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CONSTRUCTION OF ENTIRE MODULAR FORMS OF WEIGHTS 5 AND 6 FOR THE CONGRUENCE GROUP $\Gamma_0(4N)$

G. LOMADZE

ABSTRACT. Two classes of entire modular forms of weight 5 and two of weight 6 are constructed for the congruence subgroup $\Gamma_0(4N)$. The constructed modular forms as well as the modular forms from [1] will be helpful in the theory of representation of numbers by the quadratic forms in 10 and 12 variables.

The present paper is a direct continuation of [1] whose notation will be preserved here.

1.

Lemma 1. *For a given N let*

$$\begin{aligned}\Psi_3(\tau) = \Psi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; c_1, \dots, c_4; N_1, \dots, N_4) = \\ = \Big\{ \frac{1}{N_1} \vartheta''_{g_1h_1}(\tau; c_1, 2N_1) \vartheta_{g_2h_2}(\tau; c_2, 2N_2) - \\ - \frac{1}{N_2} \vartheta_{g_1h_1}(\tau; c_1, 2N_1) \vartheta''_{g_2h_2}(\tau; c_2, 2N_2) \Big\} \times \\ \times \vartheta'_{g_3h_3}(\tau; c_3, 2N_3) \vartheta_{g_4h_4}(\tau; c_4, 2N_4) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned}\Psi_4(\tau) = \Psi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; c_1, \dots, c_4; N_1, \dots, N_4) = \\ = \prod_{k=1}^{3} \vartheta'_{g_kh_k}(\tau; c_k, 2N_k) \vartheta_{g_4h_4}(\tau; c_4, 2N_4), \end{aligned} \tag{1.2}$$

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where

$$2|g_k,\quad N_k|N\quad (k=1,2,3,4),\quad 4\Big|N\sum_{k=1}^{4}\frac{h_k}{N_k}. \tag{1.3}$$

For all substitutions from Γ in the neighborhood of each rational point $\tau=-\frac{\delta}{\gamma}$ ($\gamma\neq 0$, $(\gamma,\delta)=1$), we then have

$$\begin{aligned}(\gamma\tau+\delta)^5\Psi_j(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=\\ =\sum_{n=0}^{\infty}C_n^{(j)}e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\quad (j=3,4).\end{aligned} \tag{1.4}$$

Proof. I. Taking into account (1.19) from [1], by Lemma 4 from [1], it follows for $n=2$ (with g_1, h_1, g_1', h_1', N_1, H_1 instead of g, h, g', h', N, H) and $n=0$ (with g_2, h_2, g_2', h_2', N_2, H_2 instead of g, h, g', h', N, H) that

$$\begin{aligned}\frac{1}{N_1}(\gamma\tau+\delta)^3\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)=\\ =-\frac{1}{N_1}e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\sum_{H_1\,\mathrm{mod}\,2N_1}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\times\\ \times\Big\{\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+2A_{21}\Big|_{z=0}\cdot\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}\times\\ \times\sum_{H_2\,\mathrm{mod}\,2N_2}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)=\\ =-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)\big(2|\gamma|(N_1N_2)^{1/2}\big)^{-1}\sum_{\substack{H_1\,\mathrm{mod}\,2N_1\\ H_2\,\mathrm{mod}\,2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\times\\ \times\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\Big\{\frac{1}{N_1}\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times\\ \times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-4\gamma\pi i(\gamma\tau+\delta)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times\\ \times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}.\end{aligned} \tag{1.5}$$

If in (1.5) N_1, g_1, h_1, H_1, g_1', h_1' are replaced by N_2, g_2, h_2, H_2, g_2', h_2', and vice versa, then we have

$$\begin{aligned}\frac{1}{N_2}(\gamma\tau+\delta)^3\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)=\\ =-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)\big(2|\gamma|(N_1N_2)^{1/2}\big)^{-1}\sum_{\substack{H_2\,\mathrm{mod}\,2N_2\\ H_1\,\mathrm{mod}\,2N_1}}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times\end{aligned}$$

$$\times\varphi_{g_1'g_1h_1}(\tau;0,H_1;2N_1)\times$$
$$\times\Big\{\frac{1}{N_2}\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-$$
$$-4\gamma\pi i(\gamma\tau+\delta)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times$$
$$\times\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \tag{1.6}$$

Subtracting (1.6) from (1.5), we obtain

$$(\gamma\tau+\delta)^3\Big\{\frac{1}{N_1}\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)-$$
$$-\frac{1}{N_2}\vartheta_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)\Big\}=-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)\times$$
$$\times\big(2|\gamma|(N_1N_2)^{1/2}\big)^{-1}\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times$$
$$\times\Big\{\frac{1}{N_1}\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$
$$-\frac{1}{N_2}\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}. \tag{1.7}$$

Analogously, by Lemma 4 from [1], for $n-1$ and $n=0$ we obtain

$$(\gamma\tau+\delta)^2\vartheta'_{g_3h_3}(\tau;0,2N_3)\vartheta_{g_4h_4}(\tau;0,2N_4)=$$
$$=-e\Big(\frac{1}{2}\operatorname{sgn}\gamma\Big)\big(2|\gamma|(N_3N_4)^{1/2}\big)^{-1}i\operatorname{sgn}\gamma\times$$
$$\times\sum_{\substack{H_3 \bmod 2N_3\\ H_4 \bmod 2N_4}}\varphi_{g_3'g_3h_3}(0,H_3;2N_3)\varphi_{g_4'g_4h_4}(0,H_4;2N_4)\times$$
$$\times\vartheta'_{g_3'h_3'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)\vartheta_{g_4'h_4'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_4,2N_4\Big). \tag{1.8}$$

Multiplying (1.7) by (1.8), on account of (1.1), we obtain

$$(\gamma\tau+\delta)^5\Psi_3(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=$$
$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^4N_k\Big)^{1/2}\Big)^{-1}i\operatorname{sgn}\gamma\times$$
$$\times\sum_{\substack{H_k \bmod 2N_k\\ (k=1,2,3,4)}}\prod_{k=1}^4\varphi_{g_k'g_kh_k}(0,H_k;2N_k)\times$$

$$\times\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_1,\dots,g'_4;h'_1,\dots,h'_4;H_1,\dots,H_4;N_1,\dots,N_4\Big). \quad (1.9)$$

Further, applying the same reasoning as in [1, Lemma 5, pp. 62-63], we obtain (1.4) if $j=3$.

II. As in Subsection I, by Lemma 4 from [1], for $n=1$ (with g_k, h_k, N_k, g'_k, h'_k, H_k for all $1\le k\le 4$ instead of g, h, N, g', h', H) and $n=0$ (with g_4, h_4, N_4, g'_4, h'_4, H_4 instead of g, h, N, g', h', H), we obtain

$$(\gamma\tau+\delta)^5\prod_{k=1}^{3}\vartheta'_{g_kh_k}(\tau;0,2N_k)\vartheta_{g_4h_4}(\tau;0,2N_4)=$$
$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}i\operatorname{sgn}\gamma\times$$
$$\times\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\prod_{k=1}^{4}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\prod_{k=1}^{3}\vartheta'_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\vartheta_{g'_4h'_4}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_4,2N_4\Big).$$

Hence, according to (1.2), it follows that

$$(\gamma\tau+\delta)^5\Psi_4(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=$$
$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}i\operatorname{sgn}\gamma\times$$
$$\times\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\prod_{k=1}^{4}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_1,\dots,g'_4;h'_1,\dots,h'_4;H_1,\dots,H_4;N_1,\dots,N_4\Big). \quad (1.10)$$

Further, applying the same reasoning as in [1, Lemma 5], we obtain (1.4) if $j=4$. □

Theorem 1. *For a given N the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ with $c_1=c_2=c_3=c_4=0$ are entire modular forms of weight 5 and character $\chi(\delta)=\operatorname{sgn}\gamma(\frac{-\Delta}{|\delta|})$ (Δ is the determinant of an arbitrary positive quadratic form in 10 variables) for the group $\Gamma_0(4N)$ if the following conditions hold:*

1) $2|g_k,N_k|N \qquad (k=1,2,3,4),$ (1.11)

2) $4\Big|N\sum_{k=1}^{4}\frac{h_k^2}{N_k},\ 4\Big|\sum_{k=1}^{4}\frac{g_k^2}{4N_k},$ (1.12)

3) *for all* α *and* δ *with* $\alpha\delta\equiv 1 \pmod{4N}$

$$\Big(\frac{\prod_{k=1}^{4}N_k}{|\delta|}\Big)\Psi_j(\tau;\alpha g_1,\dots,\alpha g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=$$
$$=\operatorname{sgn}\delta\Big(\frac{-\Delta}{|\delta|}\Big)\Psi_j(\tau;g_1,...,g_4;h_1,...,h_4;0,...,0;N_1,...,N_4) \tag{1.13}$$
$$(j=3,4).$$

Proof. I. As in the case of Theorem 1 from [1], the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ with $c_1=c_2=c_3=c_4=0$ satisfy the condition 1) and, by Lemma 1, also the condition 4) of the definition from [1, p. 53–54].

II. From (1.12), since $2\nmid\delta$, it follows that

$$4\Big|N\delta^2\sum_{k=1}^{4}\frac{h_k^2}{N_k},\ \ 4\Big|\sum_{k=1}^{4}\frac{g_k^2}{4N_k}\delta^{2\varphi(2N_k)-2}. \tag{1.14}$$

Taking into account (1.11), by Lemma 3 from [1], for $n=2$ and $n=0$ (with g_r, h_r, N_r and g_s, h_s, N_s instead of g, h, N), we obtain for each substitution from $\Gamma_0(4N)$

$$\vartheta''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta_{g_sh_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big)=i^{3\eta(\gamma)(sgn\delta-1)}i^{1-|\delta|}\times$$
$$\times\Big(\frac{N_rN_s}{|\delta|}\Big)(\gamma\tau+\delta)^3e\Big(\frac{\beta\delta}{4}\Big(\frac{g_r^2}{4N_r}\delta^{2\varphi(2N_r)-2}+\frac{g_s^2}{4N_s}\delta^{2\varphi(2N_s)-2}\Big)\Big)\times$$
$$\times e\Big(-\frac{\alpha\gamma\delta^2}{4}\Big(\frac{h_r^2}{4N_r}+\frac{h_s^2}{4N_s}\Big)\Big)\times$$
$$\times\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s) \tag{1.15}$$

for $r=1,\ s=2$ and $r=2,\ s=1$.

Analogously, by Lemma 3 from [1], for $n=1$ and $n=0$, we have

$$\vartheta'_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)\vartheta_{g_4h_4}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_4\Big)=\operatorname{sgn}\delta\, i^{2\eta(\gamma)(sgn\delta-1)}\times$$
$$\times i^{1-|\delta|}\Big(\frac{N_3N_4}{|\delta|}\Big)(\gamma\tau+\delta)^2e\Big(\frac{\beta\delta}{4}\Big(\frac{g_3^2}{4N_3}\delta^{2\varphi(2N_3)-2}+\frac{g_4^2}{4N_4}\delta^{2\varphi(2N_4)-2}\Big)\Big)\times$$
$$\times e\Big(-\frac{\alpha\gamma\delta^2}{4}\Big(\frac{h_3^2}{4N_3}+\frac{h_4^2}{4N_4}\Big)\Big)\times$$
$$\times\vartheta'_{\alpha g_3,h_3}(\tau;0,2N_3)\vartheta_{\alpha g_4,h_4}(\tau;0,2N_s). \tag{1.16}$$

Hence, by (1.1), (1.15), (1.16), and (1.14), we obtain

$$\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)}(-1)^{1-|\delta|}\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)(\gamma\tau+\delta)^5\times$$

$$\times\Psi_3(\tau;\alpha g_1,\dots,\alpha g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4),$$

from which according to (1.13) it follows for all α and δ with $\alpha\delta\equiv 1 \pmod{4N}$ that

$$\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\Big)=$$

$$=\operatorname{sgn}\delta\Big(\frac{-\Delta}{|\delta|}\Big)(\gamma\tau+\delta)^5\Psi_3(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4).$$

Analogously, by the just mentioned Lemma 3, for $n=1$ and $n=0$ (with g_k, h_k, N_k $(k=1,2,3,4)$ instead of g, h, N), according to (1.11) and (1.14), we find for all substitutions from $\Gamma_0(4N)$ that

$$\prod_{k=1}^{3}\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)\vartheta_{g_4h_4}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_4\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)}(-1)^{1-|\delta|}\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)(\gamma\tau+\delta)^5\times$$

$$\times\prod_{k=1}^{3}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\vartheta'_{\alpha g_4,h_4}(\tau;0,2N_4). \tag{1.17}$$

Hence, by (1.2), (1.17), and (1.13), for all α and δ with $\alpha\delta\equiv 1 \pmod{4N}$ we have

$$\Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\Big)=$$

$$=\operatorname{sgn}\delta\Big(\frac{-\Delta}{|\delta|}\Big)(\gamma\tau+\delta)^5\Psi_4(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4).$$

Thus the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ with $c_1=c_2=c_3=c_4=0$ satisfy the condition 2) of the definition from [1].

III. According to (13) from [1] we have

$$1)\ \ \vartheta''_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_sh_s}(\tau;0,2N_s)=$$

$$=-\pi^2\sum_{m_r,m_s=-\infty}^{\infty}(-1)^{h_rm_r+h_sm_s}(4N_rm_r+g_r)^2e(\Lambda_1\tau) \tag{1.18}$$

for $r = 1$, $s = 2$ and $r = 2$, $s = 1$, where

$$\Lambda_1 = \sum_{k=1}^{2} \frac{1}{4N_k}(2N_k m_k + g_k/2)^2 = \\ = \sum_{k=1}^{2} \left(N_k m_k^2 + \frac{1}{2} m_k g_k\right) + \frac{1}{4}\sum_{k=1}^{2} g_k^2/4N_k; \tag{1.19}$$

2) $\vartheta'_{g_3 h_3}(\tau; 0, 2N_3)\vartheta_{g_4 h_4}(\tau; 0, 2N_4) =$

$$= \pi i \sum_{m_3, m_4=-\infty}^{\infty} (-1)^{h_3 m_3 + h_4 m_4}(4N_3 m_3 + g_3)e(\Lambda_2 \tau), \tag{1.20}$$

where

$$\Lambda_2 = \sum_{k=3}^{4} \frac{1}{4N_k}(2N_k m_k + g_k/2)^2 = \\ = \sum_{k=3}^{4} \left(N_k m_k^2 + \frac{1}{2} m_k g_k\right) + \frac{1}{4}\sum_{k=3}^{4} g_k^2/4N_k; \tag{1.21}$$

3) $\prod_{k=1}^{3} \vartheta'_{g_k h_k}(\tau; 0, 2N_k)\vartheta_{g_4 h_4}(\tau; 0, 2N_4) =$

$$= -\pi^3 i \sum_{m_1,\ldots,m_4=-\infty}^{\infty} (-1)^{\sum_{k=1}^{4} h_k m_k} \prod_{k=1}^{3}(4N_k m_k + g_k)e(\Lambda \tau),$$

where

$$\Lambda = \sum_{k=1}^{4} \frac{1}{4N_k}(2N_k m_k + g_k/2)^2 = \\ = \sum_{k=1}^{4} \left(N_k m_k^2 + \frac{1}{2} m_k g_k\right) + \frac{1}{4}\sum_{k=1}^{4} g_k^2/4N_k. \tag{1.22}$$

$\Lambda_1 + \Lambda_2$ and Λ are integers by virtue of (1.11) and (1.12). Therefore the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ satisfy the condition 3) of the definition from [1]. $\square$

2.

Lemma 2. *For a given N let*

$$\Phi_3(\tau)=\Phi_3(\tau;g_1,\dots,g_4;h_1,\dots,h_4;c_1,\dots,c_4;N_1,\dots,N_4)=$$
$$=\Big\{\frac{1}{N_1}\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)-$$
$$-\frac{1}{N_2}\vartheta_{g_1h_1}(\tau;c_1,2N_1))\vartheta''_{g_2h_2}(\tau;c_2,2N_2)\Big\}\prod_{k=3}^{4}\vartheta'_{g_kh_k}(\tau;c_k,2N_k) \quad (2.1)$$

and

$$\Phi_4(\tau)=\Phi_4(\tau;g_1,\dots,g_4;h_1,\dots,h_4;c_1,\dots,c_4;N_1,\dots,N_4)=$$
$$=\prod_{k=1}^{4}\vartheta'_{g_kh_k}(\tau;c_k,2N_k), \quad (2.2)$$

where

$$2\big|g_k,\ N_k\big|N\ (k=1,2,3,4),\ \ 4\big|N\sum_{k=1}^{4}\frac{h_k}{N_k}. \quad (2.3)$$

For all substitutions from Γ in the neighborhood of each rational point $\tau=-\frac{\delta}{\gamma}$ ($\gamma\neq0$, $(\gamma,\delta)=1$), we then have

$$(\gamma\tau+\delta)^6\Phi_j(\tau)=\sum_{n=0}^{\infty}D_n^{(j)}e\Big(\frac{n}{4N}\ \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\qquad (j=3,4). \quad (2.4)$$

Proof. I. As in the proof of Lemma 1, by Lemma 4 from [1], it follows for $n=1$ that

$$(\gamma\tau+\delta)^3\vartheta'_{g_3h_3}(\tau;0,2N_3)\vartheta'_{g_4h_4}(\tau;0,2N_4)=$$
$$=-e\big(\frac{3}{4}\operatorname{sgn}\gamma\big)\big(2|\gamma|(N_3N_4)^{1/2}\big)^{-1}\times$$
$$\times\sum_{\substack{H_3 \bmod 2N_3\\ H_4 \bmod 2N_4}}\prod_{k=1}^{4}\varphi_{g'_3g_3h_3}(0,H_3;2N_3)\varphi_{g'_4g_4h_4}(\tau;0,H_4;2N_4)\times$$
$$\times\vartheta'_{g'_3h'_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)\vartheta'_{g'_4h'_4}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_4,2N_4\Big). \quad (2.5)$$

Multiplying (1.7) by (2.5), by virtue of (2.1), we obtain

$$(\gamma\tau+\delta)^6\Phi_3(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=$$

$$= e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4} N_k\Big)^{1/2}\Big)^{-1} \sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Phi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g'_1, \dots, g'_4; h'_1, \dots, h'_4; H_1, \dots, H_4; N_1, \dots, N_4\Big). \quad (2.6)$$

Further, applying the same reasoning as in [1, Lemma 5], we obtain (2.4) if $j = 3$.

II. By Lemma 4 from [1], for $n = 1$ we have

$$(\gamma\tau+\delta)^6 \prod_{k=1}^{4} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) = e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4} N_k\Big)^{1/2}\Big)^{-1} \times$$

$$\times \sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}} \prod_{k=1}^{4} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \prod_{k=1}^{4} \vartheta'_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big).$$

Hence, according to (2.2), it follows that

$$(\gamma\tau+\delta)^6 \Phi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) =$$

$$= e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4} N_k\Big)^{1/2}\Big)^{-1} \sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}} \prod_{k=1}^{4} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Phi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g'_1, \dots, g'_4; h'_1, \dots, h'_4; H_1, \dots, H_4; N_1, \dots, N_4\Big).$$

Further, applying the same reasoning as in [1, Lemma 5], we obtain (2.4) if $j = 4$. □

Theorem 2. *For a given N the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ are entire modular forms of weight 6 and character $\chi(\delta) = (\frac{\Delta}{|\delta|})$ (Δ is the determinant of an arbitrary positive quadratic form in 12 variables) for the group $\Gamma_0(4N)$ provided that the following conditions hold:*

1) $2|g_k, N_k|N \qquad (k = 1, 2, 3, 4),$ (2.7)

2) $4\Big|N \sum_{k=1}^{4} \frac{h_k^2}{N_k},\ 4\Big| \sum_{k=1}^{4} \frac{g_k^2}{4N_k},$ (2.8)

3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\Big(\frac{\prod_{k=1}^{4} N_k}{|\delta|}\Big)\Phi_j(\tau; \alpha g_1, \dots, \alpha g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) =$$

$$= \Big(\frac{\Delta}{|\delta|}\Big)\Phi_j(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4) \tag{2.9}$$
$$(j=3,4).$$

Proof. I. As in the case of Theorem 1, the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1=c_2=c_3=c_4=0$ satisfy the condition 1) and, on account of Lemma 2, also the condition 4) of the definition from [1].

II. Since $2\nmid\delta$, from (2.8) it follows that

$$4\Big|N\delta^2\sum_{k=1}^4\frac{h_k^2}{N_k},\ 4\Big|\sum_{k=1}^4\frac{g_k^2}{4N_k}\delta^{2\varphi(2N_k)-2}. \tag{2.10}$$

Taking into account (2.7), by Lemma 3 from [1], for $n=1$ we find that for all substitutions from $\Gamma_0(4N)$ we have

$$\prod_{k=3}^4\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=i^{3\eta(\gamma)(sgn\delta-1)}i^{1-|\delta|}\Big(\frac{N_3N_4}{|\delta|}\Big)\times$$
$$\times(\gamma\tau+\delta)^3e\Big(\frac{\beta\delta}{4}\Big(\frac{g_3^2}{4N_3}\delta^{2\varphi(2N_3)-2}+\frac{g_4^2}{4N_4}\delta^{2\varphi(2N_4)-2}\Big)\Big)\times$$
$$\times e\Big(-\frac{\alpha\gamma\delta^2}{4}\Big(\frac{h_3^2}{4N_3}+\frac{h_4^2}{4N_4}\Big)\Big)\prod_{k=3}^4\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k). \tag{2.11}$$

Taking into account (2.10) and (2.9), by (2.1), (1.15), and (2.11), for all α and δ with $\alpha\delta\equiv1 \pmod{4N}$ we obtain

$$\Phi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\Big)=$$
$$=i^{2\eta(\gamma)(\operatorname{sgn}\delta-1)}(-1)^{1-|\delta|}\Big(\frac{\prod_{k=1}^4N_k}{|\delta|}\Big)(\gamma\tau+\delta)^6\times$$
$$\times\Phi_3(\tau;\alpha g_1,\dots,\alpha g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=$$
$$=\Big(\frac{\Delta}{|\delta|}\Big)(\gamma\tau+\delta)^6\Phi_3(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4).$$

Analogously, taking into account (2.7) and (2.10), by Lemma 3 from [1], for $n=1$, we have

$$\prod_{k=1}^4\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$
$$=\Big(\frac{\prod_{k=1}^4N_k}{|\delta|}\Big)(\gamma\tau+\delta)^6\prod_{k=1}^4\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k) \tag{2.12}$$

for all substitutions from $\Gamma_0(4N)$.

Hence, by (2.2), (2.12), and (2.9), for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$, we have

$$\Phi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g_1,\dots,g_4; h_1,\dots,h_4; 0,\dots,0; N_1,\dots,N_4\Big) =$$
$$= \Big(\frac{\Delta}{|\delta|}\Big)(\gamma\tau+\delta)^6\Phi_4(\tau; g_1,\dots,g_4; h_1,\dots,h_4; 0,\dots,0; N_1,\dots,N_4).$$

Thus the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the condition 2) of the definition from [1].

III. According to (13) from [1], we have

1) (1.18) with Λ_1 defined by (1.19);

2)
$$\prod_{k=3}^{4}\vartheta'_{g_kh_k}(\tau;0,2N_k) =$$
$$= -\pi^2\sum_{m_3,m_4=-\infty}^{\infty}(-1)^{h_3m_3+h_4m_4}\prod_{k=3}^{4}(4N_km_k+g_k)e(\Lambda_2\tau),$$

where Λ_2 is defined by (1.20);

3)
$$\prod_{k=1}^{4}\vartheta'_{g_kh_k}(\tau;0,2N_k) =$$
$$= \pi^4\sum_{m_1,\dots,m_4=-\infty}^{\infty}(-1)^{\sum_{k=1}^{4}h_km_k}\prod_{k=3}^{4}(4N_km_k+g_k)e(\Lambda\tau),$$

where Λ_2 is defined by (1.21).

Thus the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the condition 3) of the definition from [1]. □

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Author's address:
Faculty of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Republic of Georgia