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KNESER-TYPE OSCILLATION CRITERIA FOR SELF-ADJOINT TWO-TERM DIFFERENTIAL EQUATIONS

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ABSTRACT. It is proved that the even-order equation $y^{(2n)}+p(t)y=0$ is (n,n) oscillatory at ∞ if

$$\lim_{t\to\infty}(-1)^n \log t\int_t^\infty s^{2n-1}\left(p(s)+\frac{\mu_{2n}}{s^{2n}}\right)ds<-K_n,$$

where $K_n=(-1)^{n-1}\frac{1}{2}\frac{d^2}{d\lambda^2}P_{2n}(\lambda)|_{\lambda=\frac{2n-1}{2}}$, $P(\lambda)=\lambda(\lambda-1)\dots(\lambda-2n+1)$, $\mu_{2n}=P\left(\frac{2n-1}{2}\right)$.

1. INTRODUCTION

In this paper we deal with the oscillation properties of two-term differential equation of even order

$$y^{(2n)}+p(t)y=0, \tag{1.1}$$

where $t\in I=[1,\infty),\ p(t)\in C(I)$. The literature dealing with this problem is very voluminous; recall the monographs [1–5] and the references given therein.

If we study the oscillation properties of (1.1) from the point of view of the calculus of variations, the following definition plays an important role.

Definition 1.1. Two points t_1,t_2 are said to be (n,n) *conjugate relative to* (1.1) if there exists a nontrivial solution of (1.1) such that $y^{(i)}(t_1)=0=y^{(i)}(t_2)$, $i=0,\dots,n-1$.

The oscillation properties of linear equations related to this definition are studied in [3, 5], and recent references concerning this topic may be found in the survey paper [6].

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If one is interested in factorization of the differential operator on the left-hand-side of (1.1) and similar problems, another definition of disconjugacy of (1.1) introduced by Levin and Nehari has to be considered.

Definition 1.2. Equation (1.1) is said to be disconjugate on an interval $I_0 \subseteq I$ whenever any nontrivial solution of (1.1) has at most $(2n-1)$ zeros on I_0. Equation (1.1) is said to be eventually disconjugate if there exists $c \in I$ such that (1.1) is disconjugate on (c, ∞).

To distinquish between the oscillation properties defined by Definition 1.1 and the disconjugacy, oscillation, etc. defined by Definition 1.2, we shall refer to the latter as LN-disconjugacy, LN-oscillation and to the former as (n,n) disconjugacy, (n,n) oscillation, etc. Clearly, if (1.1) is LN-disconjugate on an interval $I_0 \subseteq I$ it is also (n,n)-disconjugate on this interval. In this paper the principal concern is the oscillation behavior of (1.1) in the sense of Definition 1.1, but if the function $p(t)$ does not change sign for large t, the oscillation properties of (1.1) according to Definition 1.1 are very close to that given by Definition 1.2; for more details see [1].

Recall that Kneser-type oscillation criteria for (1.1) are criteria which compare equation (1.1) with the Euler equation

$$y^{(2n)} - \frac{\mu_{2n}}{t^{2n}} y = 0, \tag{1.2}$$

where $\mu_{2n} = P_{2n}(\frac{2n-1}{2})$ and

$$P_{2n}(x) = x(x-1)\dots(x-2n+1)\,. \tag{1.3}$$

Criteria of this kind for (1.1) and a partial differential equation

$$(-\Delta)^n u + p(\mathbf{x})u = 0,$$

where $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ and Δ denotes the Laplace operator, have been studied in [7–9], among others.

The paper is organized as follows. In the next section we summarize the properties of solutions of self-adjoint, even-order, differential equations and their relation to the linear Hamiltonian systems (LHS). The main result of this paper – the Kneser-type oscillation criterion for (1.1) – is given in Section 3. Section 4 is devoted to remarks and comments concerning the results, and the last section contains some technical lemma needed in the proofs of all the statements given in this paper.

2. PRELIMINARY RESULTS

First of all, recall the basic properties of the Euler equation (1.2). The algebraic equation $P_{2n}(x) = 0$ has $2n$ real roots $x_i = i-1$, $i = 1, \dots, 2n$. The function $y = P_{2n}(x)$ has exactly n local minima and $(n-1)$ local maxima and its graph is symmetric with respect to the line $x = \frac{2n-1}{2}$. The equation

$$P_{2n}(\lambda) = P_{2n}\left(\frac{2n-1}{2}\right) \tag{2.1}$$

has exactly $2n-2$ simple roots; denote them by $\lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < 2n-1-\lambda_{n-1} < 2n-1-\lambda_{n-2} < \cdots < 2n-1-\lambda_1$ and one double root $\lambda_n = \frac{2n-1}{2}$. The solutions of (1.2) are of the form $y_i = t^{\lambda_i}, i = 1, \dots, n-1$, $y_n = t^{\frac{2n-1}{2}}, y_{n+1} = t^{\frac{2n-1}{2}} \log t, y_{n+i+1} = t^{2n-1-\lambda_i}$, $i = 1, \dots, n-1$. Observe that these solutions form the so-called *Markov system* of solutions on $I_0 = (1, \infty)$, which means that the Wronskians

$$W(y_1, \dots, y_k) = \begin{vmatrix} y_1 & \dots & y_k \\ \vdots & \vdots & \\ y_1^{(k-1)} & \dots & y_k^{(k-1)} \end{vmatrix},$$

$k = 1, \dots, 2n$, are positive throughout I_0. Moreover, these solutions form the so-called *hierarchical* system of functions, i.e., $y_i = o(y_{i+1})$ as $t \to \infty$, $i = 1, \dots, 2n-1$.

Equation (1.1) is the special form of the self-adjoint even-order differential equation

$$\sum_{k=0}^{n} (-1)^k \left(p_k(t) y^{(k)}\right)^{(k)} = 0, \tag{2.2}$$

which is closely related to the linear Hamiltonian system

$$u' = Au + B(t)v, \quad v' = C(t)u - A^T v, \tag{2.3}$$

where $u, v : I \to \mathbb{R}^n$, $A, B, C : I \to \mathbb{R}^{n \times n}$, the superscript T stands for the transpose of the matrix indicated and the matrices B, C are symmetric, i.e., $B = B^T, C = C^T$. More precisely, let y be a solution of (2.2). Set $u_i = y^{(i-1)}$, $i = 1, \dots, n$, $v_n = p_n y^{(n)}$, $v_{n-i} = -v'_{n-i+1} + p_{n-i} y^{(n-i)}$, $i = 1, \dots, n-1$, $u = (u_1, \dots, u_n)^T$, $v = (v_1, \dots, v_n)^T$. The n-dimensional vectors u, v are solutions of the LHS of (2.3), where

$$\begin{aligned} B(t) &= \operatorname{diag}\{0, \dots, 0, p_n^{-1}(t)\}, \\ C(t) &= \operatorname{diag}\{p_0(t), \dots, p_{n-1}(t)\}, \\ A &= \begin{cases} 1 & \text{for } j = i+1,\ i = 1, \dots, n-1 \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

We say that the solution y of (2.2) *generates* the solution (u, v) of (2.3).

Now consider the matrix analogy of (2.3)

$$U' = AU + B(t)V, \quad V' = C(t)U - A^T V. \tag{2.4}$$

A *self-conjugate solution* (U, V) of (2.4) (i.e. $U^T(x)V(x) \equiv V^T(x)U(x)$; alternative terminology is *self-conjoined* [11] or *isotropic* [2]; our terminology is due to [13]) is said to be *principal* (*nonprincipal*) at a point b if the matrix U is nonsingular near b and

$$\lim_{x\to b}\left(\int_d^x U^{-1}(s)B(s)U^{T-1}(s)ds\right)^{-1} = 0$$
$$\left(\lim_{x\to b}\left(\int_d^x U^{-1}(s)B(s)U^{T-1}(s)ds\right)^{-1} = M\right),$$

M being a nonsingular $n\times n$ matrix, for some $d \in I$ which is sufficiently close to b. A principal solution of (2.4) at b is determined uniquely up to a right multiple by a nonsingular $n \times n$ matrix. Let $y_1, \dots, y_n$ be solutions of (2.2) and let $(u_1, v_1), \dots, (u_n, v_n)$ be the solutions of (2.3) generated by $y_1, \dots, y_n$. If the vectors $u_1, \dots, u_n$, $v_1, \dots, v_n$ form the columns of the solution (U, V) of (2.4) we say that this solution is generated by the solutions $y_1, \dots, y_n$ of (2.2). Solutions $y_1, \dots, y_n$ of (2.2) are said to form the *principal system of solutions* if the solution (U, V) of the associated LHS generated by $y_1, \dots, y_n$ is principal.

Using the concept of principal system of solutions of a self-adjoint even order differential equation, the following oscillation criterion was proved in [10].

Theorem A. *Let $y_1, \dots, y_n$ be a principal system of solutions of the equation*

$$\left(r(t)y^{(n)}\right)^{(n)} = 0 \tag{2.5}$$

at b and let (U, V) be the solution of the matrix linear Hamiltonian system corresponding to (2.5) generated by $y_1, \dots, y_n$. If there exists $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$ such that for $h = c_1y_1 + \dots + c_ny_n$

$$\limsup_{t\to b}\frac{\int_t^b qh^2}{c^T(\int^t U^{-1}BU^{T-1})^{-1}c} < -1 \tag{2.6}$$

where $B = \operatorname{diag}\{0, \dots, 0, r^{-1}(t)\}$, then the equation

$$(-1)^n\left(r(t)y^{(n)}\right)^{(n)} + q(t)y = 0 \tag{2.7}$$

is (n, n)-oscillatory at the right end point b of the interval (a, b).

(Recall that equation (2.7) is said to be (n,n)-oscillatory at b if in any neighborhood of b there exists at least one pair of conjugate points.) In this paper we prove an oscillation criterion for (1.1) which is principally similar to the criterion for equation (2.7) given by Theorem A. In Theorem A equation (2.7) is viewed as a perturbation one-term equation (2.5), and it is proved that if the function q is sufficiently negative (i.e., (2.6) holds), then (2.7) is oscillatory. Here we apply this idea in a modified form to equation (1.1); this equation is considered as a perturbation of the two-term Euler equation (1.2). Comparing Theorem A with our criterion, here we are able to compute explicitly the term whose analog in Theorem A is the term $c^T\left(\int^t U^{-1}BU^{T-1}\right)^{-1}c$, hence our criterion is more practical.

3. OSCILLATION CRITERION

The key idea of the proof of the following oscillation criterion for (1.1) consists in application of the variational principle given in Lemma 5.1. In particular, for arbitrarily large $t_0 \in \mathbb{R}$, we construct a nontrivial function $y \in W^{2,n}(t_0,\infty)$, $\operatorname{supp} y \subset (t_0,\infty)$ such that

$$I(y;t_0,\infty) = \int_{t_0}^{\infty}\left[(y^{(n)})^2 + (-1)^n p(t)y^2(t)\right]dt \le 0\,. \tag{3.1}$$

Theorem 3.1. *Suppose that*

$$\lim_{t\to\infty}(-1)^n \log t \int_t^{\infty} s^{2n-1}\left(p(s)+\frac{\mu_{2n}}{s^{2n}}\right)ds < -K_n, \tag{3.2}$$

where

$$K_n = (-1)^{n-1}\frac{1}{2}\frac{d^2}{d\lambda^2}P_{2n}(\lambda)|_{\lambda=\frac{2n-1}{2}}\,. \tag{3.3}$$

Then equation (1.1) *is* (n,n)*-oscillatory at* ∞.

Proof. Let $t_0 \in (1,\infty)$ be arbitrary and define the test function as follows

$$y(t) = \begin{cases} 0, & t\in[1,t_0],\\ f(t), & t\in[t_0,t_1],\\ t^{\frac{2n-1}{2}}, & t\in[t_1,t_2],\\ g(t), & t\in[t_2,t_3],\\ 0, & t\in[t_3,\infty),\end{cases}$$

where f,g are the solutions of (1.2) satisfying the boundary conditions

$$f^{(i)}(t_0)=0,\quad f^{(i)}(t_1)=\left(t^{\frac{2n-1}{2}}\right)^{(i)}|_{t=t_1},\quad g^{(i)}(t_2)=\left(t^{\frac{2n-1}{2}}\right)^{(i)}|_{t=t_2},$$
$$g^{(i)}(t_3)=0,\quad i=0,\dots,n-1\,.$$

These solutions exist uniquely in view of Lemma 5.3. The points $1 < t_0 < t_1 < t_2 < t_3$ will be specified later. We shall show that $I(y; t_0, \infty) < 0$ if t_1, t_2, t_3 are sufficiently large.

Let $\lambda_i, i = 1, \dots, n-1, \lambda_n = \frac{2n-1}{2}$ be the first n roots (ordered by size) of the equation (2.1). Denote $y_i = t^{\lambda_i}, i = 1, \dots, n-1,\ y_n = t^{\frac{2n-1}{2}}$, and

$$U = \begin{pmatrix} y_1 & \dots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix},$$

$$V = \begin{pmatrix} (-1)^{n-1} y_1^{(2n-1)} & \dots & (-1)^{n-1} y_n^{(2n-1)} \\ \vdots & & \vdots \\ -y_1^{(n+1)} & \dots & -y_n^{(n+1)} \\ y_1^{(n)} & \dots & y_n^{(n)} \end{pmatrix}.$$

Then by Lemma 5.6 $y_1, \dots, y_n$ form the principal system of solutions of (1.2) at ∞, and (U, V) is the principal solution of the LHS of the associated matrix LHS. By Lemma 5.2

$$\begin{aligned} u_1(t) &= U(t) \int_{t_0}^{t} U^{-1} B U^{T-1} ds \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} ds \right)^{-1} e_n, \\ v_1(t) &= \left(V(t) \int_{t_0}^{t} U^{-1} B U^{T-1} ds + U^{T-1}(t) \right) \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} ds \right)^{-1} e_n, \\ u_2(t) &= U(t) \int_{t}^{t_3} U^{-1} B U^{T-1} ds \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} ds \right)^{-1} e_n, \\ v_2(t) &= \left(V(t) \int_{t}^{t_3} U^{-1} B U^{T-1} ds - U^{T-1}(t) \right) \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} ds \right)^{-1} e_n, \end{aligned} \tag{3.4}$$

where $e_n = (0, \dots, 0, 1)^T \in \mathbb{R}^n$, are solutions of the vector LHS corresponding to (1.2) and according to Lemma 5.2 $f(t) = e_1^T u_1(t),\ g(t) = e_1^T u_2(t)$, $e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^n$.

Using Lemma 5.4 we have

$$\begin{aligned} \int_{t_0}^{t_1} \left[(f^{(n)}(t))^2 - (-1)^n \frac{\mu_{2n}}{t^{2n}} f^2(t) \right] dt = v_1^T(t_1) u_1(t_1) = \\ = e_n^T V(t_1) U(t_1) e_n + e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} ds \right)^{-1} e_n \end{aligned}$$

and by Lemma 5.9

$$e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} ds \right)^{-1} e_n = \frac{K_n}{\log t_1 + M},$$

where M is a positive real constant (its value may be computed explicitly, but it is not important and K_n is given by (3.3)). Similarly,

$$\int_{t_1}^{t_2} \left[(y_n^{(n)}(t))^2 - (-1)^n \frac{\mu_{2n}}{t^{2n}} y_n^2(t) \right] dt = e_n^T V(t_2) U(t_2) e_n - e_n^T V(t_1) U(t_1) e_n\,,$$

$$\int_{t_2}^{t_3} \left[(g^{(n)}(t))^2 - (-1)^n \frac{\mu_{2n}}{t^{2n}} g^2(t) \right] dt =$$

$$= e_n^T \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} ds \right)^{-1} e_n - e_n^T V(t_2) U(t_2) e_n\,.$$

Computing the integrals

$$\int_{t_0}^{t_1} \left[p(t) + \frac{\mu_{2n}}{t^{2n}} \right] f^2(t)\, dt, \quad \int_{t_2}^{t_3} \left[p(t) + \frac{\mu_{2n}}{t^{2n}} \right] g^2(t)\, dt,$$

we proceed as follows. The function f is a solution of (1.2), hence it can be expressed in the form $f = c_1 y_1 + \cdots + c_n y_n + c_{n+1} y_{n+1} + \cdots + c_{2n} y_{2n}$, $c_i \in \mathbb{R},\ i = 1, \ldots, 2n$. It follows that

$$\begin{aligned} \left(\frac{f}{y_n} \right)' &= c_1 \left(\frac{y_1}{y_n} \right)' + \cdots + c_{n-1} \left(\frac{y_{n-1}}{y_n} \right)' + \\ &+ c_{n+1} \left(\frac{y_{n+1}}{y_n} \right)' + \cdots + c_{2n} \left(\frac{y_{2n}}{y_n} \right)' = \\ &= c_1 \Big(\lambda_1 - \frac{2n-1}{2} \Big) t^{\lambda_1 - \frac{2n+1}{2}} + \cdots + \\ &+ c_{n-1} \Big(\lambda_{n-1} - \frac{2n-1}{2} \Big) t^{\lambda_{n-1} - \frac{2n+1}{2}} + \\ &+ c_{n+1} (\log t)' + \cdots + c_{2n} \Big(\frac{2n-1}{2} - \lambda_1 \Big) t^{\frac{2n-3}{2} - \lambda_1}. \end{aligned}$$

Since the functions $(y_1/y_n)', \ldots, (y_{n-1}/y_n)', (y_{n+1}/y_n)', \ldots, (y_{2n}/y_n)'$ form the Markov system of solutions of certain $(2n-1)$-order linear differential equation, by Lemma 5.5 this equation is LN-disconjugate on $(1, \infty)$. As $(f/y_n)'$ has zeros of multiplicity $(n-1)$ both at $t = t_0$ and $t = t_1$, this function does not vanish in the interval (t_0, t_1), i.e., the function f/y_n is increasing in this interval. By the second mean value theorem of integral calculus there exists $\xi_1 \in (t_0, t_1)$ such that

$$\int_{t_0}^{t_1} \Big(p(t) + \frac{\mu_{2n}}{t^{2n}} \Big) f^2(t)\, dt = \int_{t_0}^{t_1} \Big(p(t) + \frac{\mu_{2n}}{t^{2n}} \Big) y_n^2(t) \left(\frac{f}{y_n(t)} \right)^2 dt =$$

$$= \int_{\xi_1}^{t_1} \left(p(t) + \frac{\mu_{2n}}{t^{2n}}\right) y_n^2(t)\, dt.$$

Similarly, the function g/y_n is decreasing on (t_2, t_3) and there exists $\xi_2 \in (t_2, t_3)$ such that

$$\int_{t_2}^{t_3} \left(p(t) + \frac{\mu_{2n}}{t^{2n}}\right) g^2(t)\, dt = \int_{t_2}^{t_3} \left(p(t) + \frac{\mu_{2n}}{t^{2n}}\right) y_n^2(t) \left(\frac{g(t)}{y_n(t)}\right)^2 dt = \\ = \int_{t_2}^{\xi_2} \left(p(t) + \frac{\mu_{2n}}{t^{2n}}\right) y_n^2(t)\, dt.$$

Using these computations and Lemma 5.9 we get

$$I(y; 1, \infty) = I(y; t_0, t_3) = \int_{t_0}^{t_1} \left[(f^{(n)}(t))^2 - (-1)^n \frac{\mu_{2n}}{t^{2n}} f^2(t)\right] dt + \\ + \int_{t_1}^{t_2} \left[(y_n^{(n)}(t))^2 - (-1)^n \frac{\mu_{2n}}{t^{2n}} y_n^2(t)\right] dt + \int_{t_2}^{t_3} \left[(g^{(n)}(t))^2 - (-1)^n \frac{\mu_{2n}}{t^{2n}} g^2(t)\right] dt + \\ + (-1)^n \int_{t_0}^{t_1} \left[p(t) + \frac{\mu_{2n}}{t^{2n}}\right] f^2(t)\, dt + (-1)^n \int_{t_1}^{t_2} \left[p(t) + \frac{\mu_{2n}}{t^{2n}}\right] y_n^2(t)\, dt + \\ + (-1)^n \int_{t_2}^{t_3} \left[p(t) + \frac{\mu_{2n}}{t^{2n}}\right] g^2(t)\, dt = e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} dt\right)^{-1} e_n + \\ + e_n^T \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} dt\right)^{-1} e_n + (-1)^n \int_{\xi_1}^{\xi_2} \left[p(t) + \frac{\mu_{2n}}{t^{2n}}\right] y_n^2(t)\, dt = \\ = e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} dt\right)^{-1} e_n \left[1 + (-1)^n \frac{\int_{\xi_1}^{\xi_2} \left(p(t) + \frac{\mu_{2n}}{t^{2n}}\right) y_n^2(t)\, dt}{e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1}\, dt\right)^{-1} e_n} + \right. \\ \left. + \frac{e_n^T \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1}\, dt\right)^{-1} e_n}{e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1}\, dt\right)^{-1} e_n}\right] \le e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} dt\right)^{-1} e_n \times \\ \times \left[1 + (-1)^n \frac{\int_{\xi_1}^{\xi_2} (p(t) + \frac{\mu_{2n}}{t^{2n}})\, y_n^2(t)\, dt}{e_n^T \left(\int_{t_0}^{\xi_1} U^{-1} B U^{T-1}\, dt\right)^{-1} e_n} + \frac{e_n^T \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} dt\right)^{-1} e_n}{e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} dt\right)^{-1} e_n}\right] = \\ = \frac{K_n}{\log t_1 + M} \left[1 + (-1)^n \frac{\log \xi_1 + M}{\log \xi_1} \frac{\log \xi_1}{K_n} \int_{\xi_1}^{\xi_2} \left(p(t) + \frac{\mu_{2n}}{t^{2n}}\right) y_n^2(t)\, dt + \right. \\ \left. + \frac{\log t_1 + M}{K_n} e_n^T \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1}\, dt\right)^{-1} e_n\right] = \frac{K_n}{\log t_1 + M} \frac{\log \xi_1 + M}{\log \xi_1} \times$$

$$\times \left[\frac{\log \xi_1}{\log \xi_1 + M} + (-1)^n \frac{\log \xi_1}{K_n} \int_{\xi_1}^{\xi_2} \left(p(t) + \frac{\mu_{2n}}{t^{2n}} \right) y_n^2(t)\, dt + \right.$$
$$\left. + \frac{\log t_1 + M}{\log \xi_1 + M} \cdot \frac{\log \xi_1}{K_n} \cdot e_n^T \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} dt \right)^{-1} e_n \right].$$

The inequality in this computation is justified by the fact that according to (3.2)

$$(-1)^n \int_{\xi_1}^{\xi_2} \left[p(t) + \frac{\mu_{2n}}{t^{2n}} \right] y_n^2(t)\, dt < 0$$

if ξ_1 and ξ_2 are sufficiently large and hence

$$\frac{(-1)^n \int_{\xi_1}^{\xi_2} \left[p(t) + \frac{\mu_{2n}}{t^{2n}} \right] y_n^2(t)\, dt}{e_n^T \left(\int_{t_0}^{t_1} U^{-1} B U^{T-1} dt \right)^{-1} e_n} \leq \frac{(-1)^n \int_{\xi_1}^{\xi_2} \left[p(t) + \frac{\mu_{2n}}{t^{2n}} \right] dt}{e_n^T \left(\int_{t_0}^{\xi_1} U^{-1} B U^{T-1} dt \right)^{-1} e_n}$$

for $\xi_1 \leq t_1$. Now let $\epsilon > 0$ be such that the limit in (3.2) is less than $-K_n - 4\epsilon$. Since $\lim_{t\to\infty} (K_n \log t + M)/K_n \log t = 1$, we have

$$\frac{\log \xi_1}{\log \xi_1 + M} < 1 + \epsilon$$

if t_0 is sufficiently large. According to (3.2) $t_2 > t_1$ can be chosen such that

$$\frac{\log \xi_1}{K_n} \int_{\xi_1}^{\xi_2} \left[p(t) + \frac{\mu_{2n}}{t^{2n}} \right] y_n^2(t)\, dt < -1 - 2\epsilon,$$

whenever $\xi_2 > t_2$. Finally, since $\lim_{t_3\to\infty} \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} dt \right)^{-1} = 0$, t_3 can be chosen such that

$$\frac{\log t_1 + M}{\log \xi_1 + M} \cdot \frac{\log \xi_1}{K_n} \cdot e_n^T \left(\int_{t_2}^{t_3} U^{-1} B U^{T-1} ds \right)^{-1} e_n < \epsilon\,.$$

Summarizing all estimates, we have

$$I(y; t_0, t_3) < \frac{K_n}{\log t_1 + M} \cdot \frac{\log \xi_1 + M}{\log \xi_1} (1 + \epsilon - 1 - 2\epsilon + \epsilon) \leq 0$$

and according to Lemma 5.1 equation (1.1) is (n, n)-oscillatory at ∞. $\square$

4. REMARKS

i) In [11] we studied the problem of existence of (n,n)-conjugate points in an interval (a,b) and proved the following theorem.

Theorem B. *Let $y_1,\dots,y_m$, $1\le m\le n$, be solutions of* (2.5) *which are contained in principal systems of solutions both at a and b. If there exist $c_1,\dots,c_m\in\mathbb{R}$ such that*

$$\limsup_{t_1\downarrow a, t_2\uparrow b}\int_{t_1}^{t_2} q(t)\left(c_1y_1(t)+\cdots+c_my_m(t)\right)^2dt<0$$

then equation (2.7) *is (n,n)-conjugate on (a,b).*

A slight modification of the proof of this theorem applies also to equation (1.1) considered as a perturbation of (1.2) for $t\in(0,\infty)$. Observe that $y_n=t^{\frac{2n-1}{2}}$ is the only solution (up to a multiple by a nonzero real constant) of (1.2) which is contained in the principal systems of solutions both at $t=0$ and $t=\infty$; hence we have the following statement.

Theorem 4.1. *Suppose that*

$$\limsup_{t_1\downarrow 0, t_2\uparrow\infty}(-1)^n\int_{t_1}^{t_2} t^{\frac{2n-1}{2}}\left[p(t)+\frac{\mu_{2n}}{t^{2n}}\right]dt<0.$$

Then (1.1) *is (n,n) conjugate on the interval $(0,\infty)$.*

(ii) Taking into consideration more general test functions which are linear combinations of the principal solutions, we have the following more general statement whose proof is analogous to that of Theorem 3.1.

Theorem 4.2. *Let $\lambda_1,\dots,\lambda_{n-1}$ be the first (ordered by size) $(n-1)$ roots of* (2.1),

$$h(t)=c_1t^{\lambda_1}+\cdots+c_{n-1}t^{\lambda_{n-1}}+c_nt^{\frac{2n-1}{2}},\tag{4.1}$$

where $c_1,\dots,c_n\in\mathbb{R}$ and $l=max\{j\in\{1,\dots,n\},c_j\neq 0\}$. If

$$\lim_{t\to\infty}(-1)^nt^{2n-1-2\lambda_l}\int_t^\infty\left[p(s)+\frac{\mu_{2n}}{s^{2n}}\right]h^2(s)\,ds<-\tilde{K}_n,$$

where

$$\tilde{K}_n=\frac{1}{2}\left(\frac{2n-1}{2}-\lambda_l\right)\prod_{k=1}^{n-1}(2n-1-\lambda_k-\lambda_l)^2$$

in the case $l<n$, and

$$\lim_{t\to\infty}(-1)^n\log t\int_t^\infty\left[p(s)+\frac{\mu_{2n}}{s^{2n}}\right]h^2(s)\,ds<-K_n,$$

where K_n *is the same as in Theorem* 3.1 *for* $l = n$, *then* (1.1) *is* (n, n)-*oscillatory at* ∞.

(iii) In [1] the following LN-oscillation criterion for (1.1) has been proved.

Theorem C [1, Theorem 2.3, Theorem 2.4]. *Let one of the following two conditions be satisfied:*

(i)
$$\lim_{t\to\infty} \log t \int_t^\infty s^{k-1} \left[-p(s) + \frac{m_k^*}{s^k}\right]_- ds = \infty, \tag{4.2$_1$}$$

$$\int_1^\infty s^{k-1} \log^2 s \left[-p(s) + \frac{m_k^*}{s^k}\right]_+ ds < \infty, \tag{4.2$_2$}$$

(ii)
$$\lim_{t\to\infty} \log t \int_t^\infty s^{k-1} \left[-p(s) - \frac{m_{*k}}{s^k}\right]_+ ds = \infty, \tag{4.3$_1$}$$

$$\int_1^\infty s^{k-1} \log^2 s \left[-p(s) - \frac{m_{*k}}{s^k}\right]_- ds < \infty, \tag{4.3$_2$}$$

where m_k^*, m_{*k} *are the least local maxima of the polynomials*

$$P^*(\lambda) = -\lambda(\lambda-1)\dots(\lambda-k), \quad P_*(\lambda) = \lambda(\lambda-1)\dots(\lambda-k+1)$$

respectively, and $[f(t)]_\pm = max\{\pm f(t), 0\}$. *Then the equation* $y^{(k)} + p(t)y = 0$ *is LN-oscillatory.*

If $k = 2n$, Theorem 3.1 gives a sufficient condition even for the (n, n)-oscillation of (1.1) which is weaker than given by Theorem C. Indeed, for n even $\mu_{2n} = P_*\left(\frac{2n-1}{2}\right)$ is the least local minimum of $P_*(\lambda)$ and for n odd μ_{2n} is the greatest local minimum of $P_{2n}(\lambda) = -P^*(\lambda)$, i.e., the least local maximum of $P^*(\lambda) = -P_{2n}(\lambda)$. Hence, for n even (4.2$_1$) reads

$$\begin{aligned} &\lim_{t\to\infty} \log t \int_t^\infty s^{2n-1} \left[-p(s) - \frac{\mu_{2n}}{s^{2n}}\right]_- ds = \\ &= \lim_{t\to\infty} \log t \int_t^\infty s^{2n-1} \left[p(s) + \frac{\mu_{2n}}{s^{2n}}\right]_+ ds = \infty\,. \end{aligned} \tag{4.4}$$

On the other hand, (4.2$_2$) gives

$$\infty > \int_1^\infty s^{2n-1} \log^2 s \left[-p(s) - \frac{\mu_{2n}}{s^{2n}}\right]_+ ds > \log t \int_t^\infty s^{2n-1} \left[p(s) + \frac{\mu_{2n}}{s^{2n}}\right]_- ds.$$

Consequently, this inequality and (4.4) give

$$\lim_{t\to\infty} \log t \int_t^\infty s^{2n-1} \left[p(s) + \frac{\mu_{2n}}{s^{2n}}\right] ds = -\infty$$

which is a stronger condition than (3.2). (Under the assumption $p(t) + \frac{\mu_{2n}}{t^{2n}} \leq 0$ for large t, this condition is proved to be sufficient for oscillation of (1.1) in [3].) If n is odd, we get a similar conclusion from (4.3_1), $(4._2)$. Note also that if the function $p(t)$ does not change sign for large t, the LN-oscillation properties and (n,n)-oscillation properties of (1.1) are essentially the same (see [1]).

(iv) Ideas similar to those used here for ordinary differential equations may be used in a modified form in order to investigate oscillation and spectral properties of the singular differential operators associated with the partial differential equation

$$(-\Delta)^n u + a(\mathbf{x})u = 0,$$

where $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $\Delta = \sum_i^m \frac{\partial^2}{\partial x_i^2}$; cf. [7–9, 12]. We hope to explore this idea elsewhere.

5. Technical Lemmas

In this section we give some technical lemmas needed in the previous sections. We start with the fundamental variational lemma.

Lemma 5.1 ([7]). *Equation* (2.2) *is conjugate on* $I_0 = (c,d) \subseteq I$ *if and only if there exists a nontrivial function* $y \in \overset{\circ}{W}{}^2_n(I_0)$ ($\overset{\circ}{W}{}^2_n$ *is the Sobolev space of functions for which* $y, \dots, y^{(n-1)}$ *are absolutely continuous on* I_0, $y_n \in \mathcal{L}^2(I_0)$ *and* supp $y \subset I_0$) *such that*

$$I(y; c, d) = \int_c^d \left[\sum_{k=0}^n p_k(t)(y^{(k)}(t))^2 \right] dt \le 0\,.$$

Lemma 5.2 ([2]). *Let* (U,V) *be a self-conjugate solution of* (2.4) *such that* U *is nonsingular on some subinterval* $I_0 \subseteq I$. *Then*

$$(U_1, V_1) = \left(U \int_d^t U^{-1} B U^{T-1}\, dt,\ V \int_d^t U^{-1} B U^{T-1}\, dt + U^{T-1} \right),\ d \in I,$$

is also a self-conjugate solution of (2.3) *and* $W = VU^{-1}$ *is a solution of the Riccati matrix differential equation*

$$W' + A^T W + WA + WB(t)W - C(t) = 0\ .$$

Lemma 5.3 ([2]). *Let* (1.1) *be disconjugate on* $I_0 = (c,d) \subset I$ *and let* $t_1, t_2 \in I_0$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ *be arbitrary. There exists a unique solution* y *of* (2.2) *such that*

$$y^{(i-1)}(t_1) = \alpha_i\ , \quad y^{(i-1)}(t_2) = \beta_i\ , \quad i = 1, \dots, n\ .$$

Lemma 5.4 ([5]). *Let y be a solution of* (2.2) *and (u,v) be the solution of the associated* LHS *of* (2.3) *generated by y. Then*

$$\int_a^b \left[\sum_{k=0}^n p_k(t)\left(y^{(k)}(t)\right)^2\right] dt = u^T(b)v(b) - u^T(a)v(a)\,.$$

Now recall briefly the oscillation properties of the linear differential equation

$$y^{(n)} + q_{n-1}(t)y^{(n-1)} + \cdots + q_0(t)y = 0\,. \tag{5.1}$$

The proofs of these statements may be found in [2, Chap. III].

Lemma 5.5. *Equation* (5.1) *is LN-disconjugate on $I=(b,\infty)$ if and only if there exists a Markov system of solutions of* (5.1) *on I. This system can be found in such a way that it satisfies the additional conditions*

$$\begin{aligned} &y_i > 0 \quad \textit{for large} \quad t, \quad i=1,\dots,n\,,\\ &y_{k-1} = o(y_k) \quad \textit{for} \quad t\to\infty\,, \quad k=2,\dots,n, \end{aligned} \tag{5.2}$$

i.e., it forms a hierarchical system as $t\to\infty$.

Lemma 5.6. *Let equation* (2.2) *be eventually LN-disconjugate at ∞ and let $y_1,\dots,y_{2n}$ be a Markov system of solutions of this equation satisfying* (5.2) (*with n replaced by $2n$*). *Then $y_1,\dots,y_n$ form a principal system of solutions of* (2.2) *at ∞.*

Lemma 5.7. *Let $y_1,\dots,y_n \in C^n$, $r\in C^n$, and $r\neq 0$. Then*

$$W(ry_1,\dots,ry_n) = r^n W(y_1,\dots,y_n)\,.$$

In particular, if $y_1\neq 0$, we have

$$W(y_1,\dots,y_n) = y_1^n W\left((y_2/y_1)',\dots,(y_n/y_1)'\right).$$

Lemma 5.8. *Let $u_1=t^{\alpha_1},\dots,u_n=t^{\alpha_n}$, $\alpha_i\in\mathbb{R}$, $i=1,\dots,n$. Then*

$$W(y_1,\dots,y_n) = \prod_{1\le i<j}^n (\alpha_j-\alpha_i)t^{\sum\limits_{k=1}^n \alpha_k - \frac{n(n-1)}{2}}\,. \tag{5.3}$$

Proof. Using Lemma 5.7 we have

$$\begin{aligned} W\left(t^{\alpha_1},\dots,t^{\alpha_n}\right) &= t^{n\alpha_1} W\left(\left(t^{\alpha_2-\alpha_1}\right)',\dots,\left(t^{\alpha_n-\alpha_1}\right)'\right)' = \\ &= t^{n\alpha_1-(n-1)}(\alpha_2-\alpha_1)\dots(\alpha_n-\alpha_1)W\left(t^{\alpha_2-\alpha_1},\dots,t^{\alpha_n-\alpha_1}\right) \end{aligned}$$

and repeating the same argument $(n-1)$-times we get (5.3). □

We finish this paper with an evaluation of the constants $K_n, \tilde{K}_n$ in Theorems 3.1 and 4.2. Let $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n \in \mathbb{R}$, $\mu_i + \nu_j \neq 0$, $i, j = 1, \dots, n$. Denote

$$D(\mu_1, \dots, \mu_n; \nu_1, \dots, \nu_n) = \begin{vmatrix} (\mu_1+\nu_1)^{-1} & \dots & (\mu_1+\nu_n)^{-1} \\ \vdots & \vdots & \\ (\mu_n+\nu_1)^{-1} & \dots & (\mu_n+\nu_n)^{-1} \end{vmatrix}.$$

Then by a direct computation we have

$$D(\mu_1, \dots, \mu_n; \nu_1, \dots, \nu_n) = \frac{\prod\limits_{1 \le k < l \le n} (\mu_k - \mu_l)(\nu_k - \nu_l)}{\prod\limits_{k,l=1}^{n} (\mu_k + \nu_l)}. \tag{5.4}$$

Lemma 5.9. *Let $y_1 = t^{\lambda_1}, \dots, y_{n-1} = t^{\lambda_{n-1}}$, $y_n = t^{\frac{2n-1}{2}}$, i.e., $y_1, \dots, y_n$ is the principal system of solutions of* (1.2) *at ∞ and let $B = \operatorname{diag}\{0, \dots, 0, 1\} \in \mathbb{R}^{n \times n}$. If U denotes the Wronski matrix of $y_1, \dots, y_n$, then*

$$\left(\int^t U^{-1} B U^{T-1} ds\right)^{-1}_{n,n} = \frac{(-1)^{n-1} \frac{1}{2} \frac{d^2}{d\lambda^2} P_{2n}(\lambda)|_{\lambda = \frac{2n-1}{2}}}{\log t + M}$$

$$\left(\int^t U^{-1} B U^{T-1} ds\right)^{-1}_{i,i} \sim t^{2n-1-2\lambda_i} \, \frac{2n-1-2\lambda_i}{4} \prod_{k=1}^{n-1} (2n-1-\lambda_k-\lambda_i)^2$$

as $t \to \infty$. Here the relation $f(t) \sim g(t)$ as $t \to \infty$ means $\lim\limits_{t \to \infty} \frac{f(t)}{g(t)} = 1$.

Proof. We have $(U^{-1} B U^{T-1})_{i,j} = (U^{-1})_{i,n} (U^{-1})_{j,n}$, and using the rule for computation of the entries of the inverse matrix, we obtain

$$(U^{-1})_{i,n} = (-1)^{n+i} \frac{W(y_1, \dots, \hat{y}_i, \dots, y_n)}{W(y_1, \dots, y_n)}, \tag{5.5}$$

where the circumflex $\hat{}$ means that the denoted component is missing, i.e., $W(y_1, \dots, \hat{y}_i, \dots, y_n) = W(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$. Substituting $y_1 = t^{\lambda_1}, \dots, y_{n-1} = t^{\lambda_{n-1}}$, $y_n = t^{\frac{2n-1}{2}}$ into (5.5) and using Lemma 5.8, we have

$$\begin{aligned}(U^{-1} B U^{T-1})_{i,j} &= (-1)^{i+j} \frac{W(y_1, \dots, \hat{y}_i, \dots, y_n) W(y_1, \dots, \hat{y}_j, \dots, y_n)}{[W(y_1, \dots, y_n)]^2} = \\ &= (-1)^{i+j} \frac{t^{2n-2-\lambda_i-\lambda_j}}{\prod\limits_{\substack{1 \le k \le n \\ k \neq i}} |\lambda_i - \lambda_k| \prod\limits_{\substack{1 \le k \le n \\ k \neq j}} |\lambda_j - \lambda_k|}.\end{aligned}$$

Denote

$$A_i = \prod_{\substack{1\le k\le n\\ k\ne i}} |\lambda_i - \lambda_k| .$$

Then

$$\left(\int^t U^{-1}BU^{T-1}ds\right)_{i,j} = \begin{cases} A_n^{-2}\log t, & \text{if} \quad i+j=2n\\ \frac{(-1)^{i+j}t^{2n-1-\lambda_i-\lambda_j}}{A_iA_j(2n-1-\lambda_i-\lambda_j)}, & \text{if} \quad i+j<2n \end{cases}$$

(since $\int^t U^{-1}BU^{T-1}ds \to \infty$ as $t\to\infty$, for computation of its inverse the lower limit in the integral is immaterial). Further denote

$$a_i = \frac{2n-1}{2} - \lambda_i, \quad \gamma = \sum_{k=1}^{n-1} a_k, \quad A = \prod_{k=1}^{n} A_k .$$

We have

$$D = \left|\int^t U^{-1}BU^{T-1}ds\right| = \frac{t^\gamma}{A^2}\begin{vmatrix} \frac{1}{2a_1} & \frac{-1}{a_1+a_2} & \dots & \frac{(-1)^{n+1}}{a_1+a_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{(-1)^{n+1}}{a_n+a_1} & \frac{(-1)^{n+2}}{a_n+a_2} & \dots & \log t \end{vmatrix}.$$

Let

$$\tilde D(\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n) = \begin{vmatrix} \frac{1}{\mu_+\nu_1} & \frac{-1}{\mu_1+\nu_2} & \dots & \frac{(-1)^{n+1}}{\mu_1+\nu_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{(-1)^{n+1}}{\mu_n+\nu_1} & \frac{(-1)^{n+2}}{\mu_n+\nu_2} & \dots & \frac{1}{\mu_n+\nu_n} \end{vmatrix}$$

and let $\tilde D(\mu_1,\dots,\hat\mu_i,\dots,\mu_n;\nu_1,\dots,\hat\nu_j,\dots,\nu_n)$ denote the subdeterminant of $\tilde D(\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n)$, where the ith row and the jth column are supressed. Then

$$\tilde D(\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n) = D(\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_n),$$

$$\tilde D(\mu_1,..,\hat\mu_i,..,\mu_n;\nu_1,..,\hat\nu_j,..,\nu_n) = (-1)^{i+j}D(\mu_1,..,\hat\mu_i,..,\mu_n;\nu_1,..,\hat\nu_j,..,\nu_n)$$

and

$$D = \frac{t^\gamma}{A}\left[\log t\, D(a_1,\dots,a_{n-1};a_1,\dots,a_{n-1}) + \tilde M\right]$$

(the precise value of the constant $\tilde M$ is not important).

Now compute the entries of the matrix $(\int^t U^{-1}BU^{T-1}ds)^{-1}$.

$$\left(\int^t U^{-1}BU^{T-1}ds\right)^{-1}_{n,n} = \frac{1}{D}\det\left(\int^t U^{-1}BU^{T-1}ds\right)^{n-1}_{i,j=1} =$$

$$= \frac{t^{\gamma} \prod_{k=1}^{n-1} A_k^{-2} D(a_1, \dots, a_{n-1}; a_1, \dots, a_{n-1})}{t^{\gamma} A^{-2} (D(a_1, \dots, a_{n-1}; a_1, \dots, a_{n-1}) \log t + \tilde{M})} = \frac{A_n^2}{\log t + M} .$$

Further,

$$\left(\int^t U^{-1} B U^{T-1} ds \right)^{-1}_{i,n} = \left(\int^t U^{-1} B U^{T-1} ds \right)_{n,i} =$$
$$= \frac{1}{D} t^{\gamma - a_i} \frac{A_i A_n}{A} \tilde{D}(a_1, \dots, a_{n-1}; a_1, \dots, \hat{a}_i, \dots, a_n) =$$
$$= A_i A_n t^{\lambda_i - \frac{2n-1}{2}} (-1)^{n+i} \frac{D(a_1, \dots, a_{n-1}; a_1, \dots, \hat{a}_i, \dots, a_n)}{D(a_1, \dots, a_{n-1}; a_1, \dots, a_{n-1}) \log t + M}$$

and for $i, j < n$

$$\left(\int^t U^{-1} B U^{T-1} ds \right)^{-1}_{i,j} = \frac{1}{D} \frac{t^{\gamma - a_i - a_j} A_i A_j}{A} \times$$
$$\times \begin{vmatrix} \frac{1}{2a_1} & \dots & \frac{(-1)^j}{a_1 + a_{j-1}} & \frac{(-1)^{j+2}}{a_1 + a_{j+1}} & \dots & \frac{(-1)^{n+1}}{a_1 + a_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^i}{a_{i-1} + a_1} & \dots & & & \dots & \frac{(-1)^{n+i-1}}{a_{i-1} + a_n} \\ \frac{(-1)^{i+2}}{a_{i+1} + a_1} & \dots & & & \dots & \frac{(-1)^{n+i+1}}{a_{i+1} + a_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^{n+1}}{a_n + a_1} & \dots & \frac{(-1)^{n+j-1}}{a_n + a_{j-1}} & \frac{(-1)^{n+j+1}}{a_n + a_{j+1}} & \dots & \log t \end{vmatrix} =$$
$$= (-1)^{i+j} A_i A_j t^{\lambda_i + \lambda_j - 2n + 1} \times$$
$$\times \frac{\log t \, D(a_1, \dots, \hat{a}_i, \dots, a_{n-1}; a_1, \dots, \hat{a}_j, \dots, a_{n-1}) + L}{\log t \, D(a_1, \dots, a_{n-1}; a_1, \dots, a_{n-1}) + M} .$$

For computation of the diagonal entries, using (5.4) we have

$$A_i^2 \frac{D(a_1, \dots, \hat{a}_i, \dots, a_{n-1}; a_1, \dots, \hat{a}_i, \dots, a_n)}{D(a_1, \dots, a_{n-1}; a_1, \dots, a_{n-1})} = \frac{\prod_{k=1}^{n-1} (a_k + a_i)^2 A_i^2}{2a_i \prod_{\substack{1 \le k \le n-1 \\ k \ne i}} (a_k - a_i)^2} =$$
$$= \frac{1}{2} \frac{a_i^2}{2a_i} \prod_{k=1}^{n-1} (a_k + a_i)^2 = \frac{1}{2} \left(\frac{2n-1}{2} - \lambda_i \right) \prod_{k=1}^{n-1} (2n - 1 - \lambda_k - \lambda_i)^2 .$$

Finally,

$$A_n^2 = \prod_{k=1}^{n-1} |a_n - a_k|^2 = \prod_{k=1}^{n-1} \left(\frac{2n-1}{2} - \lambda_k \right)^2 =$$
$$= (-1)^{n-1} \prod_{k=1}^{n-1} (\lambda - \lambda_k)(\lambda - 2n + 1 + \lambda_k)|_{\lambda = \frac{2n-1}{2}} =$$
$$= (-1)^{n-1} \lim_{\lambda \to \frac{2n-1}{2}} \frac{P_{2n}(\lambda) - \mu_{2n}}{(\lambda - \frac{2n-1}{2})^2} = \frac{1}{2} \frac{d^2}{d\lambda^2} P_{2n}(\lambda)|_{\lambda = \frac{2n-1}{2}} . \qquad \square$$

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