GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 2, 1996, 177-200

ASYMPTOTIC BEHAVIOR OF EIGENFUNCTIONS AND EIGENFREQUENCIES OF OSCILLATION BOUNDARY VALUE PROBLEMS OF THE LINEAR THEORY OF ELASTIC MIXTURES

M. SVANADZE

ABSTRACT. The asymptotic behavior of eigenoscillation and eigenvector-function is studied for the internal boundary value problems of oscillation of the linear theory of a mixture of two isotropic elastic media.

INTRODUCTION

The wide application of composite materials has stimulated an intensive investigation of mathematical models of elastic mixtures. Many interesting results of theoretical and applied nature, presented mainly in the monographs [1–3] and the papers [4–8], have been obtained of late for these models.

Problems of the existence of frequencies of eigenoscillations are studied in [9–11] for internal boundary value problems of the diffusion and shift models of the linear theory of elastic mixtures. It is proved that by the diffusion model eigenoscillations do not arise in some composites, while in other composites there is a discrete spectrum of eigenoscillation frequencies. By the shift model all internal problems of oscillation have a discrete spectrum of eigenoscillation frequencies.

Lorentz's well-known postulate that "asymptotic distribution of eigenoscillation frequencies does not depend on a shape of the body but depends on its volume" was proved by Weyl [12, 13] for two- and three-dimensional membranes. The same formulas were obtained by Courant [14] by means

1991 *Mathematics Subject Classification.* 73C15, 73D99, 73K20, 35E05, 35J55.

Key words and phrases. Boundary value problems for elastic mixtures, asymptotic behavior of eigenfunctions and frequencies, fundamental solution, Green tensors.

1072-947X/96/0300-0177$09.50/0

of the variational method and by Carleman [15] who used the properties of Green functions and Tauber type theorems. The formula of asymptotic behavior of eigenfunctions was derived in [15] for a three-dimensional membrane.

Weyl [16] proved Lorentz's postulate for an isotropic three-dimensional elastic body and developed the law of asymptotic distribution of eigenoscillation frequencies. Plejel [17] proved this law by generalizing Carleman's method and obtained formulas for asymptotic behavior of eigenvector-functions and potential energy density. Niemeyer [18] proved the same formulas by a different method and obtained the best estimate of the second term of these formulas.

Using Plejel's method Burchuladze [19–20] proved Lorentz's postulate for isotropic, orthotropic, and anisotropic plane elastic bodies, while Dikhamindzhia [21] for two- and three-dimensional isotropic bodies in couple-stress elasticity. These authors derived asymptotic formulas for eigenoscillation and eigenvector-function frequencies.

Asymptotic formulas of eigenfrequencies and eigenfunctions for problems of electromagnetic oscillation were obtained by Müller and Niemeyer in [22, 23].

Using V. Avakumović's method asymptotic formulas of eigenfrequencies and eigenfunctions were obtained in [24, 25] for the first boundary value problem of different elliptic systems.

In this paper, Plejel's method is used to prove Lorentz's postulate for internal homogeneous oscillation boundary value problems in the shift model of the linear theory of a mixture of two isotropic elastic materials, and asymptotic formulas are derived for eigenfrequencies and eigenvector-functions.

1. Formulation of Boundary Value Problems

Let the finite domain Ω of the three-dimensional Euclidean space R^3 be bounded by the surface S, $S \in L_2(\varepsilon)$, $0 < \varepsilon \leq 1$ [26], $\overline{\Omega} = \Omega \cup S$. It will be assumed that Ω is filled with a mixture of two isotropic elastic materials [1–3, 7]. The scalar product of vectors $f = (f_1, f_2, \dots, f_k)$ and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_k)$ will be denoted by $f \cdot \varphi = \sum_{j=1}^k f_j \overline{\varphi}_j$, where $\overline{\varphi}_j$ is the complex-conjugate to the number φ_j, $f \times \varphi = \|\psi_{ml}\|_{k\times k}$, $\psi_{ml} = f_m \varphi_l$, $m, l = \overline{1, k}$, $|f| = (\sum_{l=1}^k f_l^2)^{1/2}$. The product of the matrix $A = \|A_{ml}\|_{p\times k}$ by f denotes the vector $Af = (\sum_{j=1}^k A_{1j} f_j, \sum_{k=1}^k A_{2j} f_j, \dots, \sum_{j=1}^k A_{pj} f_j)$; A^T is the transposition of the matrix A. The trace of the square matrix $A = \|A_{ml}\|_{k\times k}$ will be denoted by $\operatorname{Sp} A = \sum_{j=1}^k A_{jj}$.

In the absence of mass force, the system of stationary oscillation equations in the shift model of the linear theory of two-component elastic mixtures has the form [3, 5, 6, 8, 27]

$$\begin{aligned} & a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' - \\ & \qquad -\alpha(u'-u'') + \omega^2\rho_{11}u' - \omega^2\rho_{12}u'' = 0, \\ & c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' + \\ & \qquad +\alpha(u'-u'') - \omega^2\rho_{12}u' + \omega^2\rho_{22}u'' = 0, \end{aligned} \tag{1.1}$$

where $u'=(u_1',u_2',u_3')$, $u''=(u_1'',u_2'',u_3'')$ are partial displacements, Δ is the three-dimensional Laplace operator, ω is an oscillation frequency, $\omega>0$, $\alpha\geq 0$,

$$\begin{aligned} & a_j=\mu_j-\lambda_5, \quad b_j=\mu_j+\lambda_j+\lambda_5+\frac{(-1)^j\rho_{3-j}\alpha_2}{\rho_1+\rho_2}, \\ & \rho_{jj}=\rho_j+\rho_{12}, \quad j=1,2, \quad c=\mu_3+\lambda_5, \\ & d=\mu_3+\lambda_3+\lambda_5-\frac{\rho_1\alpha_2}{\rho_1+\rho_2}=\mu_3+\lambda_4-\lambda_5+\frac{\rho_2\alpha_2}{\rho_1+\rho_2}, \quad \alpha_2=\lambda_3-\lambda_4, \end{aligned}$$

$\mu_1,\mu_2,\mu_3,\lambda_1,\lambda_2,\dots,\lambda_5$ are elastic constants of the mixture [1, 7].

In the sequel it will be assumed that the following conditions are fulfilled [1, 7]:

$$\begin{gathered} \mu_1>0, \quad \mu_1\mu_2>\mu_3^2, \quad \lambda_1-\frac{\rho_2\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\,\mu_1>0, \quad \lambda_5\leq 0, \\ \rho_{11}>0, \quad \rho_{11}\rho_{22}>\rho_{12}^2, \\ \Big(\lambda_1-\frac{\rho_2\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\,\mu_1\Big)\Big(\lambda_2+\frac{\rho_1\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\,\mu_2\Big)> \\ >\Big(\lambda_3-\frac{\rho_1\alpha_2}{\rho_1+\rho_2}+\frac{2}{3}\,\mu_3\Big)^2. \end{gathered} \tag{1.2}$$

System (1.1) can be rewritten in the matrix form as

$$A(D_x,\omega^2)U(x)\equiv\big[A(D_x)+\alpha E_0+\omega^2E\big]U(x)=0, \tag{1.3}$$

where

$$\begin{aligned} & U=(U_1,U_2,\dots,U_6)=(u',u''), \quad x=(x_1,x_2,x_3)\in\Omega, \\ & D_x=\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\Big), \\ & A(D_x)=\begin{Vmatrix} A^{(1)}(D_x) & A^{(2)}(D_x) \\ A^{(2)}(D_x) & A^{(3)}(D_x) \end{Vmatrix}_{6\times 6}, \quad A^{(j)}(D_x)=\|A_{kl}^{(j)}(D_x)\|_{3\times 3}, \\ & A_{kl}^{(1)}(D_x)=a_1\Delta\delta_{kl}+b_1\,\frac{\partial^2}{\partial x_k\partial x_l}, \quad A_{kl}^{(2)}(D_x)=c\Delta\delta_{kl}+d\,\frac{\partial^2}{\partial x_k\partial x_l}, \end{aligned}$$

$$A^{(3)}_{kl}(D_x) = a_2\Delta\delta_{kl} + b_2\,\frac{\partial^2}{\partial x_k\partial x_l},\quad j,k,l=1,2,3,$$
$$E_0 = \begin{Vmatrix} -I & I \\ I & -I \end{Vmatrix}_{6\times 6},\quad E = \begin{Vmatrix} \rho_{11}I & -\rho_{12}I \\ -\rho_{12}I & \rho_{22}I \end{Vmatrix}_{6\times 6},\quad I = \|\delta_{kl}\|_{3\times 3},$$

δ_{kl} is the Kronecker symbol.

A vector-function U is called regular in the domain Ω if $U_j \in C^2(\Omega)\cap C^1(\overline{\Omega})$ $(j=\overline{1,6})$.

The internal homogeneous oscillation boundary value problems of the linear theory of elastic mixtures are formulated as follows:

Problem (K) (K=I, II, II, IV). Find a vector U regular in Ω satisfying system (1.3) and the homogeneous boundary condition

$$\overset{(\mathrm{K})}{B}(D_z,n(z))U(z) \equiv \lim_{\Omega\ni x\to z\in S} \overset{(\mathrm{K})}{B}(D_x,n(z))U(x) = 0,$$

where

$$\overset{(\mathrm{K})}{B}(D_x,n(z)) = \begin{cases} \mathcal{I} & \text{for K=I}, \\ \mathcal{P}(D_x,n(z)) & \text{for K=II}, \\ \mathcal{P}(D_x,n(z)) + \sigma\mathcal{I} & \text{for K=III}, \\ \begin{Vmatrix} I & I \\ \mathcal{P}^{(1)}+\mathcal{P}^{(2)} & \mathcal{P}^{(2)}+\mathcal{P}^{(3)} \end{Vmatrix}_{6\times 6}, & \text{for K=IV}, \end{cases}$$

$\mathcal{I} = \|\delta_{kl}\|_{6\times 6}$, $\sigma > 0$, $\mathcal{P}(D_x,n(z))$ is the stress operator in the theory of elastic mixtures [1],

$$\mathcal{P}(D_x,n(z)) = \begin{Vmatrix} \mathcal{P}^{(1)} & \mathcal{P}^{(2)} \\ \mathcal{P}^{(2)} & \mathcal{P}^{(3)} \end{Vmatrix}_{6\times 6},\quad \mathcal{P}^{(j)} = \|\mathcal{P}^{(j)}_{kl}\|_{3\times 3},$$
$$\mathcal{P}^{(1)}_{kl}(D_x,n(z)) = (\mu_1-\lambda_5)\delta_{kl}\frac{\partial}{\partial n} + (\mu_1+\lambda_5)n_l\frac{\partial}{\partial x_k} + \Big(\lambda_1 - \frac{\rho_2\alpha_2}{\rho_1+\rho_2}\Big)n_k\frac{\partial}{\partial x_l},$$
$$\mathcal{P}^{(2)}_{kl}(D_x,n(z)) = (\mu_3+\lambda_5)\delta_{kl}\frac{\partial}{\partial n} + (\mu_3-\lambda_5)n_l\frac{\partial}{\partial x_k} + \Big(\lambda_3 - \frac{\rho_1\alpha_2}{\rho_1+\rho_2}\Big)n_k\frac{\partial}{\partial x_l},$$
$$\mathcal{P}^{(3)}_{kl}(D_x,n(z)) = (\mu_2-\lambda_5)\delta_{kl}\frac{\partial}{\partial n} + (\mu_2+\lambda_5)n_l\frac{\partial}{\partial x_k} + \Big(\lambda_2 - \frac{\rho_2\alpha_2}{\rho_1+\rho_2}\Big)n_k\frac{\partial}{\partial x_l},$$

$n=(n_1,n_2,n_3)$ is the unit normal vector at a point $z\in S$.

The internal pseudooscillation boundary value problems of the linear theory of elastic mixtures are formulated as follows:

Problem $(\mathbf{K})^*_f$ (K=I, II, III, IV). Find a vector U regular in Ω satisfying the system of equations

$$A(D_x,-\varkappa^2)U(x) = 0,\quad x\in\Omega,$$

and the boundary condition

$$\overset{(\mathrm{K})}{B}(D_z, n(z))U(z) = f(z), \quad z \in S,$$

where $\varkappa > 0$, f is a given six-component vector.

We have

Lemma 1.1. *If* $f \in C^{1,\delta_1}$, $0 < \delta_1 \leq 1$, *then problem* $(\mathrm{K})_f^*$ *has a unique regular solution* (K=I,II,III,IV).

One can easily prove the uniqueness of a regular solution of problem $(\mathrm{K})_f^*$ by the Green formula [1]

$$\int_\Omega \big[U(x) \cdot A(D_x)U(x) + W(U,U)\big]\, dx = \int_S U(z) \cdot \mathcal{P}(D_z, n(z))U(z)\, d_z S, \tag{1.4}$$

where $W(U,U)$ is the doubled density of potential energy in the theory of elastic mixtures [1]. By virtue of conditions (1.2) $W(U,U)$ is a nonnegative function for an arbitrary regular vector U [1]. The existence of solutions is proved by the potential method and the theory of singular integral equations [1, 26].

We introduce the notation

$$\begin{gathered} a = a_1 + b_1, \ \ b = a_2 + b_2, \ \ c_0 = c + d, \ \ d_1 = ab - c_0^2, \\ d_2 = a_1 a_2 - c^2, \ \ \rho_0 = \rho_{11}\rho_{22} - \rho_{12}^2, \ \ \rho_3 = \rho_{11} + \rho_{22} - 2\rho_{12}, \\ q_1 = a + b + 2c_0, \ \ q_2 = a\rho_{22} + b\rho_{11} + 2c_0\rho_{12}, \\ q_3 = a_1 + a_2 + 2c, \ \ q_4 = a_1\rho_{22} + a_2\rho_{11} + 2c\rho_{12}. \end{gathered} \tag{1.5}$$

On account of (1.2) we obtain by (1.5)

$$\begin{gathered} a_j > 0, \ \ a > 0, \ \ b > 0, \ \ d_j > 0, \\ \rho_0 > 0, \ \ \rho_3 > 0, \ \ q_l > 0, \ \ j = 1,2, \ \ l = \overline{1,4}. \end{gathered} \tag{1.6}$$

Remark 1.1. If $\alpha = 0$ then (1.1) implies

$$\begin{gathered} \big(\Delta + k_1^2\big)\big(\Delta + k_2^2\big) \begin{pmatrix} \operatorname{div} u' \\ \operatorname{div} u'' \end{pmatrix} = 0, \\ \big(\Delta + k_3^2\big)\big(\Delta + k_4^2\big) \begin{pmatrix} \operatorname{rot} u' \\ \operatorname{rot} u'' \end{pmatrix} = 0, \end{gathered} \tag{1.7}$$

where k_1^2, k_2^2 and k_3^2, k_4^2 are the roots of the square equations $d_1\xi^2 - \omega^2 q_2 \xi + \omega^4 \rho_0 = 0$ and $d_2\xi^2 - \omega^2 q_4 \xi + \omega^4 \rho_0 = 0$, respectively. By virtue of conditions (1.6) we have $k_j^2 > 0$ ($j = \overline{1,4}$). Assume that $k_j > 0$ ($j = \overline{1,4}$). By (1.7) it is clear that k_1, k_2 and k_3, k_4 are the wave numbers of longitudinal and transverse waves, respectively [3]. Let

$$c_j = \omega k_j^{-1}, \quad j = \overline{1,4}. \tag{1.8}$$

The constants c_1 and c_2 will be velocities of longitudinal waves, while c_3 and c_4 the velocities of transverse waves [3]. Clearly, c_1^2, c_2^2 and c_3^2, c_4^2 are the roots of the equations

$$\rho_0\xi^2 - q_2\xi + d_1 = 0, \tag{1.9}$$

$$\rho_0\xi^2 - q_4\xi + d_2 = 0. \tag{1.10}$$

Remark 1.2. As is well known [28], by the classical theory of elasticity, in an isotropic body one longitudinal wave propagates with the velocity $v_1 = \sqrt{(\lambda+2\mu)\rho^{-1}}$, and two transverse waves with the equal velocities $v_2 = \sqrt{\mu\rho^{-1}}$ (here λ, μ are the Lamé constants, ρ is the body density). By the shift model, in mixture of two isotropic elastic materials the number of waves increases twofold [3]. For $\alpha = 0$ two longitudinal waves propagate in the mixture with the velocities c_1 and c_2, and four transverse waves with the velocities c_3 and c_4 (two pairs of waves having equal velocities) [3].

2. Fundamental Solution Matrix and Some of Its Properties

The fundamental solution matrix of the equation $A(D_x, -\varkappa^2)U(x) = 0$ is constructed in [27] in terms of elementary functions. It has the form

$$\begin{gathered}\Gamma(x,-\varkappa^2) = \begin{Vmatrix}\Gamma^{(1)}(x,-\varkappa^2) & \Gamma^{(2)}(x,-\varkappa^2)\\ \Gamma^{(2)}(x,-\varkappa^2) & \Gamma^{(3)}(x,-\varkappa^2)\end{Vmatrix}_{6\times 6},\\ \Gamma^{(j)} = \|\Gamma_{kl}^{(j)}\|_{3\times 3}, \quad j=1,2,3,\end{gathered} \tag{2.1}$$

$$\begin{aligned}\Gamma_{kl}^{(1)}(x,-\varkappa^2) =& \Big\{\frac{1}{d_2}(a_2\Delta - \beta_3)(\Delta-\varkappa_1^2)(\Delta-\varkappa_2^2)\delta_{kl} + \\ &+ \big(\overset{(1)}{r_1}\Delta^2 + \overset{(1)}{r_2}\Delta + \overset{(1)}{r_3}\big)\frac{\partial^2}{\partial x_k \partial x_l}\Big\}\gamma,\\ \Gamma_{kl}^{(2)}(x,-\varkappa^2) =& \Big\{-\frac{1}{d_2}(c\Delta + \beta_2)(\Delta-\varkappa_1^2)(\Delta-\varkappa_2^2)\delta_{kl} + \\ &+ \big(\overset{(2)}{r_1}\Delta^2 + \overset{(2)}{r_2}\Delta + \overset{(2)}{r_3}\big)\frac{\partial^2}{\partial x_k \partial x_l}\Big\}\gamma,\\ \Gamma_{kl}^{(3)}(x,-\varkappa^2) =& \Big\{\frac{1}{d_2}(a_1\Delta - \beta_1)(\Delta-\varkappa_1^2)(\Delta-\varkappa_2^2)\delta_{kl} + \\ &+ \big(\overset{(3)}{r_1}\Delta^2 + \overset{(3)}{r_2}\Delta + \overset{(3)}{r_3}\big)\frac{\partial^2}{\partial x_k \partial x_l}\Big\}\gamma,\\ &k,l = 1,2,3,\end{aligned} \tag{2.2}$$

where

$$\beta_1=\alpha+\varkappa^2\rho_{11},\quad \beta_2=\alpha+\varkappa^2\rho_{12},\quad \beta_3=\alpha+\varkappa^2\rho_{22},$$
$$\overset{(1)}{r}_1=\frac{b}{d_1}-\frac{a_2}{d_2},\quad \overset{(2)}{r}_1=-\frac{c_0}{d_1}+\frac{c}{d_2},\quad \overset{(3)}{r}_1=\frac{a}{d_1}-\frac{a_1}{d_2},$$
$$\overset{(1)}{r}_2=\frac{1}{d_1d_2}\big[2\beta_2(a_2d-cb_2)+\beta_3(2a_2b_1-2cd-b_1b_2-d^2)\big],$$
$$\overset{(2)}{r}_2=\frac{1}{d_1d_2}\big[\beta_1(a_1d-b_2c)+\beta_2(a_2b_1-2cd+ab_2-d^2)+\beta_3(a_1d-b_1c)\big],$$
$$\overset{(3)}{r}_2=\frac{1}{d_1d_2}\big[\beta_1(2a_1b_2-2cd+b_1b_2-d^2)+2\beta_2(a_1d-b_1c)\big],$$
$$\overset{(1)}{r}_3=-\frac{1}{d_1d_2}\big(b_1\beta_3^2+2d\beta_2\beta_3+b_2\beta_2^2\big),$$
$$\overset{(2)}{r}_3=-\frac{1}{d_1d_2}\big(b_1\beta_2\beta_3+b_2\beta_1\beta_2+d\beta_1\beta_3+d\beta_2^2\big),$$
$$\overset{(3)}{r}_3=-\frac{1}{d_1d_2}\big(b\beta_2^2+2d\beta_1\beta_2+b_2\beta_1^2\big),\quad \gamma=\sum_{j=1}^{4}\eta_j\gamma_j,$$
$$\eta_j=\prod_{\substack{l=1\\ l\neq j}}^{4}(\varkappa_j^2-\varkappa_l^2)^{-1},\ \ \gamma_j(x,-\varkappa^2)=-\frac{e^{-\varkappa_j|x|}}{4\pi|x|},\ \ j=\overline{1,4},$$

$\varkappa_1^2$, $\varkappa_2^2$ and $\varkappa_3^2$, $\varkappa_4^2$ are, respectively, the roots of the following square equations

$$d_1\xi^2-(a\beta_3+b\beta_1+2c_0\beta_2)\xi+\beta_1\beta_3-\beta_2^2=0$$

and

$$d_2\xi^2-(a_1\beta_3+a_2\beta_1+2c\beta_2)\xi+\beta_1\beta_3-\beta_2^2=0.$$

By virtue of (1.6) we have $\varkappa_j^2>0$ $(j=\overline{1,4})$. Assume that $\varkappa_j>0$ $(j=\overline{1,4})$ and introduce the notation

$$\begin{aligned}
&\overset{(0)}{q}_j=\big[d_1\varkappa_j^2(\varkappa_j^2-\varkappa_{3-j}^2)\big]^{-1},\ \ \overset{(0)}{q}_l=\big[d_2\varkappa_l^2(\varkappa_l^2-\varkappa_{7-l}^2)\big]^{-1},\\
&\overset{(1)}{q}_j=(b\varkappa_j^2-\beta_3)\overset{(0)}{q}_j,\ \ \overset{(2)}{q}_j=-(c_0\varkappa_j^2+\beta_2)\overset{(0)}{q}_j,\ \ \overset{(3)}{q}_j=(a\varkappa_j^2-\beta_1)\overset{(0)}{q}_j,\\
&\overset{(1)}{q}_l=(a_2\varkappa_j^2-\beta_3)\overset{(0)}{q}_l,\ \ \overset{(2)}{q}_l=-(c\varkappa_l^2+\beta_2)\overset{(0)}{q}_l,\ \ \overset{(3)}{q}_l=(a_1\varkappa_l^2-\beta_1)\overset{(0)}{q}_l,\\
&p_k=\rho_{11}\overset{(1)}{q}_k-2\rho_{12}\overset{(2)}{q}_k+\rho_{22}\overset{(3)}{q}_k,\ \ j=1,2,\ \ l=3,4,\ \ k=\overline{1,4}.
\end{aligned}\tag{2.3}$$

We have

Lemma 2.1. *The matrix $\Gamma(x,-\varkappa^2)$ has the following properties:*
(a) $\Gamma^T(x,-\varkappa^2)=\Gamma(x,-\varkappa^2)$;
(b) $\Gamma(-x,-\varkappa^2)=\Gamma(x,-\varkappa^2)$;

(c) *$A(D_x,-\varkappa^2)\Gamma(x,-\varkappa^2)=\delta(x)\mathcal{I}$, where $\delta(x)$ is the Dirac distribution function;*

(d) *for $x\neq 0$ the matrices $\Gamma^{(1)}$, $\Gamma^{(2)}$, $\Gamma^{(3)}$ have the form*

$$\Gamma^{(k)}(x,-\varkappa^2)=\operatorname{grad}\operatorname{div}\sum_{j=1}^{2}\overset{(k)}{q_j}\gamma_j(x,-\varkappa^2)-$$
$$-\operatorname{rot}\operatorname{rot}\sum_{l=3}^{4}\overset{(k)}{q_l}\gamma_l(x,-\varkappa^2),\quad k=1,2,3; \tag{2.4}$$

(e) $$\lim_{x\to 0}\operatorname{Sp}\big\{\big[\Gamma(x,-\varkappa^2)-\Gamma(x,-\varkappa_0^2)\big]E\big\}=\frac{1}{4\pi}\,\varkappa M+O(\sqrt{\varkappa}), \tag{2.5}$$
$$\varkappa>\varkappa_0>0,\quad \varkappa\to\infty;$$

(f) $$\Big|\frac{\partial^s}{\partial x_1^{s_1}\partial x_2^{s_2}\partial x_3^{s_3}}\,\Gamma_{kl}(x,-\varkappa^2)\Big|<\frac{c^*e^{-\alpha_0\varkappa|x|}}{|x|^{1+s}},$$

where $c^*=\text{const}>0$, $s=s_1+s_2+s_3$, s_1,s_2,s_3 are nonnegative integer numbers, α_0 is a positive number not depending on $\varkappa$ and x,

$$M=c_1^{-3}+c_2^{-3}+2(c_3^{-3}+c_4^{-3}). \tag{2.6}$$

Proof. The validity of properties (a), (b), (c) is proved immediately by verification.

Let us prove property (d). Taking into account the relations

$$\Delta\gamma_j(x,-\varkappa^2)=\varkappa_j^2\gamma_j(x,-\varkappa^2),$$
$$I\gamma_j(x,-\varkappa^2)=\frac{1}{\varkappa_j^2}\big(\operatorname{grad}\operatorname{div}-\operatorname{rot}\operatorname{rot}\big)\gamma_j(x,-\varkappa^2),\quad x\neq 0,\quad j=\overline{1,4},$$

we find by (2.2) that

$$\Gamma^{(1)}(x,-\varkappa^2)=\sum_{j=1}^{4}\eta_j\Big\{\Big[\frac{1}{d_2\varkappa_j^2}\,(a_2\varkappa_j^2-\beta_3)(\varkappa_j^2-\varkappa_1^2)(\varkappa_j^2-\varkappa_2^2)+$$
$$+\overset{(1)}{r_1}\varkappa_j^4+\overset{(1)}{r_2}\varkappa_j^2+\overset{(1)}{r_3}\Big]\operatorname{grad}\operatorname{div}-$$
$$-\frac{1}{d_2\varkappa_j^2}\,(a_2\varkappa_j^2-\beta_3)(\varkappa_j^2-\varkappa_1^2)(\varkappa_j^2-\varkappa_2^2)\operatorname{rot}\operatorname{rot}\Big\}\gamma_j(x,-\varkappa^2). \tag{2.7}$$

Using the relations

$$\frac{1}{d_2\varkappa_j^2}\,(a_2\varkappa_j^2-\beta_3)(\varkappa_j^2-\varkappa_1^2)(\varkappa_j^2-\varkappa_2^2)+\overset{(1)}{r_1}\varkappa_j^4+\overset{(1)}{r_2}\varkappa_j^2+\overset{(1)}{r_3}=$$
$$=\frac{1}{d_1\varkappa_j^2}\,(b\varkappa_j^2-\beta_3)(\varkappa_j^2-\varkappa_3^2)(\varkappa_j^2-\varkappa_4^2),\quad j=\overline{1,4},$$

which are easy to verify, from (2.7) we obtain

$$\Gamma^{(1)}(x,-\varkappa^2)=\sum_{j=1}^{4}\eta_j\Big[\frac{1}{d_1\varkappa_j^2}\,(b\varkappa_j^2-\beta_3)(\varkappa_j^2-\varkappa_3^2)(\varkappa_j^2-\varkappa_4^2)\operatorname{grad}\operatorname{div}-$$
$$-\frac{1}{d_2\varkappa_j^2}\,(a_2\varkappa_j^2-\beta_3)(\varkappa_j^2-\varkappa_1^2)(\varkappa_j^2-\varkappa_2^2)\operatorname{rot}\operatorname{rot}\Big]\gamma_j(x,-\varkappa^2)=$$
$$=\sum_{j=1}^{2}\frac{1}{d_1\varkappa_j^2}\,(b\varkappa_j^2-\beta_3)(\varkappa_j^2-\varkappa_3^2)(\varkappa_j^2-\varkappa_4^2)\eta_j\operatorname{grad}\operatorname{div}\gamma_j(x,-\varkappa^2)-$$
$$-\sum_{l=3}^{4}\frac{1}{d_2\varkappa_j^2}\,(a_2\varkappa_j^2-\beta_3)(\varkappa_l^2-\varkappa_1^2)(\varkappa_l^2-\varkappa_2^2)\eta_l\operatorname{rot}\operatorname{rot}\gamma_l(x,-\varkappa^2)=$$
$$=\operatorname{grad}\operatorname{div}\sum_{j=1}^{2}\overset{(1)}{q_j}\gamma_j(x,-\varkappa^2)-\operatorname{rot}\operatorname{rot}\sum_{l=3}^{4}\overset{(1)}{q_l}\gamma_l(x,-\varkappa^2).$$

For $l=2,3$ the validity of (2.4) is proved similarly to the above.

We shall now prove property (d). Let $\Phi(x,-\varkappa^2)=\Gamma(x,-\varkappa^2)E$. By (2.1), (2.2) the matrix Φ has the form

$$\Phi=\begin{Vmatrix}\Phi^{(1)} & \Phi^{(2)}\\ \Phi^{(3)} & \Phi^{(4)}\end{Vmatrix}_{6\times6},\quad \Phi^{(l)}=\|\Phi_{kj}^{(l)}\|_{3\times3},\quad l=\overline{1,4},$$
$$\Phi^{(1)}=\rho_{11}\Gamma^{(1)}-\rho_{12}\Gamma^{(2)},\quad \Phi^{(2)}=-\rho_{12}\Gamma^{(1)}+\rho_{22}\Gamma^{(2)},$$
$$\Phi^{(3)}=\rho_{11}\Gamma^{(2)}-\rho_{12}\Gamma^{(3)},\quad \Phi^{(4)}=-\rho_{12}\Gamma^{(2)}+\rho_{22}\Gamma^{(3)}.$$

Clearly, $\operatorname{Sp}\Phi=\operatorname{Sp}(\rho_{11}\Gamma^{(1)}-2\rho_{12}\Gamma^{(2)}+\rho_{22}\Gamma^{(3)})$. Hence on account of (2.3), (2.4) we obtain

$$\operatorname{Sp}\Phi(x,-\varkappa^2)=\operatorname{Sp}\Big[\operatorname{grad}\operatorname{div}\sum_{j=1}^{2}p_j\gamma_j(x,-\varkappa^2)-\operatorname{rot}\operatorname{rot}\sum_{l=3}^{4}p_l\gamma_l(x,-\varkappa^2)\Big]=$$
$$=\Delta\Big[\sum_{j=1}^{2}p_j\gamma_j(x,-\varkappa^2)+2\sum_{l=3}^{4}p_l\gamma_l(x,-\varkappa^2)\Big]=$$
$$=\sum_{j=1}^{2}p_j\varkappa_j^2\gamma_j(x,-\varkappa^2)+2\sum_{l=3}^{4}p_l\varkappa_l^2\gamma_l(x,-\varkappa^2). \tag{2.8}$$

Using the equalities $\sum_{j=1}^{2}p_j\varkappa_j^2=q_2d_1^{-1}$, $\sum_{l=3}^{4}p_l\varkappa_l^2=q_4d_2^{-1}$, from (2.8) we have

$$\operatorname{Sp}\Phi(x,-\varkappa^2)=-\frac{1}{4\pi|x|}\Big[\sum_{j=1}^{2}p_j\varkappa_j^2+2\sum_{l=3}^{4}p_l\varkappa_l^2-p_5(\varkappa)|x|\Big]+$$

$$+O(|x|) = -\frac{1}{4\pi|x|}\Big[\frac{q_2}{d_1} + \frac{2q_4}{d_2} - p_5(\varkappa)|x|\Big] + O(|x|), \quad |x| \ll 1, \quad (2.9)$$

where $p_5(\varkappa) = \sum_{j=1}^{2} p_j \varkappa_j^3 + 2\sum_{l=3}^{4} p_l \varkappa_l^3$. The relation (2.9) readily implies

$$\lim_{x\to 0} \mathrm{Sp}\,\big[\Phi(x,-\varkappa^2) - \Phi(x,-\varkappa_0^2)\big] = \frac{1}{4\pi}\big[p_5(\varkappa) - p_5(\varkappa_0)\big], \quad \varkappa_0 > 0. \quad (2.10)$$

Let us now calculate the difference $p_5(\varkappa) - p_5(\varkappa_0)$ for $\varkappa > \varkappa_0 > 0$, $\varkappa \to \infty$. Assume that τ_1^2, τ_2^2 and τ_3^2, τ_4^2 are the roots of the square equations $d_1\xi^2 - \varkappa^2 q_2 \xi + \varkappa^4 \rho_0 = 0$ and $d\xi^2 - \varkappa^2 q_4 \xi + \varkappa^4 \rho_0 = 0$, respectively. By (1.6) we have $\tau_j^2 > 0$ $(j = \overline{1,4})$. Assuming that $\tau_j > 0$ $(j = \overline{1,4})$, we obtain $\tau_j = \varkappa c_j^{-1}$ $(j = \overline{1,4})$, where c_j $(j = \overline{1,4})$ are defined by (1.8). Taking into account the relations

$$\begin{gathered}
\tau_1^2 + \tau_2^2 = \varkappa^2 q_2 d_1^{-1}, \quad \tau_1^2 \tau_2^2 = \varkappa^4 \rho_0 d_1^{-1}, \\
\varkappa_1^2 + \varkappa_1\varkappa_2 + \varkappa_2^2 = \tau_1^2 + \tau_1\tau_2 + \tau_2^2 + O(\varkappa), \\
\tau_1 + \tau_2 = (\varkappa_1 + \varkappa_2)\big[1 + O(\varkappa^{-1/2})\big], \quad \varkappa \to \infty,
\end{gathered}$$

we find by (2.3) that

$$\begin{aligned}
\sum_{j=1}^{2} p_j \varkappa_j^3 &= \frac{1}{\varkappa_1 + \varkappa_2}\Big[\frac{q_2}{d_1}\,(\varkappa_1^2 + \varkappa_1\varkappa_2 + \varkappa_2^2) - \frac{2\varkappa^2\rho_0}{d_1} - \frac{\alpha\rho_3}{d_1}\Big] = \\
&= \frac{1}{\varkappa^2}\,\frac{\tau_1 + \tau_2}{\varkappa_1 + \varkappa_2}\,(\tau_1^3 + \tau_2^3) + \frac{\alpha\rho_3}{d_1(\varkappa_1 + \varkappa_2)} = \\
&= \varkappa(c_1^{-3} + c_2^{-3}) + O(\sqrt{\varkappa}), \quad \varkappa \to \infty.
\end{aligned}$$

Similarly, one can show that $\sum_{l=3}^{4} p_l \varkappa_l^2 = \varkappa(c_3^{-3} + c_4^{-3}) + O(\sqrt{x})$, $x \to \infty$. Therefore the equality

$$p_5(\varkappa) - p_5(\varkappa_0) = \varkappa M + O(\sqrt{\varkappa}), \quad \varkappa > \varkappa_0 > 0, \quad \varkappa \to \infty, \quad (2.11)$$

is fulfilled. Putting (2.11) in (2.10) gives (2.5).

The validity of property (e) can be proved by Plejel's method used to investigate fundamental solutions of oscillation equations of the classical theory of elasticity in [17]. □

3. Some Properties of Green Tensors

The matrix

$$\overset{(\mathrm{K})}{G}(x,y,-\varkappa^2) = -\Gamma(x-y,-\varkappa^2) + \overset{(\mathrm{K})}{g}(x,y,-\varkappa^2)$$

will be called the Green tensor of problem (K) (K=I, II, III, IV) if the matrix $\overset{(\mathrm{K})}{g}$ satisfies the homogeneous equation

$$A(D_x,-\varkappa^2)\overset{(\mathrm{K})}{g}(z,y,-\varkappa^2)=0,\quad x,y\in\Omega,$$

and the boundary condition

$$\overset{(\mathrm{K})}{B}(D_z,n(z))\overset{(\mathrm{K})}{g}(z,y,-\varkappa^2)=\overset{(\mathrm{K})}{B}(D_z,n(z))\Gamma(z-y,-\varkappa^2),\ \ z\in S,\ \ y\in\Omega.$$

The matrix $\overset{(\mathrm{K})}{g}$ (K=I, II, III, IV) is a solution of problem $(\mathrm{K})^*_f$ with a special boundary value $f=\overset{(\mathrm{K})}{B}\Gamma$. The existence and uniqueness of $\overset{(\mathrm{K})}{g}$ follows from Lemma 1.1.

Let $\widetilde{u}'=(\widetilde{u}'_1,\widetilde{u}'_2,\widetilde{u}'_3)$, $\widetilde{u}''=(\widetilde{u}''_1,\widetilde{u}''_2,\widetilde{u}''_3)$, $\widetilde{U}=(\widetilde{u}',\widetilde{u}'')$. Assume that $U,\widetilde{U}$ are the vector-functions with real components. Introduce the notation [1, 7]

$$\begin{aligned}
&\varepsilon'_{lj}=\frac12\Big(\frac{\partial u'_j}{\partial x_l}+\frac{\partial u'_l}{\partial x_j}\Big),\ \ \varepsilon''_{lj}=\frac12\Big(\frac{\partial u''_j}{\partial x_l}+\frac{\partial u''_l}{\partial x_j}\Big),\\
&\widetilde{\varepsilon}'_{lj}=\frac12\Big(\frac{\partial \widetilde{u}'_j}{\partial x_l}+\frac{\partial \widetilde{u}'_l}{\partial x_j}\Big),\ \ \widetilde{\varepsilon}'_{lj}=\frac12\Big(\frac{\partial \widetilde{u}''_j}{\partial x_l}+\frac{\partial \widetilde{u}''_l}{\partial x_j}\Big),\\
&h_{lj}=\frac12\Big(\frac{\partial u'_j}{\partial x_l}-\frac{\partial u'_l}{\partial x_j}+\frac{\partial u''_j}{\partial x_l}-\frac{\partial u''_l}{\partial x_j}\Big),\\
&\widetilde{h}_{lj}=\frac12\Big(\frac{\partial \widetilde{u}'_j}{\partial x_l}-\frac{\partial \widetilde{u}'_l}{\partial x_j}+\frac{\partial \widetilde{u}''_j}{\partial x_l}-\frac{\partial \widetilde{u}''_l}{\partial x_j}\Big),\ \ l,j=1,2,3,
\end{aligned}$$

$$\begin{aligned}
W(U,\widetilde{U})=\sum_{l,j=1}^{3}\Big[&(\lambda_1\varepsilon'_{ll}+\lambda_3\varepsilon''_{ll})\widetilde{\varepsilon}'_{jj}+2\mu_1\varepsilon'_{lj}\widetilde{\varepsilon}'_{lj}+2\mu_3\varepsilon''_{lj}\widetilde{\varepsilon}'_{lj}+\\
&+(\lambda_4\varepsilon'_{ll}+\lambda_2\varepsilon''_{ll})\widetilde{\varepsilon}''_{jj}+2\mu_3\varepsilon'_{lj}\widetilde{\varepsilon}''_{lj}+2\mu_2\varepsilon''_{lj}\widetilde{\varepsilon}''_{lj}+\\
&+\frac{\alpha_2}{\rho_1+\rho_2}(\rho_2\varepsilon'_{ll}+\rho_1\varepsilon''_{ll})(\widetilde{\varepsilon}''_{jj}-\widetilde{\varepsilon}'_{jj})-2\lambda_5h_{lj}\widetilde{h}_{lj}\Big].
\end{aligned}$$

Consider the functions

$$\mathcal{L}[U,\widetilde{U}]=\int_\Omega\big[W(U,\widetilde{U})-\alpha UE_0\widetilde{U}+\varkappa^2UE\widetilde{U}\big]\,dx, \tag{3.1}$$

$$\overset{(\mathrm{K})}{L}[U]=\begin{cases}\int_\Omega[W(U,U)-\alpha UE_0\widetilde{U}+\varkappa^2UE\widetilde{U}]dx\equiv L[U] & \text{for K=I,}\\ L[U]-2\int_S U(z)\cdot\mathcal{P}\Gamma_j(z-y,-\varkappa^2)d_zS & \text{for K=II,}\\ \overset{(\mathrm{II})}{L}[U]+\sigma\int_S[U(z)-\Gamma_j(z-y,-\varkappa^2)]^2d_zS & \text{for K=III,}\\ L[U]-2\int_S u''(z)\cdot\widetilde{\mathcal{P}}\Gamma_j(z-y,-\varkappa^2)d_zS & \text{for K=IV,}\end{cases} \tag{3.2}$$

where Γ_j is the jth column of the matrix Γ, j is some fixed number, $j=\overline{1,6}$, $\widetilde{\mathcal{P}}=\|\mathcal{P}^{(1)}+\mathcal{P}^{(2)},\ \mathcal{P}^{(2)}+\mathcal{P}^{(3)}\|_{6\times 6}$, $\sigma>0$.

It is assumed that

$$\overset{(\mathrm{K})}{R}=\big\{U: U\in C^2(\Omega)\cap C^1(\overline{\Omega}),\ \overset{(\mathrm{K})}{B}U(z)= \\ =\overset{(\mathrm{K})}{B}\Gamma_j(z-y,-\varkappa^2),\ z\in S,\ y\in\Omega\big\} \tag{3.3}$$

is the set of admissible vector-functions for the functional $\overset{(\mathrm{K})}{L}$ (K=I,II,III,IV).

Let [17]

$$\widetilde{\Gamma}(x-y,-\varkappa^2)=\Big\{1-\Big[1-\Big(\frac{r}{\rho_y(x)}\Big)^m\Big]^l\Big\}\Gamma(x-y,-\varkappa^2), \tag{3.4}$$

where $r=|x-y|$, $\rho_y(x)=\max\{r,\ell_y\}$, ℓ_y is the distance from the point y to S, while m, l are natural numbers.

The following lemmas hold.

Lemma 3.1. *The matrix $\widetilde{\Gamma}$ has the properties:*

a) $\widetilde{\Gamma}_j\in\overset{(\mathrm{K})}{R}$, $j=\overline{1,6}$, K=I,II,III,IV*;*

b) $\widetilde{\Gamma}_j(x-y,-\varkappa^2)=\Gamma_j(x-y,-\varkappa^2)$ *for* $r\geq\ell_y$*;*

c) $\Big|\dfrac{\partial^s}{\partial x_1^{s_1}\partial x_2^{s_2}\partial x_3^{s_3}}\widetilde{\Gamma}_{kj}(x-y,-\varkappa^2)\Big|<\dfrac{\text{const}}{\ell_y^m}e^{-\alpha_0\varkappa r}r^{m-s-1}$ *for* $r<\ell_y$,

$m\geq s+1$, $l\geq s+1$, $s=s_1+s_2+s_3$, s_1,s_2,s_3 *are negative numbers,* α_0 *is a positive number not depending on* $\varkappa$, x, y, *and* $\widetilde{\Gamma}_j$ *is the* j*th column of the matrix* $\widetilde{\Gamma}$, $j,k=\overline{1,6}$.

The validity of properties (a) and (b) immediately follows from (3.3), (3.4). Property (c) is easily proved by virtue of Lemma 2.1 and formula (3.4).

Lemma 3.2. *If* U *and* $\widetilde{U}$ *are vector-functions, regular in* Ω, *then the functionals* $\mathcal{L}$ *and* L *have the properties:*

(a) $L[U]\geq 0$. *If* $L[U]=0$, *then* $U(x)\equiv 0$, $x\in\Omega$*;*

(b) $\mathcal{L}[U,\widetilde{U}]=\mathcal{L}[\widetilde{U},U]$*;*

(c) $\mathcal{L}[U,U]=L[U]$*;*

(d) $L[U+\widetilde{U}]=L[U]+2\mathcal{L}[U,\widetilde{U}]+L[\widetilde{U}]$*;*

(e) $$\int_\Omega\widetilde{U}\cdot A(D_x,-\varkappa^2)U\,dx+\mathcal{L}[U,\widetilde{U}]=\int_S\widetilde{U}\cdot\mathcal{P}U\,dS; \tag{3.5}$$

(f) $$\mathcal{L}[\widetilde{U},\overset{(\mathrm{K})}{g}_j]=\int_S\widetilde{U}\cdot\mathcal{P}\overset{(\mathrm{K})}{g}_j\,dS,\ \text{K=I,II,III,IV}; \tag{3.6}$$

(g) $\mathcal{L}[\widetilde{U},\overset{(\mathrm{I})}{g}_j]=0$ *for* $\widetilde{U}=U-\overset{(\mathrm{I})}{g}_j$, $U\in\overset{(\mathrm{I})}{R}$, $j=\overline{1,6}$.

Proof. The validity of properties (a), (b), (c), (d) is obvious by (3.1). Relation (3.5) follows from the Green formula [1]

$$\int_\Omega \big[\widetilde{U}\cdot A(D_x)U + W(U,\widetilde{U})\big]\,dx = \int_S \widetilde{U}\cdot \mathcal{P}U\,dS.$$

If $U = \overset{(\mathrm{K})}{g_j}$, then from (3.5) we obtain equality (3.6). If $U \in \overset{(\mathrm{I})}{R}$, then the vector $\widetilde{U} = U - \overset{(\mathrm{I})}{g_j}$ satisfies the boundary condition $\widetilde{U}(z) = 0$, $z \in S$. Therefore (3.6) implies $\mathcal{L}[\widetilde{U}, \overset{(\mathrm{I})}{g_j}] = 0$. □

Lemma 3.3. *The functional* $\overset{(\mathrm{K})}{L}$ *has a minimum value only on the vector* $\overset{(\mathrm{K})}{g_j}$, *i.e.,*

$$\min_{U\in \overset{(\mathrm{K})}{R}} \overset{(\mathrm{K})}{L}[U] = \overset{(\mathrm{K})}{L}[\overset{(\mathrm{K})}{g_j}],\quad \mathrm{K} = \mathrm{I,II,III,IV}. \tag{3.7}$$

Proof. If $U \in \overset{(\mathrm{I})}{R}$, $\widetilde{U} \equiv U - \overset{(\mathrm{I})}{g_j}$, then by Lemma 3.2 we have

$$L[U] = [L[\overset{(\mathrm{I})}{g_j} + \widetilde{U}] = L[\overset{(\mathrm{I})}{g_j}] + 2\mathcal{L}[\overset{(\mathrm{I})}{g}_j, \widetilde{U}] + L[\widetilde{U}] =$$
$$= L[\overset{(\mathrm{I})}{g_j}] + L[\widetilde{U}] \geq L[\overset{(\mathrm{I})}{g_j}].$$

Let $U \in \overset{(\mathrm{III})}{R}$, $g_j = \overset{(\mathrm{III})}{g_j}$, $\widetilde{U} = U - g_j$. Then

$$\overset{(\mathrm{III})}{L}[U] = L[g_j] + 2\mathcal{L}[g_j, \widetilde{U}] + L[\widetilde{U}] - 2\int_S g_j \mathcal{P}\Gamma_j dS - 2\int_S \widetilde{U}\cdot \mathcal{P}\Gamma_j dS +$$
$$+\sigma\int_S [g_j - \Gamma_j]^2 dS + 2\sigma\int_S (g_j - \Gamma_j)\cdot \widetilde{U} dS + \sigma\int_S \widetilde{U}^2 dS. \tag{3.8}$$

By virtue of (3.2), (3.6) and the boundary condition $\mathcal{P}g_j + \sigma g_j = \mathcal{P}\Gamma_j + \sigma\Gamma_j$ we find from (3.8) that

$$\overset{(\mathrm{III})}{L}[U] \equiv \overset{(\mathrm{III})}{L}[g_j] + L[\widetilde{U}] + \sigma\int_S \widetilde{U}^2 dS \geq \overset{(\mathrm{III})}{L}[g_j].$$

Quite similarly, equality (3.7) is proved for K=II,IV. Property (a) of Lemma 3.2 implies that the functional in the set $\overset{(\mathrm{K})}{R}$ has a minimum only on $\overset{(\mathrm{K})}{g_j}$ (K=I,II,III,IV). □

Lemma 3.4. *The vector-functions* $\overset{(\mathrm{I})}{g_j}$, $\overset{(\mathrm{II})}{g_j}$, $\overset{(\mathrm{III})}{g_j}$, $\overset{(\mathrm{IV})}{g_j}$ *satisfy the following relations:*

$$\text{(a)}\quad \overset{\text{(II)}}{L}[\overset{\text{(III)}}{g_j}] \le \overset{\text{(III)}}{L}[\overset{\text{(III)}}{g_j}], \tag{3.9}$$

$$\text{(b)}\quad \overset{\text{(I)}}{g}_{jj}(y,y,-\varkappa^2) = -L[\overset{\text{(I)}}{g_j}]+ \\ +\int_S \Gamma_j(z-y,-\varkappa^2)\cdot \mathcal{P}\Gamma_j(z-y,-\varkappa^2)\,d_zS; \tag{3.10}$$

$$\text{(c)}\quad \overset{\text{(K)}}{g}_{jj}(y,y,-\varkappa^2) = -\overset{\text{(K)}}{L}[\overset{\text{(K)}}{g_j}]- \\ -\int_S \Gamma_j(z-y,-\varkappa^2)\cdot \mathcal{P}\Gamma_j(z-y,-\varkappa^2)\,d_zS,\ \text{K=II,III}; \tag{3.11}$$

$$\text{(d)}\quad \overset{\text{(IV)}}{g}_{jj}(y,y,-\varkappa^2) = -\overset{\text{(IV)}}{L}[\overset{\text{(IV)}}{g_j}]+\int_S [\Gamma'_j P'_j - 2\Gamma''_j P'_j - \Gamma''_j P''_j]\,d_zS; \tag{3.12}$$

$$\text{(e)}\quad \overset{\text{(I)}}{g}_{jj}(y,y,-\varkappa^2) = -\overset{\text{(III)}}{L}[\overset{\text{(I)}}{g_j}]- \\ -\int_S \Gamma_j(z-y,-\varkappa^2)\cdot \mathcal{P}\Gamma_j(z-y,-\varkappa^2)\,d_zS; \tag{3.13}$$

$$\text{(f)}\quad \overset{\text{(I)}}{g}_{jj}(y,y,-\varkappa^2) \le \overset{\text{(III)}}{g}_{jj}(y,y,-\varkappa^2) \le \overset{\text{(II)}}{g}_{jj}(y,y,-\varkappa^2), \tag{3.14}$$

$$\overset{\text{(I)}}{g}_{jj}(y,y,-\varkappa^2) \le \overset{\text{(IV)}}{g}_{jj}(y,y,-\varkappa^2), \tag{3.15}$$

where $y \in \Omega$, $\Gamma'_j = (\Gamma_{1j},\Gamma_{2j},\Gamma_{3j})$, $\Gamma''_j = (\Gamma_{4j},\Gamma_{5j},\Gamma_{6j})$, $P'_j = (\mathcal{P}^{(1)} + \mathcal{P}^{(2)})\Gamma'_j$, $P''_j = (\mathcal{P}^{(2)} + \mathcal{P}^{(3)})\Gamma''_j$.

Proof. The validity of property (a) is obvious by virtue of (3.2).

(b) It is easy to show that any regular vector U can be represented in the form

$$U(y) = \int_S \big\{\overset{\text{(K)}}{G}(y,z,-\varkappa^2)\mathcal{P}U(z) - \big[\mathcal{P}\overset{\text{(K)}}{G}(z,y,-\varkappa^2)\big]^T U(z)\big\}d_zS - \\ -\int_\Omega \overset{\text{(K)}}{G}(y,z,-\varkappa^2)A(D_z,-\varkappa^2)U(z)\,dz, \\ \varkappa>0,\quad y\in\Omega,\quad \text{K=I,II,III,IV}. \tag{3.16}$$

If $U = \overset{\text{(I)}}{g_j}$, then (3.16) implies

$$\overset{\text{(I)}}{g}_{jj}(y,y,-\varkappa^2) = -\int_S \Gamma_j(z-y,-\varkappa^2)\cdot \mathcal{P}\overset{\text{(I)}}{G}_j(z,y,-\varkappa^2)\,d_zS. \tag{3.17}$$

On the other hand, by the Green formula (1.4) we have

$$\int_\Omega U\cdot A(D_x,-\varkappa^2)U\,dx + L[U] = \int_S U\cdot \mathcal{P}U\,d_zS. \tag{3.18}$$

Hence for $U = \overset{(\mathrm{K})}{g_j}$ we have

$$L[\overset{(\mathrm{K})}{g_j}] = \int_S \overset{(\mathrm{K})}{g_j}(z, y, -\varkappa^2) \cdot \mathcal{P}\overset{(\mathrm{K})}{g_j}(z, y, -\varkappa^2)\, d_z S \tag{3.19}$$

which on account of (3.17) gives

$$\begin{aligned} L[\overset{(\mathrm{I})}{g_j}] &= \int_S \overset{(\mathrm{I})}{g_j}(z, y, -\varkappa^2) \cdot \big(\mathcal{P}\overset{(\mathrm{I})}{G_j} + \mathcal{P}\Gamma_j\big)\, d_z S = \\ &= -\overset{(\mathrm{I})}{g}_{jj} + \int_S \Gamma_j(z-y, -\varkappa^2) \cdot \mathcal{P}\Gamma_j(z-y, -\varkappa^2)\, d_z S. \end{aligned}$$

(c) Formula (3.16) implies for $\overset{(\mathrm{II})}{g_j}$ that

$$\begin{aligned} \overset{(\mathrm{II})}{g_j}(y, y, -\varkappa^2) &= \int_S \overset{(\mathrm{II})}{G_j}(y, z, -\varkappa^2) \cdot \mathcal{P}\overset{(\mathrm{II})}{g_j}(z, y, -\varkappa^2)\, d_z S = \\ &= \int_S \overset{(\mathrm{II})}{G_j}(y, z, -\varkappa^2) \cdot \mathcal{P}\Gamma_j(z-y, -\varkappa^2)\, d_z S. \end{aligned}$$

By relation (3.19), from (3.2) we have

$$\begin{aligned} \overset{(\mathrm{II})}{L}[\overset{(\mathrm{II})}{g_j}] &= L[\overset{(\mathrm{II})}{g_j}] - 2\int_S \overset{(\mathrm{II})}{g_j} \cdot \mathcal{P}\Gamma_j\, d_z S = \int_S \overset{(\mathrm{II})}{g_j} \cdot \mathcal{P}\overset{(\mathrm{II})}{g_j}\, d_z S - \\ &- 2\int_S \overset{(\mathrm{II})}{g_j} \cdot \mathcal{P}\Gamma_j\, d_z S = -\int_S \overset{(\mathrm{II})}{g_j} \cdot \mathcal{P}\Gamma_j\, d_z S = \\ &= -\int_S \big(\overset{(\mathrm{II})}{G_j} + \Gamma_j\big) \cdot \mathcal{P}\Gamma_j\, d_z S = -\overset{(\mathrm{II})}{g}_{jj} - \int_S \Gamma_j \cdot \mathcal{P}\Gamma_j\, d_z S. \end{aligned}$$

The validity of formulas (3.12), (3.13), and (3.11) for K=III is proved similarly.

(f) Using Lemma 3.3 and relations (3.9) and (3.13), we find from (3.11) that

$$\begin{aligned} -\overset{(\mathrm{II})}{g}_{jj} &\le \overset{(\mathrm{II})}{L}[\overset{(\mathrm{III})}{g_j}] + \int_S \Gamma_j \cdot \mathcal{P}\Gamma_j\, dS \le \overset{(\mathrm{III})}{L}[\overset{(\mathrm{III})}{g_j}] + \int_S \Gamma_j \cdot \mathcal{P}\Gamma_j\, dS = \\ &= -\overset{(\mathrm{III})}{g} \le \overset{(\mathrm{III})}{L}[\overset{(\mathrm{I})}{g}] + \int_S \Gamma_j \cdot \mathcal{P}\Gamma_j\, dS = -\overset{(\mathrm{I})}{g}_{jj}. \end{aligned}$$

By Lemma 3.3 and equalities (3.2) and (3.10), from (3.12) we have

$$\begin{aligned} -\overset{(\mathrm{IV})}{g}_{jj} &\le \overset{(\mathrm{IV})}{L}[\overset{(\mathrm{I})}{g_j}] - \int_S \big[\Gamma'_j P'_j - 2\Gamma''_j P'_j - \Gamma''_j P''_j\big]\, dS = \\ &= L[\overset{(\mathrm{I})}{g}_j] - \int_S \big[2\Gamma''_j(P'_j + P''_j) + \Gamma'_j P'_j - 2\Gamma''_j P'_j - \Gamma''_j P''_j\big]\, dS = \end{aligned}$$

$$= L[\overset{(\mathrm{I})}{g}_j] - \int_S \left(\Gamma'_j P'_j + 2\Gamma''_j P'_j\right) dS = -\overset{(\mathrm{I})}{g}_{jj}. \qquad \square$$

Lemma 3.5. *The function* $\overset{(\mathrm{K})}{g}_{jj}$ *satisfies the relation*

$$\left|\overset{(\mathrm{K})}{g}_{jj}(y,y,-\varkappa^2)\right| < \frac{\text{const}}{\ell_y^{1+\delta}}, \quad y \in \Omega, \quad \delta > 0, \quad \text{K=I,II,III,IV}. \tag{3.20}$$

Proof. First we prove the validity of the formula

$$\int_{\mathcal{B}(y,\ell_y)} \widetilde{\Gamma}_j(x-y,-\varkappa^2)\cdot A(D_x,-\varkappa^2)\widetilde{\Gamma}_j(x-y,-\varkappa^2) \le \overset{(\mathrm{I})}{g}_{jj}(y,y,-\varkappa^2)\,dx \le$$
$$\le \int_S \Gamma_j(z-y,-\varkappa^2)\cdot \mathcal{P}\Gamma_j(z-y,-\varkappa^2)\,d_zS, \tag{3.21}$$

where $\mathcal{B}(y,\ell_y)$ is the ball with center y and radius ℓ_y.

By property (a) of Lemma 3.2 relation (3.10) implies

$$\overset{(\mathrm{I})}{g}_{jj}(y,y,-\varkappa^2)\,dx \le \int_S \Gamma_j(z-y,-\varkappa^2)\cdot \mathcal{P}\Gamma_j(z-y,-\varkappa^2)\,d_zS. \tag{3.22}$$

On the other hand, for $U=\widetilde{\Gamma}_j$ formula (3.18) gives

$$\int_\Omega \widetilde{\Gamma}_j\cdot A(D_x,-\varkappa^2)\widetilde{\Gamma}_j\,dx + L[\widetilde{\Gamma}_j] = \int_S \widetilde{\Gamma}\cdot\mathcal{P}\widetilde{\Gamma}\,dS.$$

Hence by Lemma 3.1 we have

$$L[\widetilde{\Gamma}_j] = -\int_\Omega \widetilde{\Gamma}_j\cdot A(D_x,-\varkappa^2)\widetilde{\Gamma}_j\,dx + \int_S \Gamma_j\cdot\mathcal{P}\Gamma_j\,dS. \tag{3.23}$$

By Lemma 3.3 and relation (3.23), from (3.10) we obtain

$$\overset{(\mathrm{I})}{g}_{jj}(y,y,-\varkappa^2)\,dx \ge -\overset{(\mathrm{I})}{L}[\widetilde{\Gamma}_j] + \int_\Omega \Gamma_j\cdot\mathcal{P}\Gamma_j\,dS =$$
$$= \int_{\mathcal{B}(y,\ell_y)} \widetilde{\Gamma}_j\cdot A(D_x,-\varkappa^2)\widetilde{\Gamma}_j\,dx. \tag{3.24}$$

(3.22) and (3.24) imply that (3.21) is valid.

Due to Lemma 3.1 we have

$$\begin{aligned}
&\left|\widetilde{\Gamma}_{mj}(x-y,-\varkappa^2)\right| \le \text{const}\cdot r^4\ell_y^{-5} \le \text{const}\cdot\ell_y^{-1},\\
&\left|\varkappa^2\widetilde{\Gamma}_{mj}(x-y,-\varkappa^2)\right| \le \text{const}\cdot(xr)^2 e^{-\alpha_0\varkappa r} r^2\ell_y^{-5} \le\\
&\qquad\qquad \le \text{const}\cdot\ell_y^{-3}, \quad m,j=\overline{1,6},\\
&\left|\widetilde{\Gamma}_j(x-y,-\varkappa^2)\cdot A(D_x,-\varkappa^2)\widetilde{\Gamma}_j(x-y,-\varkappa^2)\right| \le\\
&\qquad\qquad \le \text{const}\cdot r^2\ell_y^{-6} \le \text{const}\cdot\ell_y^{-4}.
\end{aligned} \tag{3.25}$$

With (3.25) taken into account, formula (3.21) readily implies

$$\big|\overset{(\mathrm{I})}{g}_{jj}(y,y,-\varkappa^2)\big| < \frac{\mathrm{const}}{\ell_y^{1+\delta}}, \quad y\in\Omega, \quad \delta>0. \tag{3.26}$$

The inequality

$$\big|\overset{(\mathrm{K})}{g}_{jj}(y,y,-\varkappa^2)\big| < \frac{\mathrm{const}}{\ell_y^{1+\delta}}, \quad y\in\Omega, \quad \delta>0, \quad \mathrm{K{=}I,II,III,IV}, \tag{3.27}$$

is proved by the method which in [17] is used to estimate the regular part of the Green tensor from the second boundary value problem of oscillation in the classical theory of elasticity.

On account of inequalities (3.14), (3.15), (3.26), (3.27) we easily find that relation (3.20) is valid. □

In a similar manner, as in the classical theory of elasticity [26], we prove

Lemma 3.6. *The Green tensor* $\overset{(\mathrm{K})}{G}$ *has the property*

$$\overset{(\mathrm{K})}{G}{}^{T}(x,y,-\varkappa^2) = \overset{(\mathrm{K})}{G}(y,x,-\varkappa^2),$$

where $x,y\in\Omega$, $\varkappa>0$, K=I,II,III,IV.

We introduce the notation

$$F = \frac{1}{\sqrt{\rho_{11}+\rho_{22}+2\sqrt{\rho_0}}}\begin{Vmatrix}(\rho_{11}+\sqrt{\rho_0})I & -\rho_{12}I\\ -\rho_{12}I & (\rho_{22}+\sqrt{\rho_0})I\end{Vmatrix}_{6\times 6}, \quad \overset{(\mathrm{K})}{\mathcal{G}} = F\overset{(\mathrm{K})}{G}F.$$

It is obvious that the equalities $F^2=E$ and

$$\overset{(\mathrm{K})}{\mathcal{G}}{}^{T}(x,y,-\varkappa^2) = \overset{(\mathrm{K})}{\mathcal{G}}(y,x,-\varkappa^2) \tag{3.28}$$

are fulfilled.

4. Asymptotic Behavior of Eigenoscillation and Eigenvector-Function Frequencies

Let U be a regular solution of problem (K) (K=I,II,III,IV). Then in (3.16) the integral through the surface S is zero. Taking into account the equality

$$A(D_x,-\varkappa_0^2)U(y) = -(\omega^2+\varkappa_0^2)EU(y), \quad \varkappa_0>0,$$

from (3.16) we obtain

$$U(x) = (\omega^2+\varkappa_0^2)\int_\Omega \overset{(\mathrm{K})}{G}(x,y,-\varkappa_0^2)EU(y)\,dy. \tag{4.1}$$

After multiplying (4.1) from the left by the matrix F, by (3.28) we obtain a system of second-kind Fredholm integral equation with symmetric kernel

$$V(x)=(\lambda^*+\varkappa_0^2)\int_\Omega \overset{(\mathrm{K})}{\mathcal{G}}(x,y,\varkappa_0^2)V(y)\,dy, \tag{4.2}$$

where $V=FU$, $\lambda^*=\omega^2$.

One can easily verify [14] that system (4.2) has a discrete spectrum of nonnegative eigenvalues. If $\overset{(\mathrm{K})}{\omega}_1,\overset{(\mathrm{K})}{\omega}_2,\dots,\overset{(\mathrm{K})}{\omega}_m,\dots$ are the frequencies of the eigenoscillations of problem (K) (with their multiplicity taken into account), and $\overset{(\mathrm{K},1)}{U},\overset{(\mathrm{K},2)}{U},\dots,\overset{(\mathrm{K},m)}{U},\dots$ are the corresponding vector-functions, then $\{\overset{(\mathrm{K})}{\omega}{}^2_m+\varkappa_0^2\}_{m=1}^\infty$ and $\{\overset{(\mathrm{K},m)}{V}\}_{m=1}^\infty$ will be a spectrum of eigenvalues and a system of eigenvector-functions of Eq. (4.2) and vice versa. It can be assumed that $0\le\overset{(\mathrm{K})}{\omega}_1\le\overset{(\mathrm{K})}{\omega}_2\le\dots,\ \overset{(\mathrm{K})}{\omega}_1>0$, $\lim_{m\to\infty}\overset{(\mathrm{K})}{\omega}_m=\infty$, and $\{\overset{(\mathrm{K},m)}{V}\}_{m=1}^\infty$ is a system of orthonormalized vectors

$$\int_\Omega \overset{(\mathrm{K},m)}{V}(y)\cdot\overset{(\mathrm{K},l)}{V}(y)\,dy=\delta_{ml},\quad m,l=1,2,3\dots.$$

Then $\{\overset{(\mathrm{K},m)}{U}\}_{m=1}^\infty$ will be a system of orthonormalized vector-functions with weight E, i.e.,

$$\int_\Omega E\overset{(\mathrm{K},m)}{U}(y)\cdot\overset{(\mathrm{K},l)}{U}(y)\,dy=\delta_{ml},\quad m,l=1,2,3\dots,\quad \mathrm{K{=}I,II,III,IV}.$$

We have

Lemma 4.1. *The matrix* $\overset{(\mathrm{K})}{\mathcal{G}}(x,y,\mu_0-\varkappa_0^2)$ *is the resolvent of the kernel* $\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa_0^2)$ *and the equality*

$$\begin{aligned}&\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa^2)-\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa_0^2)=\\&=\mu_0\int_\Omega\overset{(\mathrm{K})}{\mathcal{G}}(x,z,-\varkappa_0^2)\overset{(\mathrm{K})}{\mathcal{G}}(z,y,-\varkappa_0^2)\,dz\end{aligned}\tag{4.3}$$

is fulfilled, where $\varkappa>\varkappa_0>0$, $\mu_0=\varkappa_0^2-\varkappa^2<0$.

Proof. Let

$$\overset{(\mathrm{K})}{H}(x,y,-\varkappa^2)=\overset{(\mathrm{K})}{G}(x,y,-\varkappa^2)F-\overset{(\mathrm{K})}{G}(x,y,-\varkappa_0^2)F.$$

Then for $x\ne y$ the equalities

$$A(D_x,-\varkappa_0^2)\overset{(\mathrm{K})}{G}(x,y,-\varkappa^2)F=(\varkappa^2-\varkappa_0^2)E\overset{(\mathrm{K})}{G}(x,y,-\varkappa^2)F,$$

$$A(D_x,-\varkappa_0^2)\overset{(\mathrm{K})}{G}(x,y,-\varkappa_0^2)F=0$$

imply

$$A(D_x,-\varkappa_0^2)\overset{(\mathrm{K})}{H}(x,y,-\varkappa^2)=(\varkappa^2-\varkappa_0^2)E\overset{(\mathrm{K})}{G}(x,y,-\varkappa^2)F. \tag{4.4}$$

Moreover, $\overset{(\mathrm{K})}{H}$ satisfies the boundary condition

$$\overset{(\mathrm{K})}{B}(D_z,n(z))\overset{(\mathrm{K})}{H}(z,y,-\varkappa^2)=0,\quad z\in S,\quad y\in\Omega,$$

and the equality

$$\begin{aligned}\overset{(\mathrm{K})}{H}(x,y,-\varkappa^2)=&-\int_\Omega\overset{(\mathrm{K})}{G}(x,z,-\varkappa_0^2)\cdot A(D_x,-\varkappa_0^2)\overset{(\mathrm{K})}{H}(z,y,-\varkappa^2)\,dz+\\&+\int_S\big\{\overset{(\mathrm{K})}{G}(x,z,-\varkappa_0^2)\cdot\mathcal{P}(D_z,n)\overset{(\mathrm{K})}{H}(z,y,-\varkappa^2)-\\&-\big[\mathcal{P}(D_z,n)\overset{(\mathrm{K})}{G}(x,z,-\varkappa_0^2)\big]^T\cdot\overset{(\mathrm{K})}{H}(x,y,-\varkappa^2)\big\}d_zS.\end{aligned}\tag{4.5}$$

In (4.5) the integral over the surface S is equal to zero. Therefore by virtue of relations (4.4) and (4.5) we have

$$\overset{(\mathrm{K})}{H}(x,y,-\varkappa^2)=(\varkappa_0^2-\varkappa^2)\int_\Omega\overset{(\mathrm{K})}{G}(x,z,-\varkappa_0^2)\cdot E\overset{(\mathrm{K})}{G}(z,y,-\varkappa^2)F\,dz. \tag{4.6}$$

By multiplying equality (4.6) from the left by the matrix F we obtain relation (4.3). □

By Lemma 4.1 the matrix $\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa^2)-\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa_0^2)$ can be decomposed into the series [14]

$$\begin{aligned}&\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa^2)-\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa_0^2)=\\&=(\varkappa_0^2-\varkappa^2)\sum_{m=1}^\infty(\overset{(\mathrm{K})}{\lambda}_m+\varkappa^2)^{-1}(\overset{(\mathrm{K})}{\lambda}_m+\varkappa_0^2)^{-1}\overset{(\mathrm{K},m)}{V}(x)\times\overset{(\mathrm{K},m)}{V}(y),\end{aligned}\tag{4.7}$$

where $\overset{(\mathrm{K})}{\lambda}_m=\overset{(\mathrm{K})}{\omega}{}^2_m$, K=I,II,III,IV, $m=1,2,3\ldots$, $x,y\in\Omega$, $\varkappa>\varkappa_0>0$. (4.7) implies

$$\begin{aligned}&\lim_{x\to y}\mathrm{Sp}\big[\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa^2)-\overset{(\mathrm{K})}{\mathcal{G}}(x,y,-\varkappa_0^2)\big]=\\&=(\varkappa_0^2-\varkappa^2)\sum_{m=1}^\infty(\overset{(\mathrm{K})}{\lambda}_m+\varkappa^2)^{-1}(\overset{(\mathrm{K})}{\lambda}_m+\varkappa_0^2)^{-1}|V(y)|^2.\end{aligned}\tag{4.8}$$

Taking into account the equalities $\operatorname{Sp} \overset{(\mathrm{K})}{\mathcal{G}} = \operatorname{Sp}(\overset{(\mathrm{K})}{G} E)$ and (2.5), from (4.8) we shall obtain

$$\frac{1}{4\pi}(\varkappa_0-\varkappa)M+\operatorname{Sp}\big\{\big[\overset{(\mathrm{K})}{g}(y,y,-\varkappa^2)-\overset{(\mathrm{K})}{g}(y,y,-\varkappa_0^2)\big]E\big\}=$$
$$=(\varkappa_0^2-\varkappa^2)\sum_{m=1}^{\infty}(\overset{(\mathrm{K})}{\lambda}_m+\varkappa^2)^{-1}(\overset{(\mathrm{K})}{\lambda}_m+\varkappa_0^2)^{-1}|\overset{(\mathrm{K},m)}{V}(y)|^2. \tag{4.9}$$

By Lemma 3.5, from (4.9) we have

$$\sum_{m=1}^{\infty}(\overset{(\mathrm{K})}{\lambda}_m+\varkappa^2)^{-1}(\overset{(\mathrm{K})}{\lambda}_m+\varkappa_0^2)^{-1}|\overset{(\mathrm{K},m)}{V}(y)|^2\sim\frac{M}{4\pi\varkappa},\quad \varkappa\to\infty. \tag{4.10}$$

By the technique (a Tauber type theorem) used in [17, 19, 21], we obtain

$$\sum_{\overset{(\mathrm{K})}{\omega}_m\le t}|\overset{(\mathrm{K},m)}{V}(y)|^2\sim\frac{M}{6\pi^2}\,t^3,\quad t\to\infty. \tag{4.11}$$

By (4.11) we can write the following formula for asymptotic behavior of eigen vector-functions of problem (K):

$$\sum_{\overset{(\mathrm{K})}{\omega}_m\le t}|F\overset{(\mathrm{K},m)}{U}(y)|^2\sim\frac{M}{6\pi^2}\,t^3,\quad t\to\infty,\quad \mathrm{K{=}I,II,III,IV}. \tag{4.12}$$

Integrating equality (4.9) in the domain Ω and applying the fact that the vectors of the system $\{\overset{(\mathrm{K},m)}{V}\}_{m=1}^{\infty}$ are orthonormal, also recalling the results of Lemma 3.5, we obtain

$$\sum_{m=1}^{\infty}\big[(\overset{(\mathrm{K})}{\lambda}_m+\varkappa^2)(\overset{(\mathrm{K})}{\lambda}_m+\varkappa_0^2)\big]^{-1}=\frac{1}{4\pi}|\Omega|M\varkappa^{-1}+O(\varkappa^{-2-2\delta}), \tag{4.13}$$
$$0<\delta<\frac{1}{2},$$

where $|\Omega|$ is the volume of Ω.

Similarly, like in the classical theory of elasticity [17, 19] and the couple-stress theory of elasticity [21], from (4.13) we obtain the formula for asymptotic distribution of eigenoscillation frequencies of problem (K):

$$\overset{(\mathrm{K})}{N}(t)\sim\frac{1}{6\pi^2}|\Omega|Mt^3,\quad t\to\infty, \tag{4.14}$$

where $\overset{(\mathrm{K})}{N}(t)=\sum_{\overset{(\mathrm{K})}{\omega}_m\le t}1$ is the number of eigenoscillation frequencies $\overset{(\mathrm{K})}{\omega}_m$ not greater than t (K=I,II,III,IV).

Thus we have proved

Theorem 4.1. *For all the internal boundary value problems of the shift model of the linear theory of a mixture of two isotropic materials, the asymptotic behavior of eigenoscillation and eigenvector-function frequencies are expressed by formulas* (4.14) *and* (4.12), *respectively.*

Remark 4.1. By virtue of (4.14) the asymptotic distribution of eigenoscillation frequencies does not depend on the form of an elastic mixture, but depends on its volume. With Remark 1.2 taken into account, the number M defined by (2.6) is the sum of the inverse cubes of velocities of plane waves, propagating in a mixture of two isotropic elastic materials, for $\alpha = 0$.

Remark 4.2. The Weyl formula for the asymptotic distribution of eigenoscillation frequencies in the classical theory of elasticity for a three-dimensional isotropic body is written as [16]

$$N(t) \sim \frac{1}{6\pi^2}\, |\Omega| M_0 t^3, \quad t \to \infty, \tag{4.15}$$

where $N(t)$ is the number of eigenoscillation frequencies not greater than t, $M_0 = \rho^{3/2}[(\lambda+2\mu)^{-3/2}+2\mu^{-3/2}] = v_1^{-3}+2v_2^{-3}$. By Remark 1.2 the number M_0 is the sum of the inverse cubes of velocities of plane waves propagating in an isotropic body. Formula (4.14) is a corollary of formula (4.14). Indeed, if we consider an isotropic body as a mixture of two isotropic materials of the same kind, then the elastic constants and densities will satisfy the conditions

$$\begin{gathered} a_1 = a_2 = \mu, \quad b_1 = b_2 = \lambda + \mu, \quad c = d = 0, \\ \rho_{11} = \rho_{22} = \rho, \quad \rho_{12} = 0. \end{gathered} \tag{4.16}$$

By virtue of (1.9), (1.10), (4.16) we have $c_1 = c_2 = v_1$, $c_3 = c_4 = v_2$, $M = 2M_0$. With conditions (4.16) taken into account, system (1.1) takes the form

$$\mu\Delta u' + (\lambda + \mu)\operatorname{grad}\operatorname{div} u' + \rho\omega^2 u' = 0, \tag{4.17}$$

$$\mu\Delta u'' + (\lambda + \mu)\operatorname{grad}\operatorname{div} u'' + \rho\omega^2 u'' = 0, \tag{4.18}$$

and the boundary condition in problem (I) becomes

$$u'(z) = 0, \tag{4.19}$$

$$u''(z) = 0, \quad z \in S. \tag{4.20}$$

Obviously, problems (4.17)–(4.20) and (4.17), (4.19) have the same eigenoscillation frequencies, but the multiplicity of these frequencies will be two times higher for problem (4.17)–(4.20) than for problem (4.17), (4.19). Thus we have

$$N(t) = \frac{1}{2}\overset{(\mathrm{I})}{N}(t) \sim \frac{1}{2}\,\frac{1}{6\pi^2}\,|\Omega| M t^3 = \frac{1}{6\pi^2}\,|\Omega| M_0 t^3, \quad t \to \infty.$$

References

1. D. G. Natroshvili, A. J. Jagmaidze, and M. Zh. Svanadze, Some problems of the theory of elastic mixtures. (Russian) *Tbilisi Univ. Press, Tbilisi,* 1986.

2. L. P. Khoroshun and N. S. Soltanov, Thermoelasticity of two-component mixtures. (Russian) *Naukova Dumka, Kiev,* 1984.

3. Ya. Ya. Rushchitskii, Elements of mixture theory. (Russian) *Naukova Dumka, Kiev,* 1991.

4. I. G. Filippov, Dynamical theory of a relative flow of multicomponent media. (Russian) *Prikl. Mekhanika* **7**(1971), No. 10, 92–99.

5. B. Lempriere, On practicability of analyzing waves in composites by the theory of mixtures. *Lockheed Palo Alto Research Laboratory. Report* No. LMSC-6-78-69-21 (1969), 76–90.

6. H. D. McNiven and Y. A. Mengi, A mathematical model for the linear dynamic behavior of two-phase periodic materials. *Int. J. Solids and Struct.* **15**(1979), No. 4, 571–580.

7. T. R. Steel, Applications of a theory of interacting continua. *Quart. J. Mech. and Appl. Math.* **20**(1967), No. 1, 57–72.

8. M. Zh. Svanadze, Representation of a general solution of the equation of steady-state oscillations of two-component elastic mixtures. (Russian) *Prikl. Mekhanika* **29**(1993), No. 12, 22–29.

9. M. Zh. Svanadze, The uniqueness of solutions of stable oscillation problems of the linear theory of a two-component elastic mixture. (Russian) *Bull. Acad. Sci. Georgia* **145**(1992), No. 1, 51–54.

10. M. Zh. Svanadze, Investigation of boundary value problems of steady-state oscillations of the theory of elastic mixtures. 10*th Conference of Problems and Methods in Math. Physics, September* 13–17, 1993, *Chemnitz, Germany, Abstracts, p.* 60.

11. M. Zh. Svanadze, Uniqueness theorem for solutions of the internal boundary value problems of steady-state oscillations of the linear theory of elastic mixtures. *Proc. I. Vekua Inst. Appl. Math. Tbilisi State Univ.* **46**(1992), 179–190.

12. H. Weyl, Über die Abhängigkeit der Eigenschwingungen einer Membran von deren Begrenzung. *J. Reine und Angew. Math.* **141**(1912), 1–11.

13. H. Weyl, Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze. *J. Reine und Angew. Math.* **143**(1913), 177–202.

14. R. Courant and D. Hilbert, Methoden der mathematischen Physik. *Julius Springer, Berlin,* B.1, 2, 1931, 1937.

15. T. Carleman, Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes. *Comptes Rendus des Mathématiciens Scandinaves á Stockholm* 14–18 *Aout* 1934, 34–44, *Lund,* 1935.

16. H. Weyl, Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalteten elastischen Körpers. *Rendiconti del Circolo Matematico di Palermo* **39**(1915), 1–49.

17. A. Plejel, Propriétés asymptotiques des fonctions et valeurs propres de certains problems de vibrations. *Arkiv för Math., Astr. och Fysik.* **27A**(1940), 1–100.

18. H. Niemeyer, Über die elastischen Eigenschwingungen endlicher Körper. *Arch. Rat. Mech. Ann.* **19**(1965), 24–61.

19. T. V. Burchuladze, To the theory of boundary value problems of oscillation for an elastic body. (Russian) *Proc. Tbilisi University, Mat. Mech. Astron.* **64**(1957), 215–240.

20. T. V. Burchuladze, On the asymptotic behavior of eigenfunctions of some boundary value problems of an anisotropic elastic body. (Russian) *Bull. Acad. Sci. Georgian SSR* **23**(1959), No. 3, 265–272.

21. R. G. Dikhamindzhia, Asymptotic distribution of eigenfunctions and eigenvalues of some basic problems of vibration of the moment theory of elasticity. (Russian) *Trudy Tbiliss. Matem. Inst. Razmadze* **73**(1983), 64–71.

22. C. Müller and H. Niemeyer, Greensche Tensoren und asymptotische Gesetze der elekromagnetischen Hohlraumschwingungen. *Arch. Rat. Mech. Ann.* **7**(1961), 305–348.

23. H. Niemeyer, Eine Verschärfung der asymptotischen Gesetze elektomagnetischen Hohlraumschwingungen. *Arch. Rat. Mech. Ann.* **7**(1961), 412–433.

24. W. Gromes, Über das asymptotische Verhalten der Spektralfunktion elliptischer Systeme. *Math. Zeitschrift* **118**(1970), No. 4, 254–270.

25. R. Brübach, Über die Spektralmatrix elliptischer Systeme. *Math. Zeitschrift* **140**(1974), No. 3, 231–244.

26. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Translated from Russian) *North-Holland series in applied mathematics and mechanics,* v.25, *North-Holland Publishing Company, Amsterdam–New York–Oxford,* 1979; Russian original: *Nauka, Moscow,* 1976.

27. M. Zh. Svanadze, Fundamental solutions of equations of stable oscillation and pseodo-oscillation of a two-component elastic mixture. (Russian) *Proc. I. Vekua Inst. Appl. Math. Tbilisi State Univ.* **39**(1990), 227–240.

28. I. G. Petrashen, Elastic wave propagation in anisotropic media. (Russian) *Nauka, Leningrad,* 1980.

(Received 06.10.1994)

Author's address:
I. Vekua Institute of Applied Mathematics
Tbilisi State University
2, University St., Tbilisi 380043
Republic of Georgia