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ON ARU-RESOLUTIONS OF UNIFORM SPACES

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Abstract. In this paper theorems which give conditions for a uniform space to have an *ARU*-resolution are proved. In particular, a finitistic uniform space admits an *ARU*-resolution if and only if it has trivial uniform shape or it is an absolute uniform shape retract.

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1. Introduction

The definition of resolution of topological spaces was introduced by S. Mardešic [8] and it was used to define and study the shapes of topological spaces [10]. More recently, J. Segal, S. Spiez and B. Günter [14] defined a uniform version of resolution and proved that every finitistic uniform space has $ANRU$-resolution. They and T. Miyata [11] showed its usefulness in uniform shape theory.

The natural question is to determine which uniform spaces admit ARU-resolutions, i.e. uniform resolutions $\mathbf{p} : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, where each X_λ is an ARU-space. In [5] A. Koyama, S. Mardešic and T. Watanabe proved that a topological space X has an AR-resolution if and only if X has trivial shape.

In this paper we prove that for every finitistic uniform space X the following assertions are equivalent:
(i) X admits an ARU-resolution.
(ii) X has trivial uniform shape.
(iii) X is an absolute uniform shape retract.

Without specific references we use results from [4], [10], [11] and [14]. A uniform space X is said to be an $A(N)RU$-space if whenever X is embedded in a uniform space Y, then there is a uniform retraction of Y (some uniform neighborhood of X in Y) onto X, equivalently, if whenever A is a uniform subspace of Y, then every uniform map $f : A \to X$ extends over Y (some uniform neighborhood of A in Y). We say that two uniform maps $f, g : X \to Y$ are semi-uniformly homotopic maps and write $f \simeq_u g$ if there exists a uniform map $H : X * I \to Y$ such that $H_0 = f$ and $H_1 = g$. Here $X * I$ is the semi-uniform product of X and $I = [0, 1]$ (see [4, p. 44]).

A uniform space X is said to be semi-uniform contractible [11] if the identity uniform map $1_X : X \to X$ and some constant uniform map $c : X \to X$ are semi-uniformly homotopic maps. It is clear that a uniform space is semi-uniform

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contractible if and only if it has the semi-uniform homotopy type of a uniform space consisting of one point.

Let **Unif** denote the category of uniform spaces and uniform maps. By **A**(**N**)**RU** we denote the full subcategory of **Unif** whose objects are $A(N)RU$-spaces. Also, we write **H**(**Unif**) and **H**(**A**(**N**)**RU**) for the semi-uniform homotopy category of **Unif** and **A**(**N**)**RU**, respectively.

There have been uniform shape theories defined by many mathematicians ([1], [2], [3], [12], [13]), based on the uniform homotopy [4]. Recently, T. Miyata [11] introduced a new uniform shape theory by the inverse system approach, using the semi-uniform homotopy which is weaker than the uniform homotopy. Here we consider T. Miyata's uniform shape theory which is convenient for our purpose. The uniform shape category **uSh** is the abstract shape category $\mathbf{Sh}_{(\mathcal{T},\mathcal{P})}$ [10], where $\mathcal{T} = \mathbf{H}(\mathbf{Unif})$ and $\mathcal{P} = \mathbf{H}(\mathbf{ANRU})$. The uniform shape morphism $F : X \to Y$ of category **uSh** is given by a triple $(\mathbf{p}, \mathbf{q}, \mathbf{f})$, where $\mathbf{p} : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \in \mathbf{pro} - \mathbf{H}(\mathbf{ANRU})$ and $\mathbf{q} : Y \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M) \in \mathbf{pro} - \mathbf{H}(\mathbf{ANRU})$ are $\mathbf{H}(\mathbf{ANRU})$-expansions of X and Y, respectively, and $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a morphism of category $\mathbf{pro} - \mathbf{H}(\mathbf{ANRU})$. Two uniform spaces X and Y are said to have the same uniform shape denoted by $ush(X) = ush(Y)$, if X and Y are isomorphic objects of category **uSh**. We say that a uniform space X has trivial uniform shape and write $ush(X) = 0$ if X has the uniform shape of one point $\{*\}$.

2. Aru-Resolution

A uniform resolution $\mathbf{p} = (p_\lambda) : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of a uniform space X consists of an inverse system $\mathbf{X}$ in **Unif** and of a morphism $\mathbf{p}$ in $\mathbf{pro} - \mathbf{Unif}$, which has the property that for every $ANRU$-space P and every uniform covering $\mathcal{V}$ of P the following two conditions are satisfied:

(UR1) For every uniform map $f : X \to P$ there exist $\lambda \in \Lambda$ and a uniform map $h : X_\lambda \to P$ such that the maps $h \cdot p_\lambda$ and f are $\mathcal{V}$-near, i.e. every $x \in X$ admits $V \in \mathcal{V}$ such that $h \cdot p_\lambda(x), f(x) \in V$.

(UR2) There exists a uniform covering $\mathcal{V}'$ of P such that if for $\lambda \in \Lambda$ and for two uniform maps $h, h' : X_\lambda \to P$ the maps $h \cdot p_\lambda$, $h' \cdot p_\lambda$ are $\mathcal{V}'$-near, then there exists $\lambda' \geq \lambda$ such that the maps $h \cdot p_{\lambda\lambda'}$, $h' \cdot p_{\lambda\lambda'}$ are $\mathcal{V}$-near.

A uniform resolution is said to be cofinite if its index set Λ is cofinite. If all the terms X_λ, $\lambda \in \Lambda$, are $ANRU(ARU)$-spaces, then we speak of an $ANRU(ARU)$-resolution.

In [14] it is proved that every finitistic uniform space admits an $ANRU$-resolution [14, Theorem 1]. A. Koyama, S. Mardešić and T. Watanabe raised the question of determining which spaces admit an AR-resolution, i.e. a resolution where each term X_λ is an absolute retract for metric spaces [5]. In this section we prove

Theorem 1. *Let X be a uniform space. Then the following assertions hold:*
(i) *If X admits an ARU-resolution, then X has trivial uniform shape.*

(ii) *If X is a finitistic space and has trivial uniform shape, then X admits an ARU-resolution.*

The proof of this theorem is based on the following lemmas.

Lemma 2. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be the inverse systems of uniform spaces. If all Y_μ are ARU-spaces, then any two morphisms (f_μ, φ), $(f'_\mu, \varphi') : \mathbf{X} \to \mathbf{Y}$ of category* **inv − Unif** *are equivalent in the sense of morphisms* [10].

Proof. For each index $\mu \in M$ there exists an index $\lambda \in \Lambda$ such that $\lambda \geq \varphi(\mu)$, $\varphi'(\mu)$. Let $Z = X_\lambda \times \{0\} \cup X_\lambda \times \{1\}$. Let $h : Z \to Y_\mu$ be a map defined by the formulas

$$h_{|X_\lambda \times \{0\}} = f_\mu \cdot p_{\varphi(\mu)\lambda}, \quad h_{|X_\lambda \times \{1\}} = f'_\mu \cdot p_{\varphi'(\mu)\lambda}.$$

It is clear that h is a uniform map. This map has a uniform extension $h' : X_\lambda * I \to Y_\mu$. Hence, $f_\mu \cdot p_{\varphi(\mu)\lambda}$ and $f'_\mu \cdot p_{\varphi'(\mu)\lambda}$ are semi-uniformly homotopic maps. Consequently, $(f_\lambda, \varphi) \sim (f'_\lambda, \varphi')$. □

Lemma 3. *Let $f : X \to Y$ be a uniform map from a uniform space X into an ANRU-space Y. If the map f is semi-uniformly homotopic to a constant map, then there exist an ARU-space Z and a uniform map $r : Z \to Y$ such that $X \subseteq Z$ and $r_{|X} = f$.*

Proof. Consider X as a uniform subspace of an *ARU*-space Z. Since f is semi-uniform homotopic to a constant map $c : X \to Y$ and since c has a uniform extension on Z, the semi-uniform homotopy extension theorem (see [Theorem 1.6][11]) proves that $f : X \to Y$ too extends to a uniform map $r : Z \to Y$. □

Lemma 4. *If a semi-uniform contractible space Y is an ANRU-space, then Y is an ARU-space.*

Proof. By the assumption of the lemma, there exist a point $y_0 \in Y$ and a semi-uniform homotopy $H : Y * I \to Y$ such that $H(y, 0) = y$ and $H(y, 1) = y_0$ for every $y \in Y$.

Let $f : A \to Y$ be a uniform map defined on a uniform subspace A of a uniform space X. Since Y is an *ANRU*-space, f has a uniform extension $f' : U \to Y$ over some uniform neighborhood U of A in X. There is a uniform covering $\mathcal{U}$ of X such that $St(A, \mathcal{U}) \cap St(X \backslash U, \mathcal{U}) = \varnothing$. By the uniform version of Urysohn's lemma there exists a uniform map $e : X \to I$ such that

$$e(x) = \begin{cases} 0, & x \in A, \\ 1, & x \in X \backslash St(A, \mathcal{U}). \end{cases}$$

Then we define a map $g : X \to Y$ by

$$g(x) = \begin{cases} H(f'(x), e(x)), & x \in U, \\ y_0, & x \in X \backslash St(A, \mathcal{U}). \end{cases}$$

Let $\mathcal{V}$ be an arbitrary uniform covering of X. There exists a uniform covering $\mathcal{W}$ of X which is a refinement of the intersection $\mathcal{U} \wedge \mathcal{V}$ of uniform coverings

$\mathcal{U}$ and $\mathcal{V}$. Let $W \in \mathcal{W}$ be an element of $\mathcal{W}$ such that $W \cap U \neq \varnothing$ and $W \cap (X \backslash St(A,\mathcal{U})) \neq \varnothing$. Now show that $W \cap (U \cap (X \backslash St(A,\mathcal{U}))) \neq \varnothing$. First assume that $W \cap (X \backslash U) = \varnothing$. Then $W \subset U$ and, consequently, $W \cap (U \cap (X \backslash St(A,\mathcal{U}))) = W \cap (X \backslash St(A,\mathcal{U})) \neq \varnothing$. Now assume that $W \cap (X \backslash U) \neq \varnothing$. There exists an element $G \in \mathcal{U}$ such that $W \subset G$. Note that $G \cap (X \backslash U) \neq \varnothing$. Hence, $G \subset St(X \backslash U,\mathcal{U})$. Since $St(A,\mathcal{U}) \cap St(X \backslash U,\mathcal{U}) = \varnothing$, we have $G \cap St(A,\mathcal{U}) = \varnothing$. Consequently, $G \subset X \backslash St(A,\mathcal{U})$. We obtained that $G \cap (X \backslash St(A,\mathcal{U})) = G$. Note that $W \cap (X \backslash St(A,\mathcal{U})) = W$. This proves that $W \cap (U \cap (X \backslash St(A,\mathcal{U}))) = W \cap U \neq \varnothing$.

By Lemma 7 of [6] the map g is a uniform map. For each $a \in A$ we have

$$g(a) = H(f'(a), e(a)) = H(f'(a), 0) = f'(a) = f(a),$$

i.e. g is an extension of a uniform map $f : A \to Y$. Hence $Y \in ARU$. □

Proof of Theorem 1. We can easily prove the first assertion by following the argument of (i)⇒(ii) in [5]. Here we use our Lemma 2 in the place of [5, Lemma 1]. Indeed, by the assumption (i) X admits an ARU-resolution $\mathbf{p} = (p_\lambda) : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$. The space $Y = \{*\}$ consisting of one point $*$ also admits an ARU-resolution $\mathbf{q} = (q_\mu) : Y \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, where $M = \{\mu_0\}$, $q_{\mu\mu'} = q_{\mu_0\mu_0} = 1_{Y_{\mu_0}} = 1_Y$, $Y_\mu = Y_{\mu_0} = Y$, $q_\mu = q_{\mu_0} = 1_Y$.

Let $f : X \to Y$ and $g : Y \to X$ be constant uniform maps. Let $(f_\mu, \varphi) : \mathbf{X} \to \mathbf{Y}$ and $(g_\lambda, \psi) : \mathbf{Y} \to \mathbf{X}$ be morphisms induced by f and g, respectively. By Lemma 3 we have

$$(f_\mu, \varphi) \cdot (g_\lambda, \psi) \sim (1_{Y_\mu}, 1_M), \quad (g_\lambda, \psi) \cdot (f_\mu, \varphi) \sim (1_{X_\lambda}, 1_\Lambda).$$

Let $F : X \to Y$ and $G : Y \to X$ be uniform shape morphisms with representatives $\mathbf{f} = [(f_\mu, \varphi)] : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{g} = [(g_\lambda, \psi)] : \mathbf{Y} \to \mathbf{X}$. It is clear that $F \cdot G = I_Y$ and $G \cdot F = I_X$, where I_X and I_Y are the identity uniform shape morphisms. Hence $ush(X) = ush(Y)$, i.e. $ush(X) = 0$.

Now we will prove the second assertion. Since X is finitistic, there is an antisymmetric cofinite $ANRU$-resolution $\mathbf{p} = (p_\lambda) : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of X [14]. Using this $ANRU$-resolution, the construction in the argument (iii)⇒(i) of [5] and our Lemma 3 in the place of [5, Lemma 2] and also Lemma 4 in next, we can define the ARU-system $\mathbf{Z} = (Z_\lambda, q_{\lambda\lambda'}, \Lambda^*)$ and the morphism $\mathbf{q} = (q_\lambda) : X \to \mathbf{Z}$.

Following the argument in [5] that the obtained resolution satisfies properties(R1) and (R2), we can easily show that $\mathbf{q}$ has properties (UR1) and (UR2). □

Remark. Following the approach by [9], we can construct the coherent semi-uniform homotopy category and, hence, uniform strong shape theory for uniform spaces by the inverse system approach based on semi-uniform homotopy. It is possible to find a condition in terms of uniform strong shape for a uniform finitistic space to have an ARU-resolution.

3. Uniform Shape Retracts

In this section we generalize the notion of an absolute shape retract [7] to the uniform spaces. Let A be a subspace of uniform space X. A uniform shape retraction is a uniform shape morphism $R : X \to A$ such that $R \cdot J = I_A$, where $J : A \to X$ is the uniform shape morphism induced by the inclusion uniform map $j : A \to X$. In this case the subspace A is called a uniform shape retract of X. Notice that if a uniform shape morphism $R : X \to A$ is induced by a uniform retraction $r : X \to A$, then R is a uniform shape retraction. If $A \subseteq Y \subseteq X$ and A is a uniform shape retract of X, then A is also a uniform shape retract of Y. Moreover, if A is a uniform shape retract of Y and Y is a uniform shape retract of X, then A is a uniform shape retract of X.

A uniform space X is called an absolute uniform shape retract provided for every uniform space Y with $X \subset Y$, X is a uniform shape retract of Y.

We have

Theorem 5. *Let X be a uniform space. Then the following assertions are equivalent:*
(i) *X is an absolute uniform shape retract.*
(ii) *X has trivial uniform shape.*

We first establish

Lemma 6. *Let X and Y be uniform spaces. If Y has trivial uniform shape, then there exists a unique uniform shape morphism $F : X \to Y$.*

Proof. Let $ush(Y) = 0$. The uniform shape morphism $C : Y \to \{*\}$, induced by the constant map $c : Y \to \{*\}$, has an inverse uniform shape morphism $G : \{*\} \to Y$, $G \cdot C = I_Y$. For each uniform shape morphism $F : X \to Y$ we have $F = G \cdot C \cdot F = G \cdot C'$, where $C' = C \cdot F : X \to \{*\}$ is the only uniform shape morphism into $\{*\}$. □

Proof of Theorem 5. Implication (i)⇒(ii). Let X be an absolute uniform shape retract. We can consider X as a uniform subspace of an *ARU*-space Z. Note that Z is a semi-uniform contractible space. Consequently, Z has the semi-uniform homotopy type of a point, and $ush(Z) = 0$. By condition, there exists a uniform shape retraction $R : Z \to X$, $R \cdot J = I_X$, where $J : X \to Z$ is the uniform shape morphism induced by the inclusion uniform map $j : X \to Z$. From Lemma 6 it follows that $J \cdot R = I_Z$. Thus $ush(X) = ush(Z) = 0$.

Implication (ii)⇒(i). Let $ush(X) = 0$ and $X \subset Y$. By Lemma 6 there exist unique uniform shape morphisms $R : Y \to X$ and $I_X : X \to X$. Therefore $R \cdot J = I_X$, i.e. X is an absolute uniform shape retract. □

Corollary 7. *Let $F : A \to Y$ be a uniform shape morphism from a uniform subspace A of a uniform space X into an absolute uniform shape retract Y. Then there exists a uniform shape morphism $F' : X \to Y$ such that $F' \cdot J = F$, where $J : A \to X$ is the inclusion uniform shape morphism induced by the inclusion uniform map $j : A \to X$.*

Proof. From Theorem 6 we obtain $ush(Y) = 0$. Then by Lemma 6 there are unique shape morphisms $F : A \to Y$ and $F' : X \to Y$. Consequently, the uniform shape morphism $F' \cdot J : A \to Y$ coincides with $F : A \to Y$. $\square$

Consider a uniform space X as a uniform subspace of an ARU-space Z. The set $\Lambda = \{U\}$ of all uniform neighborhoods of X in Z is directed by the order $\leq$ defined by inclusion:

$$U \leq U' \Leftrightarrow U' \subseteq U.$$

Let $i_U : X \to U$ and $i_{UU'} : U' \to U$ be the inclusion uniform maps. The family $\mathbf{i} = ([i_U]) : X \to \mathbf{X} = (U, [i_{UU'}], \Lambda)$ forms a morphism of category **pro-H(ANRU)**. In [11] it is proved that $\mathbf{i} : X \to \mathbf{X}$ is an **H(ANRU)**-expansion.

Theorem 8. *A uniform subspace X of an ARU-space Z has trivial uniform shape if and only if for every uniform neighborhood U of X in Z there exists a uniform neighborhood U' of X in Z such that $U' \subseteq U$ and the inclusion uniform map $i_{UU'} : U' \to U$ is semi-uniformly homotopic to a constant map.*

Proof. We can prove this as in S. Mardešic [7, Theorem 5], using Lemma 7. $\square$

Theorem 9. *For every finitistic uniform space X the following statements are equivalent:*

(i) *X admits an ARU-resolution.*
(ii) *X has trivial uniform shape.*
(iii) *X is an absolute uniform shape retract.*

Proof. This is an immediate consequence of Theorems 1 and 5. $\square$

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