Georgian Mathematical Journal
Volume 10 (2003), Number 2, 381–399

ON APPROXIMATE LARGE DEVIATIONS FOR 1D DIFFUSION

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In memory of Rezo Chitashvili

Abstract. We establish sufficient conditions under which the rate function for the Euler approximation scheme for a solution of a one-dimensional stochastic differential equation on the torus is close to that for an exact solution of this equation.

2000 Mathematics Subject Classification: 60F10, 60H10, 60H35, 65C30.
Key words and phrases: Itô equation, Euler scheme, large deviations.

1. Introduction

Let X_t satisfy a SDE

$$X_t = x + \int_0^t \sigma(X_s)\, dB_s \quad (\text{mod } 1), \quad t \geq 0, \quad X_0 = x,$$

on the torus $T^1 = [0,1]$, where B_t is a standard Wiener process, σ is bounded, Borel measurable, and non-degenerate. Another way to understand solutions (mod (1)) and the problem setting on T^1 is to say that functions σ and f (below) are periodic with period 1, while X_t is a solution on R^1. Consider the problem of large deviations for the marginal distribution of the functional

$$F_t[X] := \int_0^t f(X_s)\, ds, \quad t \to \infty.$$

Often, the answer is described via the rate function L which in "good cases" is a Fenchel–Legendre transformation of the (convex) function

$$L(\alpha) = \sup_\beta \left(\alpha\beta - H(\beta)\right),$$

where

$$H(\beta) := \lim_{t\to\infty} t^{-1} \log\, E \exp\left(\beta \int_0^t f(X_s)\, ds\right), \tag{1}$$

see [2], [10] et al. Suppose we use an approximate solution of the SDE instead of X_t itself (see [8]) with an approximate function $\tilde{H}$ (if any). The problem is whether $\tilde{H}$ is close to H; then corresponding $\tilde{L}$ should be also close to L.

For SDEs with a unit diffusion and drift, under appropriate conditions the answer turned out to be positive, see [11] (1D) and [12] (multi-dimensional case). The case with a variable diffusion is more technically involved. This is the reason why we do not consider drift here, just to simplify the calculus though a bit, and concentrate on the main new difficulty. This difficulty, which did not appear in the unit diffusion case, is a possible degeneration of the first 'Malliavin derivative' of the approximation process. To keep it positive, a stopping rule is applied. While differentiating the Girsanov identity, however, this stopping rule turns out to be a disadvantage, because of non-differentiability of indicator functions. Hence a good deal of the proof in the paper consists of explanation why Girsanov's formula still can be differentiated.

Assumptions for the main result are: σ is bounded, non-degenerate, periodic and of the class C_b^3, $f \in C_b^1$ and also periodic. Perhaps these rather strong assumptions are due to the method, however, they may also reflect the vulnerability of large deviations to small disturbances of the system under consideration.

We consider a standard Euler scheme,

$$X_t^h = x + \int_0^t \sigma(X_{[s/h]h}^h)\, dB_s, \tag{2}$$

where $[a]$ denotes the integer part of the value a. However, for the purpose of the analysis of semigroup operators, we propose a minor modification of this scheme on a time interval $[0,1]$ which includes a stopping rule. Some reason for this will be presented shortly. Notice that the quasi-generator of X_t^h is $\sigma^2(X_{[t/h]h}^h)\partial_x^2/2$. The approximate value of H is

$$H^h(\beta) := \lim_{t\to\infty} t^{-1} \log\, E \exp\left(\beta \int_0^t f(X_s^h)\, ds\right). \tag{3}$$

The main question is whether H^h, – suppose it is well-defined, – approximates H. If it does, then the approximate rate function

$$L^h(\alpha) = \sup_\beta \left(\alpha\beta - H^h(\beta)\right)$$

will be also close to $L(\alpha)$, see [11]. Later on we shall state conditions which ensure the existence of both limits (1) and (3). For the sake of simplicity, we consider h's such that $n = 1/h$ is an integer.

The approach is based on the analysis of semigroup operators A^β and $A^{h,\beta}$ (see below) which correspond to the processes X_t and X_t^h for $0 \le t \le 1$. Due to assertions (8) and (9) in Lemma 1, the double inequality (10) is crucial in establishing an approximate equality of H^h and H. Hence we present this inequality as a main result of the paper, see below Theorem 1.

We propose the following modification of the algorithm (2) as $0 \le t \le 1$: to investigate the properties of both operators; emphasize that this does not mean any improvement in the definition of the function H^h, but only concerns an analysis of operators at $t = 1$. Let X_t^h satisfy (2), $\rho := h^{2/5}$, $\gamma_n(k) := \inf(t \in [kh,(k+1)h] : |B_t - B_{kh}| > \rho)$, $\gamma_n := \inf_{k\le n-1}\gamma_n(k)$, with a standard notation $\inf\varnothing = +\infty$. Notice that $P(\gamma_n = 1) = 0$. Now we consider the process $X^h_{\min(t,\gamma_n)}$, or, in the other words, X_t^h as $t \le \gamma_n$. This changes the process X_t^h on $[0,1]$ if and only if $\gamma_n < 1$, that is, if the increments of B_t become extremely large. The probability of such an event possesses a sub-exponential bound

$$P(\gamma_n < 1) \le \exp(-n^c), \quad c > 0. \tag{4}$$

Indeed, as $n \to \infty$,

$$\begin{aligned} P(\gamma_n < 1) &\le nP(\gamma_n(1) \le h) = nP(\sup_{t\le h}|B_t| > \rho) = nP(\sup_{t\le 1}|B_t| > \rho/\sqrt{h}) \\ &= 2nP(|B_1| > \rho\sqrt{n}) \sim 4n\frac{1}{(\rho\sqrt{n})}e^{-\rho^2 n/2} = 4n^{9/10}\exp(-n^{1/5}/2). \end{aligned}$$

Possibly, the result and technique can be also helpful in investigating moderate deviations of approximate solutions of SDEs.

2. Auxiliary and Main Results

In this section we consider the process and operator on the torus T^1 and $C(T^1)$ respectively. It is known [2] that the limit H does exist and is equal to

$$H(\beta) = \log r(A^\beta),$$

where $r(A^\beta)$ is the spectral radius of the semigroup operator on the function space $C(T^1)$,

$$A^\beta\phi(x) = E_x\phi(X_1)\exp\left(\beta\int_0^1 f(X_s)\,ds\right).$$

We will state this assertion in Lemma 1 for the reader's convenience. We also need an approximate semigroup operator $A^{h,\beta}$ on $C(T^1)$,

$$A^{h,\beta}\phi(x) = E_x\phi(X_1)\exp\left(\beta\int_0^1 f(X_s^h)\,ds\right),$$

and $r(A^{\beta,h})$ denotes its spectral radius.

Recall [5] that the operator A is called positive if $f \ge 0$ implies $Af \ge 0$. The operator A is called 1-bounded if for any $g:\ g \in C(T^1),\ g \ge 0,\ g \ne 0$, with some $c = c(g)$,

$$0 < c^{-1} \le Ag(x) \le c. \tag{5}$$

Notice that for any such g, $\sqrt{g} \in L_1(T^1)$, and $\|\sqrt{g}\|_{L_1(T^1)} > 0$. Denote

$$q(A) = \inf(q:\ |\lambda| \le q\,r(A),\ \lambda \in \text{spectrum of } A);$$

this value is called the spectral gap of the operator A.

Lemma 1.

1. (*Frobenius–Krasnosel'skii:* [5], *Theorem* 11.5) *Let A be a positive compact 1-bounded operator on $C(T^1)$. Then the spectral radius $r(A^\beta)$ is an isolated simple spectrum point of the operator A, its eigenfunction is positive, and there exists $q < 1$ such that all other spectrum points λ satisfy the bound $|\lambda| \le q\, r(A^\beta)$.*

   For the next assertions, 2 *to* 6, *it is assumed that σ, σ^{-1}, f are bounded and Borel, and $\sigma > 0$.*
2. (*Krylov and Safonov* [6]) *Two operators A^β and $A^{h,\beta}$ are compact in $C(T^1)$.*
3. *For any $b > 0$, there exists C such that for any $|\beta| < b$,*
$$C^{-1} \le r(A^\beta) \le C, \tag{6}$$
   and uniformly in $h > 0$,
$$C^{-1} \le r(A^{h,\beta}) \le C. \tag{7}$$
4. *For any $b > 0$, there exists C such that for any $|\beta| < b$ and all h small enough,*
$$|\log r(A^{h,\beta}) - \log r(A^\beta)| \le C\|A^{h,\beta} - A^\beta\|_{C(T^1)}. \tag{8}$$
5. (*Freidlin–Gärtner, see* [2]) *The limit* (1) *does exist, and $H(\beta)$ coincides with $\lambda(A^\beta) := \log r(A^\beta)$. Moreover, the limit in* (3) *exists, too, and coincides with the spectral radius of the approximate semigroup operator $A^{h,\beta}$,*
$$H^h(\beta) = \log\ r(A^{h,\beta}). \tag{9}$$
6. *For any $b > 0$ there exist $\bar{q} < 1$ and $C > 0$ such that for any $|\beta| < b$,*
$$q(A^\beta) \le \bar{q}, \quad \textit{and} \quad \Delta_0 \ge C^{-1}.$$

For additional details of the proof of this lemma, which just combines several important technical results from various areas, see [11]. Now we formulate the main result of the paper.

Theorem 1. *Let $\sigma \in C_b^3$, $\sigma^{-1} > 0$, and $f \in C_b^1$. Then for any $b > 0$ there exists C such that for any $|\beta| < b$,*
$$\|A^\beta - A^{h,\beta}\|_{C(T^1)} \le \|A^\beta - A^{h,\beta}\|_{C(R^1)} \le Ch^{1/2}. \tag{10}$$

Notice that all X_t^h's and X_t are defined on a common probability space with a Wiener process.

Corollary 1. *Under the assumptions of Theorem* 1, *for any* $\inf f < \alpha < \sup f$, *there exists $\beta(\alpha)$ such that $L(\alpha) = \alpha\beta(\alpha) - H(\beta(\alpha))$, and*
$$L(\alpha) \le L^h(\alpha) + C_{\beta(\alpha)} h^{1/2} \Delta_0^{-1};$$
if, in addition, H is strictly convex at $\beta(\alpha)$, then
$$L(\alpha) \ge L^h(\alpha) - C_{\beta(\alpha)} h^{1/2} \Delta_0^{-1};$$

if $\alpha \notin [\inf f, \sup f]$ *, and* $f \neq const$*, then*

$$L(\alpha) = +\infty \quad \textit{and} \quad L^h(\alpha) = +\infty.$$

3. Proof of Theorem 1

1. We use the idea from [1]. Let $\|\phi\|_{C(R^1)} \leq 1$. We are going to compare the two expressions,

$$E_x\phi(X_1)\exp\left(\int_0^1 \beta f(X_s)ds\right) \quad \text{and} \quad E_x\phi(X_1^h)\exp\left(\int_0^1 \beta f(X_s^h)ds\right).$$

The goal is to establish a bound for their difference which may only depend on $\|\phi\|_C$, but not on the modulus of continuity of ϕ. We will use the representation $E_x\phi(X_1)\exp(\beta\int_0^1 f(X_s)ds) = u(0,x)$, where $u(t,x)$ is an appropriate solution of the problem

$$\partial_t u + Lu + \beta f u = 0, \quad u(1,x) = \phi(x),$$

and L denotes the generator of the process X_t,

$$L = a(x)\partial_x^2, \quad a(x) = \sigma^2(x)/2.$$

It is well-known that $u(0,x) = E_x\phi(X_1)\exp(\beta\int_0^1 f(X(s))ds)$. The solution exists and is unique in the class of functions

$$C([0,1]\times R^1)\bigcap_{t_0<1}\bigcap_{p>1} W^{1,2}_{p,loc}([0,t_0]\times R^1),$$

see [7] (also see [9] for an exact reference); we do not use Hölder classes (which would be appropriate) because of the lack of references concerning equations with only continuous initial data (i.e. ϕ). Embedding theorems and a priori inequalities in Sobolev spaces (cf. [7], ch. 3 and 5) imply the following for any $t_0 < 1$ (in the next three formulas $C = C(t_0,\beta)$):

$$\|u\|_{C^{0,1}([0,t_0]\times R^1)} \leq C\|\phi\|_C.$$

Moreover, since the coefficients $\sigma \in C_b^3$ and $f \in C_b^1$, we can differentiate the equation with respect to x, and by virtue of the same a priori estimates in Sobolev spaces, embedding theorems and the equation,

$$\|u\|_{C^{1,2}([0,t_0]\times R^1)} \leq C\|\phi\|_{C(R^1)}. \tag{11}$$

Also, in the maximal cylinder, $[0,1]\times R^1$,

$$\|u\|_{C([0,1]\times R^1)} \leq C\|\phi\|_{C(R^1)}. \tag{12}$$

Finally, we will use the bound (e.g., it is a straightforward consequence from [3], Ch. 9, Theorem 7),

$$\|u(t,\cdot)\|_{C^1(R^d)} \leq C(1-t)^{-1/2}\|\phi\|_C, \quad 0 \leq t < 1. \tag{13}$$

2. Now, with $h = 1/n$, we find

$$
\begin{aligned}
&E_x\phi(X_1^h)\exp\left(\int\limits_0^1 \beta f(X_s^h)ds\right) - E_x\phi(X_1)\exp\left(\int\limits_0^1 \beta f(X_s)ds\right) \\
&= E_x u(1, X_1^h)\exp\left(\int\limits_0^1 \beta f(X_s^h)ds\right) - u(0,x) \\
&= E_x\left(u(1, X_1^h)\exp\left(\int\limits_0^1 \beta f(X_s^h)ds\right) - u(0,x)\right) 1(\gamma_n < 1) \\
&\quad + E_x\left(u(1, X_1^h)\exp\left(\int\limits_0^1 \beta f(X_s^h)ds\right) - u(0,x)\right) 1(\gamma_n \geq 1).
\end{aligned}
$$

Here the first expectation is estimated due to the bound (4),

$$
\begin{aligned}
&\left| E_x\left(u(1, X_1^h)\exp\left(\int\limits_0^1 \beta f(X_s^h)ds\right) - u(0,x)\right) 1(\gamma_n < 1)\right| \\
&\leq C\|\phi\|_{C(R^1)} P(\gamma_n < 1) \leq C\|\phi\|_{C(R^1)}\exp(-n^c).
\end{aligned}
$$

Hence it remains to estimate the second term with $1(\gamma_n > 1)$ (recall that $P(\gamma_n = 1) = 0$). Denote $\Lambda_n = 1(\gamma_n > 1)$, and let us use the identity,

$$
\begin{aligned}
&E_x\left(u(1, X_1^h)\exp\left(\int\limits_0^1 \beta f(X_s^h)ds\right) - u(0,x)\right)\Lambda_n \\
&= \sum_{k=0}^{n-1} E_x\left[u\left((k+1/n), X_{(k+1)/n}^h\right)\exp\left(\int\limits_0^{(k+1)/n} \beta f(X_s^h)ds\right)\right. \\
&\quad \left. - u\left(k/n, X_{k/n}^h\right)\exp\left(\int\limits_0^{k/n} \beta f(X_s^h)\, ds\right)\right]\Lambda_n.
\end{aligned}
$$

Denote

$$
L_z = a(z)\partial_x^2.
$$

The operator $L_{X_{k/n}^h}$ is the generator of the process X_s^h on $k/n \leq s \leq (k+1)/n$. Due to the Itô formula on the set $(k+1)/n \leq \gamma_n$ which holds true for any k including $k = n-1$ (see below – at the end of the paper – about the latter case)

$$
E_x\left[u\left((k+1)/n, X_{(k+1)/n}^h\right)\exp\left(\int\limits_0^{(k+1)/n} \beta f(X_s^h)ds\right)\right.
$$

$$
\left. -u\left(k/n, X^h_{\frac{k}{n}}\right) \exp\left(\int\limits_0^{k/n} \beta f(X^h_s) ds\right)\right] \Lambda_n
$$

$$
= E_x \Lambda_n \int\limits_{k/n}^{(k+1)/n} \exp\left(\int\limits_0^t \beta f(X^h_s) ds\right) \left[\partial_t + L_{X^h_{\frac{k}{n}}} + f(X^h_t)\right] u(t, X^h_t)\, dt
$$

$$
= E_x \Lambda_n \int\limits_{k/n}^{(k+1)/n} \exp\left(\int\limits_0^t \beta f(X^h_s) ds\right) \left[-Lu(t, X^h_t) + L_{X^h_{\frac{k}{n}}} u(t, X^h_t)\right] dt
$$

$$
= \int\limits_{k/n}^{(k+1)/n} E_x \Lambda_n \exp\left(\int\limits_0^t \beta f(X^h_s) ds\right) \left[-a(X^h_t) + a(X^h_{\frac{k}{n}})\right] u_{xx}(t, X^h_t)\, dt
$$

$$
= - \int\limits_{k/n}^{(k+1)/n} E_x \Lambda_n \exp\left(\int\limits_0^t \beta f(X^h_s)\, ds\right) \left(\int\limits_{k/n}^t a(X^h_{\frac{k}{n}}) a_{xx}(X^h_s)\, ds\right) u_{xx}(t, X^h_t)\, dt
$$

$$
- \int\limits_{k/n}^{(k+1)/n} E_x \Lambda_n \exp\left(\int\limits_0^t \beta f(X^h_s) ds\right) \left(\int\limits_{k/n}^t a_x(X^h_s) \sigma(X^h_{\frac{k}{n}})\, dB_s\right) u_{xx}(t, X^h_t)\, dt
$$

$$
\equiv -I_1 - I_2.
$$

We have used the identity due to Itô's formula on the set $t \le \gamma_n$ (in the sequel, $k/n \le t \le (k+1)/n$),

$$
a(X^h_t) - a(X^h_{k/n}) = \int\limits_{k/n}^t L_{X^h_{k/n}} a(X^h_s)\, ds + \int\limits_{k/n}^t a_x(X^h_s) \sigma(X^h_{k/n})\, dB_s.
$$

3. Consider first the case $k/n \le 1/2$. We have

$$
\left| \exp\left(\int\limits_0^t \beta f(X^h_s) ds\right) \left(\int\limits_{k/n}^t L_{X^h_{k/n}} a(X^h_s)\, ds\right) u_{xx}(t, X^h_t) \right| \le Ch.
$$

Hence, the integral I_1 possesses the bound

$$
|I_1| = \left| \int\limits_{k/n}^{(k+1)/n} E_x \Lambda_n \exp\left(\int\limits_0^t \beta f(X^h_s) ds\right) \left(\int\limits_{k/n}^t L_{X^h_{k/n}} a(X^h_s)\, ds\right) u_{xx}(t, X^h_t)\, dt \right|
$$

$$
\le C\, h^2.
$$

Let us estimate I_2:

$$\left| \int\limits_{k/n}^{(k+1)/n} E_x \, \Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h) ds\right) \left(\int\limits_{k/n}^t a_x(X_s^h)\sigma(X_{\frac{k}{n}}^h)\, dB_s\right) u_{xx}(t, X_t^h)\, dt \right|$$

$$\le C \int\limits_{k/n}^{(k+1)/n} E_x \left| \int\limits_{k/n}^t a_x(X_s^h)\sigma(X_{\frac{k}{n}}^h)\, dB_s \right| dt \le Ch^{3/2}.$$

Overall, one half of the whole sum is estimated by

$$\sum_{k=0}^{n/2} \left| E_x \left[u\left((k+1/n), X_{(k+1)/n}^h\right) \exp\left(\int\limits_0^{(k+1)/n} \beta f(X_s^h) ds\right) \right.\right.$$

$$\left.\left. - u\left(k/n, X_{k/n}^h\right) \exp\left(\int\limits_0^{k/n} \beta f(X_s^h)\, ds\right)\right] \Lambda_n \right| \le Ch^{1/2}.$$

4. Consider the case $k/n > 1/2$, the term I_1. Note that generally speaking u_{xx} is not bounded as $t \to 1$, therefore, we must get rid of terms like this. We will replace it by u_x which has an integrable singularity at zero; ideally, it would be nice to fulfill a second differentiation and replace this term by the bounded u, but there are technical reasons which prevent us from this second differentiation. We use the Bismut approach to stochastic calculus, based on Girsanov's formula. Denote by $X_t^{h,\epsilon}$ the solution of the SDE

$$X_t^{h,\epsilon} = x + \int\limits_0^t \sigma(X_{[s/h]h}^{h,\epsilon})\, (dB_s + \epsilon\, ds), \tag{14}$$

$$Y_t^{h,\epsilon} = \frac{\partial X_t^{h,\epsilon}}{\partial \epsilon},\ Z_t^{h,\epsilon} = \frac{\partial Y_t^{h,\epsilon}}{\partial \epsilon}$$

(both derivative processes are understood in the classical sense, see [2]), and

$$Y_t^h = Y_t^{h,0},\ Z_t^h = Z_t^{h,0}.$$

In this stage we use essentially the factor Λ_n in all expectations, and hence can restrict ourselves to considering stopped processes, $X_{t\wedge\gamma_n}^h$, etc. We are going to show the following properties and bounds: for some $h_0 > 0$,

- $$Y_{t\wedge\gamma_n}^h > 0 \ \text{ for any } \ t > 0,\ h < h_0; \tag{15}$$
- $$\limsup_{n\to\infty} \sup_{h<h_0} \sup_{0\le t\le 1} E\Lambda_n\left((Y_t^h)^p\right) < \infty, \quad \forall p > 0; \tag{16}$$
- $$\limsup_{n\to\infty} \sup_{h<h_0} \sup_{0\le t\le 1} E\Lambda_n(Y_t^h)^{-p} < \infty, \quad \forall p > 0; \tag{17}$$

•
$$\limsup_{n\to\infty}\sup_{h<h_0}\sup_{0\le t\le 1} E\Lambda_n\left(|Z^h_{t\wedge\gamma_n}|^p\right)<\infty,\quad \forall p>0. \tag{18}$$

We always have for $t\le\gamma_n$,

$$Y^{h,\epsilon}_t=\int_0^t\sigma(X^{h,\epsilon}_{[s/h]h})\,ds+\int_0^t\sigma_x(X^{h,\epsilon}_{[s/h]h})Y^{h,\epsilon}_{[s/h]h}\,(dB_s+\epsilon\,ds).$$

So, for $\epsilon=0$,

$$Y^h_t=\int_0^t\sigma(X^h_{[s/h]h})\,ds+\int_0^t\sigma_x(X^h_{[s/h]h})Y^h_{[s/h]h}\,dB_s. \tag{19}$$

All the properties (16)–(18) mentioned above follow from this representation even though we cannot assert that Y^h_t nor $(Y^h_t)^{-1}$ is bounded. The first consequence of (19) is the bound (16). It follows from the Burkholder – Davies – Gundy inequality. (A similar bound holds true without γ_n, too; but we will not use it here.) Next, denote $b_k=\sigma(X^h_{kh})$, and $d_k=\sigma_x(X^h_{kh})$. Both random sequences are bounded, and $\inf_k\inf_\omega b_k>0$. From (19) we conclude,

$$Y^h_{(k+1)h}=Y^h_{kh}(1+d_k\Delta B_{(k+1)h})+b_kh,\quad \Delta B_{(k+1)h}:=B_{(k+1)h}-B_{kh}.$$

Hence, by induction,

$$Y^h_{kh}=\sum_{j=0}^{k-1}hb_j\prod_{i=j+1}^{k-1}(1+d_i\Delta B_{(i+1)h}).$$

A similar representation holds true for $kh\le t<(k+1)h$:

$$Y^h_t=(1+d_k(B_t-B_{kh}))\sum_{j=0}^{k-1}hb_j\prod_{i=j+1}^{k-1}(1+d_i\Delta B_{(i+1)h})+b_k(t-kh).$$

Now choose $h>0$ so small that $\rho\sup_k|d_k|<1$; it suffices that $\rho\,\|\sigma_x\|_C<1$. Then all values Y^h_t (as $t\le\gamma_n$) are positive.

5. Let us show the bound

$$\sup_{h<h_0}E\Lambda_n|Y^h_t|^{-p}<\infty.$$

We have

$$E\Lambda_n|Y^h_t|^{-p}\le\sum_{m>0}(m+1)^pP(|Y^h_{t\wedge\gamma_n}|\le 1/m;\,\gamma_n>1),$$

so that we only need to estimate the probabilities under the sum. For the sake of simplicity, we restrict the calculus to the case $0<t=kh\le 1$; the case $0<(k-1)h<t<kh\le 1$ is considered similarly although it requires more new notations. Notice that on $\{(i+1)/n\le\gamma_n\}$, $\ln(1+d_i\Delta B_{(i+1)h})=$

$d_i\Delta B_{(i+1)h} - d_i^2(\Delta B_{(i+1)h})^2/2 + o_i(h)$, where $\sup_i |o_i(h)|/h \to 0$. We can choose $h > 0$ so small that $\inf_i(n\, o_i(h)) \ge -\ln 2$. Let $\kappa^{-1} \in (0, \inf_i b_i)$. Then we get

$$
\begin{aligned}
&P(|Y^h_{kh}| \le 1/m;\, \gamma_n > 1) = P\left(\sum_{j=0}^{k-1} hb_j \prod_{i=j+1}^{k-1}(1 + d_i\Delta B_{(i+1)h}) \le 1/m;\, \gamma_n > 1\right)\\
&= P\left(\sum_{j=0}^{k-1} hb_j \exp\left(\sum_{i=j+1}^{k-1}\ln(1 + d_i\Delta B_{(i+1)h})\right) \le 1/m;\, \gamma_n > 1\right)\\
&\le P\left(\sum_{j=0}^{k-1} hb_j \exp\left(\sum_{i=j+1}^{k-1}\left[d_i\Delta B_{(i+1)h} - (d_i\Delta B_{(i+1)h})^2/2 + o_i(h)\right]\right)\right.\\
&\qquad\qquad\qquad\qquad \left.\le 1/m;\, \gamma_n > 1\right)\\
&\le P\left(\inf_{0\le j\le k-1}\exp\left(\sum_{i=j+1}^{k-1}\left[d_i\Delta B_{(i+1)h} - (d_i\Delta B_{(i+1)h})^2/2\right]\right) \le 2/(mt);\, \gamma_n > 1\right)\\
&= P\left(\inf_{0\le j\le k-1}\left(\sum_{i=j+1}^{k-1}\left[d_i\Delta B_{(i+1)h} - (d_i\Delta B_{(i+1)h})^2/2\right]\right)\right.\\
&\qquad\qquad\qquad\qquad \left.\le -\log(mt/2);\, \gamma_n > 1)\right)\\
&\le P\left(\sup_{0\le j\le k-1}\left(\sum_{i=0}^{j}\left[d_i\Delta B_{(i+1)h} - (d_i\Delta B_{(i+1)h})^2/2\right]\right)\right.\\
&\qquad\qquad\qquad\qquad \left.\ge \log(mt/2)/(2\kappa);\, \gamma_n > 1\right)\\
&\quad + P\left(-\sum_{i=0}^{k-1}\left[d_i\Delta B_{(i+1)h} - (d_i\Delta B_{(i+1)h})^2/2\right] \ge \log(mt/2)/(2\kappa);\, \gamma_n > 1\right)\\
&:= P_1 + P_2.
\end{aligned}
$$

6. Let us estimate P_2 (we use a constant $\lambda > 0$ to be fixed later):

$$
\begin{aligned}
P_2 &\le P\left(\exp\left(-\lambda\sum_{i=0}^{k-1}\left[d_i\Delta B_{(i+1)h} - (d_i\Delta B_{(i+1)h})^2/2\right]\right)\right.\\
&\qquad\qquad \left.\ge \exp\left((\lambda/(2\kappa))\log(mt/2)\right)\right)\\
&\le \exp\left(-(\lambda/(2\kappa))\log(mt/2)\right)
\end{aligned}
$$

$$\times E\exp\left(-\sum_{i=0}^{k-1}\left[\lambda d_i\Delta B_{(i+1)h} \mp \lambda^2 d_i^2 h - \lambda(d_i\Delta B_{(i+1)h})^2/2\right]\right)$$
$$\le \exp\left(-(\lambda/(2\kappa))\log(mt/2)\right)\left(E\exp\left(-2\lambda\sum_{i=0}^{k-1} d_i\Delta B_{(i+1)h} - 2\lambda^2 d_i^2 h\right)\right)^{1/2}$$
$$\times \exp(C\lambda^2)\left(E\exp\left(C\sum_{i=0}^{k-1}\lambda(\Delta B_{(i+1)h})^2\right)\right)^{1/2}.$$

Since $E\exp\left(-2\lambda\sum_{i=0}^{k-1} d_i\Delta B_{(i+1)h} - 2\lambda^2 d_i^2 h\right) = 1$, and for $h > 0$ small enough

$$E\exp\left(C\lambda\sum_{i=0}^{k-1}(\Delta B_{(i+1)h})^2\right) = \left(\frac{1}{\sqrt{1-\lambda Ch}}\right)^k \le \left(1+\frac{\lambda C}{2n}\right)^n \le e^{\lambda C/2},$$

we get

$$P_2 \le C_{\lambda,t} m^{-\lambda/2}, \qquad C_{\lambda,t} < \infty.$$

7. Now let us estimate P_1. Using the Kolmogorov–Doob inequality, we get a similar bound:

$$P_1 \le P\left(\sup_{0\le j\le k-1}\exp\left(\sum_{i=0}^{j}\left[d_i\Delta B_{(i+1)h} - (d_i\Delta B_{(i+1)h})^2/2\right]\right)\right.$$
$$\left.\ge \exp\left((\lambda/(2\kappa))\log(mt/2)\right)\right)$$
$$\le \exp(-(\lambda/(2\kappa))\log(mt/2))$$
$$\times E\exp\left(\sum_{i=0}^{k-1}\left[\lambda d_i\Delta B_{(i+1)h} \mp \lambda^2 d_i^2 h - \lambda(d_i\Delta B_{(i+1)h})^2/2\right]\right)$$
$$\le \exp\left(-(\lambda/(2\kappa))\log(mt/2) + C\lambda^2\right)$$
$$\times\left(E\exp\left(2\lambda\sum_{i=0}^{k-1} d_i\Delta B_{(i+1)h} - (4\lambda^2/2)\sum_{i=0}^{k-1} d_i^2 h\right)\right)^{1/2}$$
$$\le C_{\lambda,t} m^{-\lambda/2}, \qquad C_{\lambda,t} < \infty.$$

Hence, choosing $\lambda > 2p + 2$, we estimate,

$$E\Lambda_n|Y^h_{kh}|^{-p} \le \sum_{m>0}(m+1)^p P(|Y^h_{kh}| \le 1/m;\ \gamma_n > 1)$$
$$\le \sum_{m>0}(m+1)^p C_{\lambda,t} m^{-\lambda/2} < \infty.$$

Thus (17) is established; we remind that the general case $t \ne kh$ is considered similarly.

8. Now (18) follows from (17) and a representation

$$Z_t^{h,\epsilon} = 2\int_0^t \sigma_x(X_{[s/h]h}^{h,\epsilon})Y_{[s/h]h}^{h,\epsilon}\,ds + \int_0^t \sigma_{xx}(X_{[s/h]h}^{h,\epsilon})\left(Y_{[s/h]h}^{h,\epsilon}\right)^2(dB_s + \epsilon\,ds)$$
$$+ \int_0^t \sigma_x(X_{[s/h]h}^{h,\epsilon})Z_{[s/h]h}^{h,\epsilon}(dB_s + \epsilon\,ds).$$

At $\epsilon = 0$ this reads as

$$Z_t^h = 2\int_0^t \sigma_x(X_{[s/h]h}^h)Y_{[s/h]h}^h\,ds + \int_0^t \sigma_{xx}(X_{[s/h]h}^h)\left(Y_{[s/h]h}^h\right)^2 dB_s$$
$$+ \int_0^t \sigma_x(X_{[s/h]h}^h)Z_{[s/h]h}^h dB_s.$$

By virtue of (16) and (17) this implies (18).

9. Let $\Lambda_n^\epsilon = 1(\gamma_n^\epsilon > 1)$, where γ_n^ϵ is defined as γ_n for the process $B_t + \epsilon t,\ t \geq 0$. To get rid of u_{xx} in our integrals, consider Girsanov's transformation,

$$E_x\,\Lambda_n^\epsilon \exp\left(\int_0^t \beta f(X_s^{h,\epsilon})\,ds\right)\left(\int_{k/n}^t a(X_{k/n}^{h,\epsilon})a_{xx}(X_s^{h,\epsilon})\,ds\right)u_x(t, X_t^{h,\epsilon})$$
$$\times\left(Y_t^{h,\epsilon}\right)^{-1}\exp\left(-\epsilon B_1 - \frac{\epsilon^2}{2}\right) = const.$$

We differentiate this identity with respect to ϵ at $\epsilon = 0$ to get the following:

$$0 = \lim_{\epsilon\to 0} E_x\,\frac{\Lambda_n^\epsilon - \Lambda_n}{\epsilon}\exp\left(\int_0^t \beta f(X_s^h)\,ds\right)\left(\int_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t, X_t^h)$$
$$+ E_x\,\Lambda_n \exp\left(\int_0^t \beta f(X_s^h)\,ds\right)\left(\int_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_{xx}(t, X_t^h)$$
$$+ E_x\,\Lambda_n \exp\left(\int_0^t \beta f(X_s^h)\,ds\right)\left(\int_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t, X_t^h)$$
$$\times\left(Y_t^h\right)^{-1}\int_0^t \beta f_x(X_s^h)Y_s^h\,ds$$

$$
\begin{aligned}
&+ E_x\,\Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h)\,ds\right)\left(\int\limits_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t,X_t^h)\\
&\times \left(Y_t^h\right)^{-1}\int\limits_{k/n}^t a_x(X_{k/n}^h)Y_{k/n}^h a_{xx}(X_s^h)\,ds\\
&+ E_x\,\Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h)\,ds\right)\left(\int\limits_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t,X_t^h)\\
&\times \left(Y_t^h\right)^{-1}\int\limits_{k/n}^t a(X_{k/n}^h)a_{xxx}(X_s^h)Y_s^h\,ds\\
&+ E_x\,\Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h)\,ds\right)\left(\int\limits_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t,X_t^h)\\
&\times \left(Y_t^h\right)^{-1}\left(-Z_t^h\left(Y_t^h\right)^{-2}\right)\\
&+ E_x\,\Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^{h,\epsilon})\,ds\right)\left(\int\limits_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t,X_t^h)\\
&\times \left(Y_t^h\right)^{-1}(-B_1).
\end{aligned}
$$

Here all the terms with u_x but the first one possess the bound by absolute value, $\le C(1-t)^{-1/2}h$.

10. Let us to show the bound

$$
\limsup_{\epsilon\to 0}\left|E_x\,\frac{\Lambda_n^\epsilon-\Lambda_n}{\epsilon}\exp\left(\int\limits_0^t \beta f(X_s^h)\,ds\right)\left(\int\limits_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t,X_t^h)\right|
\le Che^{-n^c}/\sqrt{1-t}.
$$

We have,

$$
\limsup_{\epsilon\to 0}\left|E_x\,\frac{\Lambda_n^\epsilon-\Lambda_n}{\epsilon}\exp\left(\int\limits_0^t \beta f(X_s^h)\,ds\right)\left(\int\limits_{k/n}^t a(X_{k/n}^h)a_{xx}(X_s^h)\,ds\right)u_x(t,X_t^h)\right|
\le Ch(1-t)^{-1/2}\limsup_{\epsilon\to 0}E_x\left|\frac{\Lambda_n^\epsilon-\Lambda_n}{\epsilon}\right|.
$$

The random variables Λ_n and Λ_n^ϵ may be represented as follows:

$$\Lambda_n = \prod_{i=1}^n 1\left(\sup_{ih<t<(i+1)h} |B_t - B_{ih}| \le \rho\right),$$

$$\Lambda_n^\epsilon = \prod_{i=1}^n 1\left(\sup_{ih<t<(i+1)h} |B_t - B_{ih} + \epsilon(t-ih)| \le \rho\right).$$

Hence

$$\begin{aligned}
|\Lambda_n^\epsilon - \Lambda_n| \le \sum_{i=1}^n \Bigg| & 1\left(\sup_{ih<t<(i+1)h} |B_t - B_{ih} + \epsilon(t-ih)| \le \rho\right) \\
& -1\left(\sup_{ih<t<(i+1)h} |B_t - B_{ih}| \le \rho\right)\Bigg| \\
\le & \sum_{i=1}^n 1\left(\rho - \epsilon h \le \sup_{ih<t<(i+1)h} |B_t - B_{ih}| \le \rho + \epsilon h\right).
\end{aligned}$$

So, as $\epsilon \ll h$,

$$\begin{aligned}
E_x \left|\frac{\Lambda_n^\epsilon - \Lambda_n}{\epsilon}\right| &\le n\epsilon^{-1} E_x 1\left(\rho - \epsilon h \le \sup_{0<t<h} |B_t| \le \rho + \epsilon h\right) \\
&= n\epsilon^{-1} P_x\left(h^{-1/2}(\rho - \epsilon h) \le \sup_{0<t<1} |B_t| \le h^{-1/2}(\rho + \epsilon h)\right) \\
&= 4n\epsilon^{-1} \int\limits_{h^{-1/2}(\rho-\epsilon h)}^{h^{-1/2}(\rho+\epsilon h)} (2\pi)^{-1/2} \exp(-x^2/2)\, dx \\
&\le 4n\epsilon^{-1} 2h^{-1/2}\epsilon h (2\pi)^{-1/2} \exp(-(h^{-1/2}(\rho - \epsilon h))^2/2) \\
&= \frac{8}{\sqrt{2\pi}} n^{1/2} \exp\left(-(h^{-1/2}(h^{2/5} - \epsilon h))^2/2\right) \\
&\sim \frac{8}{\sqrt{2\pi}} n^{1/2} \exp\left(-n^{1/5}/2\right) = o(h), \qquad n \to \infty.
\end{aligned}$$

Thus, in fact, all the terms with u_x including the first one possess the bound by absolute value, $\le C(1-t)^{-1/2}h$. After integration with respect to t from kh to $(k+1)h$, for $k \le n-2$ this gives a bound

$$\left| E_x \Lambda_n \int\limits_{k/n}^{(k+1)/n} \exp\left(\int\limits_0^t \beta f(X_s^h)\, ds\right) \left(\int\limits_{k/n}^t a(X_{k/n}^h) a_{xx}(X_s^h)\, ds\right) u_{xx}(t, X_t^h)\, dt\right|$$
$$\le C(1-(k+1)h)^{-1/2} h^2.$$

After summation over $n/2 < k \le n-2$, we get

$$\sum_{n/2<k\le n-2} \Bigg| E_x \Lambda_n \int_{k/n}^{(k+1)/n} \exp\left(\int_0^{(k+1)/n} \beta f(X_s^h)\, ds \right) \times \left(\int_{k/n}^{(k+1)/n} a(X_{k/n}^h) a_{xx}(X_s^h)\, ds \right) u_{xx}(t, X_t^h)\, dt \Bigg| \le Ch.$$

Notice that we have used $a \in C_b^3$, $f \in C_b^1$.

11. Let us consider the case $k = n-1$. The reason why it should be estimated separately is that we cannot apply the Itô formula directly up to $t = 1$ since u_x and u_{xx} may be unbounded.

For all terms but the first one, we get an upper bound, with $1 - h < T < 1$,

$$\int_{1-h}^{T} Ch(1-t)^{-1/2}\, dt \le Ch.$$

The first term is bounded by the value

$$Ce^{-n^c} \int_{1-h}^{T} (1-t)^{-1/2}\, dt \le Ch^{1/2} e^{-n^c} \le Ch.$$

The bound is, of course, rather rough, but we cannot make better the inequality in the step (**10**). Now we put $T \to 1$ and apply the Fatou lemma to get the same bound with $T = 1$.

12. The <u>integral I_2 for $k/n > 1/2$</u> is estimated similarly. We have

$$E_x \Lambda_n^\epsilon \exp\left(\int_0^t \beta f(X_s^{h,\epsilon})\, ds \right) \left(\int_{k/n}^t \sigma(X_{k/n}^{h,\epsilon}) a_x(X_s^{h,\epsilon})\, dB_s^\epsilon \right) u_x(t, X_t^{h,\epsilon}) \times \left(Y_t^{h,\epsilon} \right)^{-1} \exp\left(-\epsilon B_1 - \frac{\epsilon^2}{2} \right) = const.$$

Now differentiate this identity with respect to ϵ, which at $\epsilon = 0$ reads,

$$0 = \lim_{\epsilon\to 0} E_x \frac{\Lambda_n^\epsilon - \Lambda_n}{\epsilon} \exp\left(\int_0^t \beta f(X_s^h)\, ds \right) \left(\int_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s \right) u_x(t, X_t^h)$$

$$+ E_x \Lambda_n \exp\left(\int_0^t \beta f(X_s^h)\, ds \right) \left(\int_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s \right) u_{xx}(t, X_t^h)$$

$$
\begin{aligned}
&+ E_x\, \Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h)\, ds\right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s\right) u_x(t, X_t^h) \\
&\quad \times \left(Y_t^h\right)^{-1} \int\limits_0^t \beta f_x(X_s^h) Y_s^h\, ds \\
&+ E_x\, \Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h)\, ds\right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s\right) u_x(t, X_t^h) \\
&\quad \times \left(Y_t^h\right)^{-1} \int\limits_{k/n}^t \sigma_x(X_{k/n}^h) Y_{k/n}^h a_x(X_s^h)\, dB_s \\
&+ E_x\, \Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h)\, ds\right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s\right) u_x(t, X_t^h) \\
&\quad \times \left(Y_t^h\right)^{-1} \int\limits_{k/n}^t \sigma(X_{k/n}^h) a_{xx}(X_s^h) Y_s^h\, dB_s \\
&+ E_x\, \Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^h)\, ds\right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s\right) u_x(t, X_t^h) \\
&\quad \times \left(Y_t^h\right)^{-1} \left(-Z_t^h \left(Y_t^h\right)^{-2}\right) \\
&+ E_x\, \Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^{h,\epsilon})\, ds\right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s\right) u_x(t, X_t^h) \\
&\quad \times \left(Y_t^h\right)^{-1} (-B_1) \\
&+ E_x\, \Lambda_n \exp\left(\int\limits_0^t \beta f(X_s^{h,\epsilon})\, ds\right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, ds\right) u_x(t, X_t^h) \left(Y_t^h\right)^{-1}.
\end{aligned}
$$

Here all the terms with u_x but the first one possess the bound by absolute value, $\le C(1-t)^{-1/2} h^{1/2}$, or better. After integration with respect to t from kh to $(k+1)h$, for $k \le n-2$ this gives a bound $\le C(1-(k+1)h)^{-1/2} h^{3/2}$. After summation over $n/2 < k \le n-2$, we therefore get a bound

$$\sum_{n/2<k\le n-2} \left| E_x\, \Lambda_n \int\limits_{k/n}^{(k+1)/n} \exp\left(\int\limits_0^{(k+1)/n} \beta f(X_s^h)\, ds \right) \right.$$
$$\left. \times \left(\int\limits_{k/n}^{(k+1)/n} \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s \right) u_{xx}(t, X_t^h)\, dt \right| \le Ch^{1/2}.$$

13. Similarly to step (**10**), one gets a bound

$$\limsup_{\epsilon\to 0} \left| E_x \frac{\Lambda_n^\epsilon - \Lambda_n}{\epsilon} \exp\left(\int\limits_0^t \beta f(X_s^h)\, ds \right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, ds \right) u_x(t, X_t^h) \right|$$
$$\le Ch^{1/2} e^{-n^c} / \sqrt{1-t}.$$

Hence all the terms with u_x including the first one possess the bound by absolute value, $\le C(1-t)^{-1/2} h^{1/2}$. After integration with respect to t from kh to $(k+1)h$, for $k \le n-2$ this gives a bound

$$\left| E_x\, \Lambda_n \int\limits_{k/n}^{(k+1)/n} \exp\left(\int\limits_0^t \beta f(X_s^h)\, ds \right) \left(\int\limits_{k/n}^t \sigma(X_{k/n}^h) a_x(X_s^h)\, dB_s \right) u_{xx}(t, X_t^h)\, dt \right|$$
$$\le C(1-(k+1)h)^{-1/2} h^{3/2}.$$

After summation over $n/2 < k \le n-2$, we get

$$\sum_{n/2<k\le n-2} \left| E_x\, \Lambda_n \int\limits_{k/n}^{(k+1)/n} \exp\left(\int\limits_0^{(k+1)/n} \beta f(X_s^h)\, ds \right) \right.$$
$$\left. \times \left(\int\limits_{k/n}^{(k+1)/n} \sigma(X_{k/n}^h) a_x(X_s^h)\, ds \right) u_{xx}(t, X_t^h)\, dt \right| \le Ch^{1/2}.$$

Here we have used $\sigma \in C_b^2, f \in C_b^1$.

14. Let us consider the case $k = n-1$. For all terms but the first one, we get, with $1-h < T < 1$,

$$\int\limits_{1-h}^{T} Ch^{1/2} (1-t)^{-1/2}\, dt \le Ch.$$

The first term is bounded by the value

$$Ce^{-n^c}\int_{1-h}^{T}(1-t)^{-1/2}\,dt \le Ch^{1/2}e^{-n^c} \le Ch.$$

Now we put $T \to 1$ and apply the Fatou lemma to get the same bound for integrals up to 1. Combining all the bounds obtained, we get

$$\|A^{h,\beta} - A^{\beta}\| \le Ch^{1/2}.$$

The theorem is proved.

Acknowledgements

First attempts in studying the problem with unit diffusion and non-trivial drift were made jointly with Jean Memin who then had to leave this project for his other duties to the author's regret as his interest and hints were very essential. This work could not have been done without discussions on positive operators with Alexander Sobolev and Mark Krasnosel'skii. The paper is written in memory of Rezo Chitashvili, a great enthusiast of stochastic analysis who shared generously his enthusiasm with others. The author is deeply thankful to all these colleagues and friends of his, including the anonymous referee.

This work was supported by grants EPSRC-GR/R40746/01, NFGRF 2301863, INTAS-99-0590 and RFBR-00-01-22000.

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(Received 19.11.2002; revised 23.02.2003)

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