GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 2, 1996, 121-132

ON A PROBLEM OF LITTLEWOOD

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ABSTRACT. The theorem on the tending to zero of coefficients of a trigonometric series is proved when the L^1-norms of partial sums of this series are bounded. It is shown that the analog of Helson's theorem does not hold for orthogonal series with respect to the bounded orthonormal system. Two facts are given that are similar to Weis' theorem on the existence of a trigonometric series which is not a Fourier series and whose L^1-norms of partial sums are bounded.

Let $S_n(x)$, $n = 1, 2, \dots$, denote the partial sums of a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx. \tag{1}$$

It is well known (see [1,Ch.4]) that if $p > 1$ and

$$\int_0^{2\pi} |S_n(x)|^p dx = O(1), \quad n = 1, 2, \dots, \tag{2}$$

then (1) is a Fourier series.

Littlewood posed the question whether (1) is a Fourier series if

$$\int_0^{2\pi} |S_n(x)| dx = O(1), \quad n = 1, 2, \dots. \tag{3}$$

Weis [4] constructed the example of the trigonometric series which is not a Fourier series and for which property (3) is fulfilled.

Katznelson [6] strengthened this result by constructing the trigonometric series which is not a Fourier series and for which

$$S_n(x) \geq 0, \quad n = 1, 2, \dots, \quad x \in [0, 2\pi]. \tag{4}$$

1991 *Mathematics Subject Classification.* 42A16.

Key words and phrases. Tending to zero of coefficients of a trigonometric series, Riesz product, Fourier–Stieltjes series.

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Helson [5] showed that the coefficients of any series (1) with condition (3) tend to zero.

It is also well known (see [1,Ch.7]) that if (3) holds for a trigonometric series $\sum_{k=-\infty}^{+\infty} c_k e^{ikx}$, then $\frac{1}{n}\sum_{k=1}^{n}|c_k| = O\big(\frac{1}{\ln n}\big)$.

The following proposition (see [3,Ch.8]) is true: If the partial sums $S_n(x)$, $n=1,2,\dots$ of any series $\sum_{k=1}^{+\infty} a_k\varphi_k(x)$ with respect to a uniformly bounded orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$, $x\in[0,1]$, satisfy the condition $\int_0^1 |S_n(x)|dx = O(1)$, then

$$\frac{1}{n}\sum_{k=1}^{n} a_k^2 = O\Big(\frac{1}{\ln n}\Big). \tag{5}$$

There naturally arise the questions:

1. Let $\{n_k\}$ be an arbitrary increasing sequence of natural numbers. What can one say about the tendency to zero of coefficients of series (1) if the condition

$$\int\limits_0^{2\pi} |S_{n_k}(x)|dx = O(1), \quad n=1,2,\dots \tag{6}$$

is fulfilled?

2. If the partial sums $S_n(x)$, $n=1,2,\dots$, of any series $\sum_{k=1}^{+\infty} a_k\varphi_k(x)$ in terms of a uniformly bounded orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$, $x\in[0,1]$, satisfy the condition $\int_0^1 |S_n(x)|dx = O(1)$, then does this imply that $a_k \to 0$?

3. The generalized problem of Littlewood: If there is a sequence of positive numbers $\{\varepsilon_n\}$ tending to zero as $n\to\infty$ such that

$$\int\limits_0^{2\pi} |S_n(x)|^{1+\varepsilon_n} dx = O(1), \tag{7}$$

then does this always imply that (1) is a Fourier series?

4. What is a nontrivial condition which together with (3) enables one to state that (1) is a Fourier series?

5. Given a function $p(x)\ge 0$, $x\in[0,2\pi]$, $\operatorname{vrai\,inf} p(x) = 0$, does there always exist a trigonometric series (1) which is not a Fourier series such that

$$\int\limits_0^{2\pi} |S_n(x)|^{1+p(x)} dx = O(1)? \tag{8}$$

This paper gives the answers to the above-posed questions.

§ 1. On the Tending to Zero of Coefficients of Trigonometric Series

Theorem 1.1. *Let* $\{n_k\}$ *be an increasing sequence of natural numbers. Then:*

(a) *If* $n_{k+1} - n_k = O(1)$, $k \to \infty$, *and the partial sums* $S_{n_k}(x)$ *of series* (1) *satisfy condition* (6), *then the coefficients of this series tend to zero.*

(b) *If* $n_{k+1} - n_k \neq O(1)$, $k \to \infty$, *then there exists a trigonometric series whose coefficients do not tend to zero but its partial sums* $S_{n_k}(x)$, $k = 1, 2, \ldots$, *are positive* (*and the more so satisfy condition* (6)).

Proof. (a) Condition (6) implies that (1) is a Fourier–Stieltjes series and therefore its coefficients are bounded. By virtue of the condition $n_{k+1}-n_k = O(1)$, $k = 1, 2, \ldots$, (3) also holds for series (1), which fact implies by Helson's theorem that the coefficients of (1) tend to zero.

(b) Let $n_1 < n_2 < \cdots < n_k < \cdots$ and $n_{k+1} - n_k \neq O(1)$, $k = 1, 2, \ldots$. We construct a new increasing sequence of natural numbers $m_1 < m_2 < \cdots < m_p < \cdots$ in the following manner:

Let $m_1 = n_1$ and $k_1 = 1$; if $m_1 < m_2 < \cdots < m_p$ and $k_1 < k_2 < \cdots < k_p$ are already defined, then there exists $k_{p+1} > k_p$ such that $n_{k_{p+1}+1} - n_{k_{p+1}} > 4(m_1 + m_2 + \cdots + m_p)$ so that it can be assumed that $m_{p+1} = \left[\frac{n_{k_{p+1}+1}+n_{k_{p+1}}}{2}\right]$, etc.

Since by the definition of the sequence $\{m_p\}$ we have $m_{p+1} > 2(m_1+m_2+\cdots + m_p)$ for any p, performing the formal multiplication in the $\prod_{p=1}^{\infty}(1 + \cos m_p x)$ we obtain the trigonometric series $1 + \sum_{\nu=1}^{\infty} \gamma_\nu \cos \nu x$. It will be shown that for this series the conditions of the theorem are fulfilled. The fact that the coefficients of this series do not tend to zero is obvious. Further, for any n_k there exists p such that

$$n_{k_p} < m_p < n_{k_p+1} \leq n_k \leq n_{k_{p+1}} < m_{p+1} < n_{k_{p+1}+1}$$

and by the definition of $\{m_p\}$ one can easily verify that

$$m_1 + m_2 + \cdots + m_p \leq n_k < m_{p+1} - m_p - m_{p-1} - \cdots - m_1.$$

Therefore

$$S_{n_k}(x) = S_{m_1+m_2+\cdots+m_p}(x) = \prod_{j=1}^{p}(1 + \cos m_j x) \geq 0. \quad \square$$

Remark. The following statement is valid: If $\{n_k\}$ is an arbitrary increasing sequence of natural numbers and (6) holds for series (1), then $a_{n_k} \to 0$, $b_{n_k} \to 0$ as $k \to \infty$. This is proved exactly in the same way as Helson's theorem (see [5]).

Theorem 1.2. *There exists an orthogonal series*

$$\sum_{k=1}^{\infty} a_k \varphi_k(x)$$

with respect to a uniformly bounded orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$, $x \in [0,1]$, *for which*

$$\int_0^1 |S_n(x)|dx = O(1),$$

but the coefficients a_k *do not tend to zero.*

Proof. It will do to construct on $[0, 2\pi]$ an orthogonal series

$$\sum_{k=1}^{\infty} a_k \varphi_k(x)$$

such that the conditions

$$\int_0^{2\pi} |S_n(x)|dx = O(1),$$

$$\|\varphi_k(x)\|_{L^\infty} \leq M, \quad k = 1, 2, \dots,$$

$$\|\varphi_k(x)\|_{L^2} \geq a > 0, \quad k = 1, 2, \dots,$$

are fulfilled and the coefficients a_k do not tend to zero.

It is well known (see [1,Ch.5]) that there is a constant $C > 1$ such that for any natural N and $x \in [0, 2\pi]$

$$\Big| \frac{1}{\sqrt{N}} \sum_{1}^{N} e^{in \ln n} e^{inx} \Big| < C.$$

We define the trigonometric polynomial

$$P_N(x) = \mathrm{Re}\Big(\frac{1}{2C\sqrt{N}} \sum_{1}^{N} e^{in \ln n} e^{inx} \Big).$$

For any N and λ (λ is also natural) we have that conditions $|P_N(\lambda x)| < \frac{1}{2}$, $x \in [0, 2\pi]$, $\|P_N(\lambda x)\|_{L^2} = \frac{\sqrt{\pi}}{2C}$, and the absolute value of any coefficient of $P_N(\lambda x)$ does not exceed $\frac{1}{\sqrt{N}}$. The first one of these properties also holds for any partial sum of $P_N(\lambda x)$.

Construct the increasing sequences of natural numbers $N_1 < N_2 < \cdots < N_k < \cdots$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$.

Let $N_1 = 1$, $\lambda_1 = 1$. If $N_1 < N_2 < \cdots < N_{k-1}$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1}$ are already defined, then λ_k is such that $\lambda_k > 2(\lambda_1 N_1 + \lambda_2 N_2 + \cdots + \lambda_{k-1}N_{k-1})$. Define N_k in a manner such that

$$\frac{1}{\sqrt{N_k}} < 2^{-k-2}\Big\| \prod_{j=1}^{k-1}(1 - P_{N_j}(\lambda_j x))\Big\|_A^{-1}, \tag{9}$$

where $\|P(x)\|_A$ denotes the sum of absolute values of the coefficients of a trigonometric polynomial $P(x)$.

By virtue of the definition of $\{N_k\}$ and $\{\lambda_k\}$ the infinite product $\prod_{k=1}^{\infty}(1-P_{N_k}(\lambda_k x))$ gives rise to the rigonometric series

$$1 + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x + b_\nu \sin \nu x. \tag{10}$$

We shall show that the partial sums of this series are positive.

Clearly,

$$S_{\lambda_1 N_1 + \cdots + \lambda_k N_k}(x) = \prod_{j=1}^{k}(1 - P_{N_j}(\lambda_j x)) > 2^{-k}.$$

If $\lambda_1 N_1 + \cdots + \lambda_k N_k < n < \lambda_1 N_1 + \cdots + \lambda_{k+1}N_{k+1}$, then we have

$$S_n(x) = P(x) + Q(x), \tag{11}$$

where $P(x)$ is the polynomial obtained by multiplying $\prod_{j=1}^{k}(1 - P_{N_j}(\lambda_j x))$ by some partial sum of the polynomial $1 - P_{N_{k+1}}(\lambda_{k+1}x)$. Therefore

$$P(x) > 2^{-k-1}. \tag{12}$$

Here $Q(x)$ is the partial sum of the trigonometric polynomial which is obtained by multiplying $\prod_{j=1}^{k}(1 - P_{N_j}(\lambda_j x))$ by some term of the polynomial $P_{N_{k+1}}(\lambda_{k+1}x)$ (i.e., by $a_m \cos m\lambda_{k+1}x + b_m \cos m\lambda_{k+1}x$ with some m and the cofficients a_m and b_m with absolute values not exceeding $1/\sqrt{N_{k+1}}$). By (9) we conclude that

$$|Q(x)| < 2^{k-2}. \tag{13}$$

The relations (11), (12) and (13) imply that $S_n(x) > 0$ $x \in [0, 2\pi]$.

For any k, $k = 1, 2, \ldots$, in series (10) all terms of the polynomial $P_{N_k}(\lambda_k x)$ lie on the segment from the number $\lambda_1 N_1 + \cdots + \lambda_{k-1}N_{k-1}$ to $\lambda_1 N_1 + \cdots + \lambda_k N_k$. On this segment we perform the permutation of the terms of (10) in a manner such that the first term of the polynomial $P_{N_k}(\lambda_k x)$ is followed by its other terms, but we do not change the mutual arrangement of the other terms. The polynomial $P_{N_k}(\lambda_k x)$ will be treated as a term of the new orthogonal series in terms of the new orthogonal system consisting of the trigonometric series terms and the polynomial $P_{N_k}(\lambda_k x)$, $k = 1, 2, \dots$. Now, taking into account the above-mentioned properties of $P_N(\lambda x)$ and using the fact that the partial sums of the initial series (10) are positive, it is easy to show that the obtained orthogonal series satisfies the conditions indicated at the beginning of the proof. □

§ 2. Generalized Problem of Littlewood

Theorem 2.1. *For any sequence* $\{\varepsilon_n\}_{n\ge 1}$, $\varepsilon_n \ge 0$, $\varepsilon_n \to 0$, *there is a trigonometric series* (1) *which is not a Fourier series but for which condition* (7) *is fulfilled.*

Proof. First, we shall prove the theorem when $1 \ge \varepsilon_n > 0$, $n = 1, 2, \dots$, and $\varepsilon_n \downarrow 0$.

Consider the trigonometric series

$$\prod_{k=1}^{\infty}\Big(1 + \frac{1}{\sqrt{k+1}}\cos n_k x\Big) = 1 + \sum_{\nu=1}^{\infty}\gamma_\nu \cos \nu x, \tag{14}$$

where $n_1 < n_2 < \cdots < n_k < \cdots$ is some increasing sequence of natural numbers for which

$$\frac{n_{k+1}}{n_k} \ge q > 3, \quad k = 1, 2, \dots.$$

Since $\sum_{k=1}^{\infty}\big(\frac{1}{\sqrt{k+1}}\big)^2 = \infty$, then this series is the zero-series (see [1,Ch.5]) and thus cannot be a Fourier series.

Our purpose is to choose numbers $n_1 < n_2 < \cdots < n_k < \cdots$ such that (7) is fulfilled for series (14).

By the condition $\varepsilon_n \downarrow 0$ there is n_1 such that

$$\Big(1 + \frac{1}{\sqrt{2}}\Big)^{\varepsilon_{n_1}} < 2.$$

Next, we assume that $n_1, \dots, n_k$ $(k \ge 1)$ are already chosen. We choose n_{k+1} such that $\frac{n_{k+1}}{n_k} \ge 4$ and

$$\Big[\Big(1 + \frac{1}{\sqrt{2}}\Big)\cdots\Big(1 + \frac{1}{\sqrt{k+1}}\ \Big)\Big(1 + \frac{1}{\sqrt{k+2}}\Big)\Big]^{\varepsilon_{\mu_{k+1}}} < 2,$$

where $\mu_{k+1} = n_{k+1} + n_k + \cdots + n_1$, and so on.

For the partial sums $S_{\mu_k}(x)$, $k = 1, 2, \dots$, of series (14) we have

$$\int_0^{2\pi} |S_{\mu_k}(x)|^{1+\varepsilon_{\mu_k}} dx = \int_0^{2\pi} \Big[\Big(1 + \frac{1}{\sqrt{2}} \cos n_1 x\Big) \cdots$$

$$\cdots \Big(1 + \frac{1}{\sqrt{k+1}} \cos n_k x\Big)\Big]^{1+\varepsilon_{\mu_k}} dx \le \int_0^{2\pi} \Big(1 + \frac{1}{\sqrt{2}} \cos n_1 x\Big) \cdots$$

$$\cdots \Big(1 + \frac{1}{\sqrt{k+1}} \cos n_k x\Big) dx \Big[\Big(1 + \frac{1}{\sqrt{2}}\Big) \cdots \Big(1 + \frac{1}{\sqrt{k+1}}\Big)\Big]^{\varepsilon_{\mu_k}} < 4\pi.$$

Therefore

$$\int_0^{2\pi} |S_{\mu_k}(x)|^{1+\varepsilon_{\mu_k}} dx = O(1), \quad k = 1, 2, \dots. \tag{15}$$

Let us show that

$$\int_0^{2\pi} |S_{\mu_{k+1}}(x) - S_{\mu_k}(x)|^2 dx < C, \quad k = 1, 2, \dots, \tag{16}$$

where the constant C does not depend on k. We have

$$\int_0^{2\pi} |S_{\mu_{k+1}}(x) - S_{\mu_k}(x)|^2 dx = \pi \sum_{\nu=\mu_k+1}^{\mu_{k+1}} \gamma_\nu^2.$$

We can readily see that each nonzero coefficient γ_ν, $\nu = \mu_k + 1$, $\mu_k + 2, \dots, \mu_{k+1}$, has the form

$$\frac{1}{2^p} \cdot \frac{1}{\sqrt{i_1+1}} \cdot \frac{1}{\sqrt{i_2+1}} \cdots \frac{1}{\sqrt{i_p+1}} \cdot \frac{1}{\sqrt{k+2}}, \quad i_1 < i_2 < \cdots < i_p < k+1,$$

and each product of this kind is encountered among the coefficients γ_ν, $\nu = \mu_k + 1, \dots, \mu_{k+1}$, 2^p-times. Now we have

$$\int_0^{2\pi} |S_{\mu_{k+1}}(x) - S_{\mu_k}(x)|^2 dx =$$

$$= \pi\Big(\frac{1}{k+2} + \sum_{i_1=1}^{k} \frac{1}{2} \frac{1}{(i_1+1)(k+2)} + \cdots +$$

$$+ \sum_{i_1<i_2<k+1} \frac{1}{2^2} \frac{1}{(i_1+1)(i_2+1)(k+2)} + \cdots +$$

$$+\sum_{i_1<\cdots<i_k<k+1}\frac{1}{2^k}\frac{1}{(i_1+1)\cdots(i_k+1)(k+2)}\Big)<$$
$$<\pi\frac{1}{k+2}\Big(1+\Big(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k+1}\Big)+$$
$$+\Big(\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k\cdot(k+1)}\Big)+$$
$$+\cdots+\frac{1}{2\cdot 3\cdots(k+1)}\Big)=\frac{\pi}{k+2}\prod_{i=2}^{k+1}\Big(1+\frac{1}{i}\Big)=\frac{\pi}{2}.$$

Let n be any natural number. Then there is k such that $\mu_k\le n<\mu_{k+1}$. Therefore

$$S_n(x)=S_{\mu_k}(x)+(S_n(x)-S_{\mu_k}(x)),$$
$$\|S_n(x)\|_{1+\varepsilon_n}\le\|S_{\mu_k}(x)\|_{1+\varepsilon_n}+\|S_n(x)-S_{\mu_k}(x))\|_{1+\varepsilon_n}.$$

Since $\varepsilon_n\downarrow 0$ and $\varepsilon_n\le 1$, by (15) and (16) we obtain

$$\|S_n(x)\|_{1+\varepsilon_n}\le 2\pi\|S_{\mu_k}(x)\|_{1+\varepsilon_{\mu_k}}+\|S_n(x)-S_{\mu_k}(x))\|_2\cdot 2\pi\le 2\pi M,$$

where M does not depend on n. Therefore

$$\int_0^{2\pi}|S_n(x)|^{1+\varepsilon_n}dx=O(1),\quad n=1,2,\dots.$$

Let now $\{\varepsilon_n\}$ be any sequence of nonnegative numbers tending to zero. Then there is a sequence $\{\varepsilon'_n\}$, monotonically converging to zero, such that starting from he number N we have $1\ge\varepsilon'_n>0$ and $0\le\varepsilon_n<\varepsilon'_n$. Then, by virtue of the facts proved above, for this sequence there is a trigonometric series which is not a Fourier series but for whose partial sums the estimate

$$\int_0^{2\pi}|S_n(x)|^{1+\varepsilon'_n}dx<K,\quad n=1,2,\dots,\tag{17}$$

where K does not depend on n, is fulfilled.

We shall show that estimate (17) (perhaps with another constant K) remains valid for the same series and the sequence $\{\varepsilon_n\}$.

Let

$$M=\max\Big\{\int_0^{2\pi}|S_n(x)|^{1+\varepsilon_n}dx\ :\ n=0,1,\dots N-1\Big\}.\tag{18}$$

If $n \geq N$, then

$$\int_0^{2\pi} |S_n(x)|^{1+\varepsilon_n} dx = \int_{\{x\in[0,2\pi]:0\leq|S_n(x)|\leq 1\}} |S_n(x)|^{1+\varepsilon_n} dx + \\ + \int_{\{x\in[0,2\pi]:|S_n(x)>1\}} |S_n(x)|^{1+\varepsilon_n} dx < \\ < 2\pi + \int_0^{2\pi} |S_n(x)|^{1+\varepsilon'_n} dx < 2\pi + K. \tag{19}$$

If $K' = 2\pi + K + M$, on account of (18) and (19) we obtain

$$\int_0^{2\pi} |S_n(x)|^{1+\varepsilon_n} dx \leq K', \quad n = 1, 2, \ldots. \quad \square$$

Remark. Let $\overline{S}_n(x)$ denote partial sums of the trigonometric series conjugated with (1). Then the condition $\int_0^{2\pi} |\overline{S}_n(x)| dx = O(1)$ together with (3) enables us to state that series (1) is a Fourier series.

To prove the latter statement we have to show that the function

$$F(z) = F(re^{ix}) = P(r, x) + iQ(r, x),$$

where $P(r, x)$ and $Q(r, x)$ denote respectively the Abelian means of (1) and of its conjugate series, belongs to the Hardy class H.

Now one can easily verify that (1) is a Fourier series of the real part of the limiting value of $F(re^{ix})$ for $r \to 1$.

Theorem 2.2. (I) *If* $p(x) \geq 0$, $x \in [0, 2\pi]$, *and* $p(x) = 0$, $x \in (a, b) \subset [0; 2\pi]$, $a < b$, *then there exists a trigonometric series* (1) *which is not a Fourier series, but condition* (8) *holds for it.*

(II) *If condition* (8) *is fulfilled for the trigonometric series for* $p(x) = x$, *then* (1) *is a Fourier series.*

(Note that vrai inf $p(x) = 0$ for $p(x) = x$.)

Proof. (I) Let $a < a' < a'' < b'' < b' < b$. It is well known that if $F(x)$ is a continuous function of the bounded variation, the Fourier–Stieltjes series $S[dF]$ is a Fourier series if and only if $F(x)$ is absolutely continuous.

It is also well known (see [1,Ch.4]) that if series (1) satisfies condition (3), then it is a Fourier–Stieltjes series of the continuous function $F(x)$.

Clearly, if $S[dF(x)]$ satisfies condition (3), then it also satisfies condition (3) for any α. Therefore by Weis' theorem there exists a function $F(x)$ which is continuous, is a function of the bounded variation, is singular on

(a'', b''), and the Fourier–Stieltjes series of $F(x) - S[dF]$ satisfies condition (3).

On $[0, 2\pi]$ we define the function $\rho(x)$ as follows: $\rho(x) = 0$, $x \in [0, a') \cup (b', 2\pi]$, $\rho(x) = 1$, $x \in (a, b)$, and $\rho(x)$ is linear on the intervals (a', a'') and (b'', b').

Clearly $\rho(x)F(x)$ is a continuous function of the bounded variation which is singular on (a'', b''). Therefore $S[d\rho F]$ is not a Fourier series,

$$S_n(x, d\rho F) = \frac{1}{\pi} \int\limits_0^{2\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} d\rho(x+t)F(x+t). \tag{20}$$

Since $\rho(x) = 0$ for $x \in [0, a'] \cup [b', 2\pi]$, there is M such that

$$|S_n(x, d\rho F)| < M, \quad x \in [0, a] \cup [b, 2\pi], \quad n = 1, 2, \dots .$$

Then we have

$$\int\limits_0^a \left|S_n\left(x, d\frac{1}{M}\rho F\right)\right|^{1+P(x)} dx + \int\limits_b^{2\pi} \left|S_n\left(x, d\frac{1}{M}\rho F\right)\right|^{1+P(x)} dx \leq 2\pi, \tag{21}$$
$$n = 1, 2, \dots$$

Let us show that $S\left[d\frac{1}{M}\rho F\right]$ satisfies the conditions of Theorem (I). For this by virtue of (21) it is sufficient to show that

$$\int\limits_a^b \left|S_n(x, d\rho F)\right| dx = O(1), \quad n = 1, 2, \dots . \tag{22}$$

From (20) we have

$$S_n(x, d\rho F) = \frac{1}{\pi} \int\limits_0^{2\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} \rho(x+t) dF(x+t) +$$

$$+ \frac{1}{\pi} \int\limits_0^{2\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} F(x+t) d\rho(x+t) =$$

$$= \frac{1}{\pi} \int\limits_0^{2\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} (\rho(x+t) - \rho(x)) dF(x+t) +$$

$$+ \frac{1}{\pi} \rho(x) \int\limits_0^{2\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} dF(x+t) +$$

$$\frac{1}{\pi}\int_0^{2\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}F(x+t)\rho_t'(x+t)dt =$$
$$= S_n^{(1)}(x)+S_n^{(2)}(x)+S_n^{(3)}(x). \tag{23}$$

By the definition of $\rho(x)$, there is C such that $|\rho(x+t)-\rho(x)|<Ct$ for any x and t. Therefore there is a constant C_1 such that

$$\left|S_n^{(1)}(x)\right|<C_1,\quad x\in[0,2\pi],\quad n=1,2,\ldots. \tag{24}$$

By the definition of $F(x)$ we have

$$\int_0^{2\pi}\left|S_n^{(2)}(x)\right|dx = O(1),\quad n=1,2,\ldots. \tag{25}$$

Since $S_n^{(3)}(x)$, $n=1,2,\ldots$, are the partial sums of the Fourier series of functions of the bounded variation, we obtain

$$\int_0^{2\pi}\left|S_n^{(3)}(x)\right|dx = O(1),\quad n=1,2,\ldots. \tag{26}$$

The relations (23)–(26) imply (22).

(II) If the condition

$$\int_0^{2\pi}\left|S_n(x,)\right|^{1+x}dx = O(1),\quad n=1,2,\ldots. \tag{27}$$

is fulfilled for series (1), then series (1) is a Fourier–Stieljes series of the function $F(x)$ which is continuous, is the function of the bounded variation, and is defined as

$$F(x)=\lim_{j\to\infty}F_{n_j}(x)=\lim_{j\to\infty}\int_0^x S_{n_j}(t)dt,$$

for some sequence $\{n_j\}$ (see [1,Ch.4]).

Then, since for any $a\in(0,2\pi]$ we have $p(x)=x\geq a$ for $x\in[a,2\pi]$, functions $F_{n_j}(x)$ $j=1,2,\ldots$, on the segment $[a,2\pi]$ will be uniformly absolutely continuous and therefore $F(x)$ will be absolutely continuous on $[a,2\pi]$ (see [1,Ch.4]).

Thus we make the following statement: $F(x)$ is absolutely continuous on $[a,2\pi]$ for any $a\in(0,2\pi)$; moreover, $F(x)$ is a function of the bounded variation and is continuous on $[0,2\pi]$. Hence we conclude that $F(x)$ is absolutely continuous on $[0,2\pi]$. $\square$

Remark. In proving part (I) of Theorem 2.2 the obtained trigonometric series $S\left[d\frac{1}{M}\rho F\right]$ satisfying the conditions of the theorem depends only on the interval (a,b) and does not depend on the value of $p(x)$ outside (a,b).

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(Received 23.06.1994; revised version 21.09.1994)

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