functions in domains with quasiconformal boundary in weighted plane integral metrics. (Russian) *Dokl. Akad. Nauk SSSR* **304**(1989), No. 1, 41–43; English translation: *Soviet. Math. Dokl.* **39**(1989), No. 1, 32–34.

3. L. N. Chikvinidze, Local approximation of analytic functions and their derivatives in integral metrics in domains with a quasi-conformal boundary. *Proc. A. Razmadze Math. Inst.* **101**(1992), 5–26.

4. L. V. Ahlfors, Lectures on quasiconformal mappings. *Van Nostrand, Princeton, etc.,* 1966.

5. J. B. Garnett, Bounded analytic functions. *Academic Press, New York,* 1981.

6. D. Bekolle and A. Bonami, Inegalites a poids pour le noyou de Bergman. *C.R. Acad. Sci., Paris* **AB286**(1978), No. 18, 775–778.

7. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal functions. *Trans. Amer. Math. Soc.* **165**(1972), 207–226.

8. I. M. Batchaev, Integral representations in domains with quasi-conformal boundary and some of their applications. (Russian) Abstract of Thesis for Cand. Sci. Degree. *Math. Inst. Acad. Sci. Azerb. SSR, Baku,* 1981.

9. L. N. Chikvinidze, On one variant of Hardy and Littlewood maximal function. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **86**(1987), 101–109.

10. L. N. Chikvinidze, On an analogue of the Muckenhoupt condition in domains with quasi-conformal boundary. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **88**(1989), 41–58.

11. V. V. Andrievsky, Geometrical properties of V. K. Dzyadyk domains. (Russian) *Ukrain. Math. Zh.* **33**(1981), No. 6, 723–727.

12. V. K. Dzyadyk, Introduction to the theory of uniform approximation of functions by polynomials. (Russian) *Nauka, Moscow,* 1977.

13. P. M. Tamrazov, Smoothness and polynomial approximations. (Russian) *Naukova Dumka, Kiev,* 1975.

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