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ON ONE THEOREM OF S. WARSCHAWSKI

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Abstract. A theorem of S. Warschawski on the derivative of a holomorphic function mapping conformally the circle onto a simply-connected domain bounded by the piecewise-Lyapunov Jordan curve is extended to domains with a non-Jordan boundary having interior cusps of a certain type.

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1. Let a simply-connected domain B be bounded by a closed piecewise-smooth curve $\gamma : z = z(s)$, $0 \le s \le S$, where s is a natural parameter. Let s_k, $k = \overline{1,n}$, be the points of discontinuity of $z'(s)$. The point $z_k = z(s_k)$, $k = \overline{1,n}$, of discontinuity of the function $z'(s)$ will be called the corner of opening $\nu_k\pi = \pi - \arg(z'(s_k+0) : z'(s_k-0))$, where $0 \le \nu_k \le 2$ and $-\pi < \arg\cdot \le \pi$.

In [1] S. Warschawski established a result describing the behavior of the derivative of the holomorphic function $\omega(z)$ which maps the domain B onto the unit circle $\mathbb{D}$ in the neighborhood of corners. Namely, it was proved that if the Jordan curve γ is piecewise-Lyapunov and $0 < \nu_k \le 2$, $k = \overline{1,n}$, then

$$\omega'(z) = \omega_0(z) \prod_{k=1}^{n} (z - z_k)^{\frac{1}{\nu_k} - 1}, \tag{1}$$

where $\omega_0(z)$ is a function holomorphic in B, continuous and non-vanishing in $\overline{B}$. An analogous representation is valid for $(\omega^{-1})'$ as well. Various aspects of this range of problems were intensively investigated in the subsequent period too. A vast list of works on this topic can be found in the monograph [2]. In [3], using the results of the theory of a discontinuous Riemann problem, the authors showed the validity of representation [1] for a piecewise-smooth Jordan curve γ and $0 < \nu_k \le 2$, $k = \overline{1,n}$. In that case the function $\omega_0(z)$ belongs to any Smirnov class E_p, $p > 0$.

In this paper a sufficiently simple way is proposed for proving one theorem of Warschawski for a simply-connected domain with a non-Jordan boundary. This proof covers the cases of both a piecewise-Lyapunov and a piecewise-smooth curve γ ($0 < \nu_k \le 1$, $k = \overline{1,n}$). In addition to the classical statement, it is proved that for a piecewise-Lyapunov boundary the function $\omega_0(z)$ satisfies the Hölder condition (condition $H(\mu)$)

$$|\omega_0(z_1) - \omega_0(z_2)| < K|z_1 - z_2|^{\mu}, \quad 0 < \mu \le 1, \tag{2}$$

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not only on smooth parts of the curve γ and in the neighborhood of corners with $\nu_k < 2$, but also in the neighborhood of cusps ($\nu_k = 2$) of a certain type. (In connection with this question see also [3].)

2. Recall that the smoothness of a curve is equivalent to the continuity of an angle formed by the tangent to the curve with a fixed direction. If however this angle as a function of the arc length satisfies the Hölder condition, then the curve is called a Lyapunov curve. Piecewise smoothness imposes on the above-said angle a condition of the existence of one-sided limits at points of discontinuity, while the property of being piecewise-Lyapunov curve implies that the Hölder condition is satisfied on each interval between points of discontinuity, including end-points.

Lemma 1. *If* $z(t) \in C^{1,\mu}[a,b]$, $0 < \mu \le 1$ *and,* $z'(a) \neq 0$, $z(t) - z(a) \neq 0$ *on* $[a,b]$, *then*

$$\arg[z(t) - z(a)] \in C^{0,\mu}[a,b].$$

By the condition we have

$$\begin{aligned} x'(t) &= \operatorname{Re} z'(t) = x'(a) + f(t)(t-a)^\mu, \\ y'(t) &= \operatorname{Im} z'(t) = y'(a) + h(t)(t-a)^\mu, \end{aligned} \tag{3}$$

where $f(t)$ and $h(t)$ are bounded on $[a,b]$. Applying mean value theorem, we obtain

$$\begin{aligned} x(t) - x(a) &= \int_a^b x'(\tau)d\tau = x'(a)(\xi - a) + x'(t)(t-a) \\ &= (x'(a) - x'(t))(\xi - a) + x'(t)(t-a), \end{aligned}$$

where $a \le \xi \le t$. Hence

$$\begin{aligned} x(t) - x(a) &= \frac{x'(a) - x'(t)}{(t-a)^\mu} \cdot \frac{\xi - a}{t - a}(t-a)^{\mu+1} + x'(t)(t-a) \\ &= x'(a)(t-a) + \frac{x'(t) - x'(a)}{(t-a)^\mu}(t-a)^{\mu+1} \\ &\quad + \frac{x'(a) - x'(t)}{(t-a)^\mu} \cdot \frac{\xi - a}{t-a}(t-a)^{\mu+1} \\ &= x'(a)(t-a) + \varphi(t)(t-a)^{\mu+1}, \end{aligned} \tag{4}$$

where

$$\varphi(t) = f(t)\Big(1 - \frac{\xi - a}{t - a}\Big) \tag{5}$$

is the function bounded on $[a,b]$.

By a similar reasoning we get

$$y(t) - y(a) = y'(a)(t-a) + \psi(t)(t-a)^{\mu+1}, \tag{6}$$

where

$$\psi(t) = h(t)\Big(1 - \frac{\eta - a}{t - a}\Big), \quad a \le \eta \le t, \tag{7}$$

is also bounded on $[a,b]$.

Using (3)–(7) we obtain

$$\begin{aligned}\frac{d}{dt}\arg[z(t)-z(a)] &= \frac{d}{dt}\operatorname{arc\,tg}\frac{y(t)-y(a)}{x(t)-x(a)}\\ &= \frac{\big(x'(a)(t-a)+\varphi(t)(t-a)^{\mu+1}\big)\big(y'(a)+h(t)(t-a)^{\mu}\big)}{\big(x'(a)(t-a)+\varphi(t)(t-a)^{\mu+1}\big)^2+\big(y'(a)(t-a)+\psi(t)(t-a)^{\mu+1}\big)^2}\\ &- \frac{\big(y'(a)(t-a)+\psi(t)(t-a)^{\mu+1}\big)\big(x'(a)+f(t)(t-a)^{\mu}\big)}{\big(x'(a)(t-a)+\varphi(t)(t-a)^{\mu+1}\big)^2+\big(y'(a)(t-a)+\psi(t)(t-a)^{\mu+1}\big)^2}\\ &= \frac{(t-a)^{\mu+1}\big(x'(a)h(t)+y'(a)\varphi(t)+h(t)\varphi(t)(t-a)^{\mu}\big)}{(t-a)^2\big[(x'(a)+\varphi(t)(t-a)^{\mu})^2+(y'(a)+\psi(t)(t-a)^{\mu})^2\big]}\\ &- \frac{(t-a)^{\mu+1}\big[(y'(a)f(t)+x'(a)\psi(t)+f(t)\psi(t)(t-a)^{\mu}\big]}{(t-a)^2\big[(x'(a)+\varphi(t)(t-a)^{\mu})^2+(y'(a)+\psi(t)(t-a)^{\mu})^2\big]} = \frac{b(t)}{(t-a)^{1-\mu}},\end{aligned}$$

where the function $b(t)$ is bounded on $[a,b]$ by virtue of the condition $|z'(a)|\neq 0$. The latter equality implies

$$\begin{aligned}\big|\arg(z(t_1)-z(a))-\arg(z(t_2)-z(a))\big| &= \Big|\int_{t_2}^{t_1}\frac{d}{dt}(\arg(z(t)-z(a))dt\Big|\\ &\leq M_1\big|(t_1-a)^{\mu}-(t_2-a)^{\mu}\big| \leq M_1|t_1-t_2|^{\mu}.\end{aligned}$$

Denote by P_β ($\beta>0$) the mapping $w=z^\beta$, and by $E_\alpha(q)$ the angle $\{z;\ -\pi\alpha<\arg(z-q)<\pi\alpha,\ \alpha<1,\ \operatorname{Im} q=0\}$. For $\beta>1$ the mapping P_β is univalent in $E_{\beta^{-1}}(0)$.

Lemma 2. *Let $\gamma_0: z=z(s)$, $\overline{s}\leq s\leq \overline{\overline{s}}$ be a piecewise-smooth arc with the corner $z(s_0)=0$, and let the positive semi-axis be the bisectrix of the interior angle of the opening $\pi\nu$, $0<\nu\leq 2$, at the point $z(s_0)$. Then the curve $P_{\frac{1}{\nu}}\circ\gamma$ is smooth.*

If we write the equation of the curve $\Gamma=P_{\frac{1}{\nu}}\circ\gamma$ in the form $w=w(s)=[z(s)]^{\frac{1}{\nu}}$, then $dw(s)=\frac{1}{\nu}[z(s)]^{\frac{1}{\nu}-1}\cdot z'(s)ds$, $d\sigma(s)=|dw(s)|=\frac{1}{\nu}|z(s)|^{\frac{1}{\nu}-1}\cdot ds$ and

$$\frac{dw(s)}{d\sigma(s)}=\exp\big[i\arg[z(s)]^{\frac{1}{\nu}-1}\cdot z'(s)\big].$$

Furthermore,

$$\lim_{s\to s_0+0}\arg z(s)=\pi\frac{\nu}{2},\qquad \lim_{s\to s_0+0}\arg z'(s)=\pi\frac{\nu}{2}-\pi,$$
$$\lim_{s\to s_0-0}\arg z(s)=-\pi\frac{\nu}{2},\qquad \lim_{s\to s_0-0}\arg z'(s)=-\pi\frac{\nu}{2}.$$

Hence we obtain

$$\lim_{s\to s_0-0}\arg[z(s)]^{\frac{1}{\nu}-1}\cdot z'(s)=\Big(\frac{1}{\nu}-1\Big)\lim_{s\to s_0-0}\arg z(s)+\lim_{s\to s_0-0}\arg z'(s)$$
$$=\Big(\frac{1}{\nu}-1\Big)\pi\,\frac{\nu}{2}+\pi\,\frac{\nu}{2}-\pi=-\frac{\pi}{2},$$
$$\lim_{s\to s_0+0}\arg[z(s)]^{\frac{1}{\nu}-1}\cdot z'(s)=\Big(\frac{1}{\nu}-1\Big)\lim_{s\to s_0+0}\arg z(s)+\lim_{s\to s_0+0}\arg z'(s)$$
$$=\Big(\frac{1}{\nu}-1\Big)\Big(-\pi\,\frac{\nu}{2}\Big)-\pi\,\frac{\nu}{2}=-\frac{\pi}{2}.$$

Corollary. *If in the conditions of Lemma 2 γ is piecewise-Lyapunov curve, then Γ is Lyapunov curve.*

On writing the equation of the curve γ for the parameter σ as $\gamma: z=[w(\sigma)]^\nu$, $\overline{\sigma}\le\sigma\le\overline{\overline{\sigma}}$, we obtain $ds=|dz(\sigma)|=\nu|w(\sigma)|^{\nu-1}d\sigma$. Let the points a_1 and a_2 lie on the curve γ so that both values s_1 and s_2 of the arc variable, which correspond to the points a_1 and a_2, occur either in the interval $[\overline{s},s_0]$ or in $[s_0,\overline{\overline{s}}]$, and let $A_j=P_{\frac{1}{\nu}}(a_j)$, $j=1,2$. Then

$$s(a_1,a_2)=\int\limits_{s_1}^{s_2}ds=\nu\int\limits_{\sigma(A_1)}^{\sigma(A_2)}|w(\sigma)|^{\nu-1}d\sigma\le M_2\Big(\int\limits_{\sigma(A_1)}^{\sigma(A_2)}d\sigma\Big)^\mu$$
$$=M_2\big[\sigma(A_1,A_2)\big]^\mu,\quad 0<\mu\le1. \tag{8}$$

By the condition and Lemma 1 we have

$$\Big|\arg\frac{dw}{d\sigma}(\sigma(s_1))-\arg\frac{dw}{d\sigma}(\sigma(s_2))\Big|\le M_3|s_1-s_2|^{\mu'},\quad 0<\mu'\le1,$$

for $\overline{s}\le s_1,\ s_2\le s_0$ or $\overline{s}_2\le s_1,\ s_2\le\overline{\overline{s}}$. From this, by virtue of (8), we obtain

$$\Big|\arg\frac{dw}{d\sigma}(\sigma_1)-\arg\frac{dw}{d\sigma}(\sigma_2)\Big|\le M_4|\sigma_1-\sigma_2|^{\mu''},\quad 0<\mu''\le1, \tag{9}$$

where $\sigma_1=\sigma(s_1)$, $\sigma_2=\sigma(s_2)$. But by Lemma 2 the curve Γ is smooth and therefore the fulfilment of condition (9) on the arcs composing Γ implies that this condition is fulfilled on the entire curve ([4], Ch. 1, §5).

Lemma 3. *For $0<\beta<1$, $0<\alpha<1$ $a>0$ $P_\beta(E_\alpha(a))\subset E_\alpha(a^\beta)$.*

After writing the equation of one of the sides of the angle $E_\alpha(a)$ as $z=a+t\exp(i\alpha\pi)$, $0\le t<\infty$, we obtain

$$\arg\big[(a+t\exp(i\alpha\pi))^\beta\big]'=\arg(a+t\exp(i\alpha\pi))^{\beta-1}+\alpha\pi$$
$$=(\beta-1)\arg(a+t\exp(i\alpha\pi))+\alpha\pi\le\alpha\pi.$$

Next,

$$\big(\arg(a+t\exp(i\alpha\pi))\big)'=\frac{(a+t\cos\alpha\pi)\sin\alpha\pi-t\cos\alpha\pi\,\sin\alpha\pi}{(\alpha+t\cos\alpha\pi)^2+t^2\sin^2\alpha\pi}$$
$$=\frac{a\sin\alpha\pi}{(a+t\cos\alpha\pi)^2+t^2\sin^2\alpha\pi}>0,$$

and therefore

$$\left[(\arg(a + t\exp(i\alpha\pi))^{\beta})'\right]' < 0.$$

Moreover,

$$\left[(\beta - 1)\arg(a + t\exp(i\alpha\pi) + \alpha\pi\right]\Big|_{t=0} = \alpha\pi,$$
$$\lim_{t\to\infty}\left[(\beta - 1)\arg(a + t\exp(i\alpha\pi) + \alpha\pi\right] = \beta\alpha\pi.$$

Thus $\arg\left[(a+t\exp(i\alpha\pi))^{\beta}\right]'$ is a decreasing function on $[0,\infty)$ from the value $\alpha\pi$ to $\beta\alpha\pi$. Repeating the above arguments for another side of the angle, we ascertain that the lemma is valid.

3. As mentioned above, the boundary γ of the simply-connected domain B is not assumed to be a Jordan curve. We will describe those properties of the boundary curve which are needed for further constructions.

Denote by $\langle\gamma\rangle$ the range of values of the mapping γ, by $B_\infty(\gamma)$ the component of the set $\overline{\mathbb{C}}\backslash\langle\gamma\rangle$ containing the point at infinity, and by $W(\gamma)$ the set of points of all other components of the set $\overline{\mathbb{C}}\backslash\langle\gamma\rangle$. The symbol $\gamma[t_1, t_2]$ will denote the arc of the parametrized curve corresponding to the variation of t from the value t_1 to t_2, including end-points, while $\gamma(t_1, t_2)$ will denote the same arc but without the end-points. If the arcs $\gamma_1 = \gamma[a, b]$ and $\gamma_2 = \gamma[c, d]$ are such that $\gamma(b) = \gamma(c)$, then

$$(\gamma_1\cdot\gamma_2)(t) = \begin{cases}\gamma(t), & a \le t \le b,\\ \gamma(t + c - b), & b \le t \le d + b - c.\end{cases}$$

Let some point of the curve γ be the impression of two different prime ends a_1 and a_2, and let $\{d_k\}_{k=1}^{\infty}$, $d_k \subset B$, $k = 1, 2, \ldots$, and $\left\{d'_j\right\}_{j=1}^{\infty}$, $d'_j \subset B$, $j = 1, 2, \ldots$, be the sequences of domains which determine these prime ends. Denote $D^+(z_0, \rho) = \{z \in \mathbb{D},\ |z - z_0| < \rho\}$ and note that for the conformal mapping f of the circle $\mathbb{D}$ onto B we have $f(D^+(e^{i\theta_1}, \rho)) \cap f(D^+(e^{i\theta_2}, \rho)) = \varnothing$ for sufficiently small ρ and for different θ_1 and θ_2. If now as θ_1 and θ_2 we take the values corresponding to the prime ends a_1 and a_2 and take into account the fact that for fixed ρ we have $f(D^+(e^{i\theta_1}, \rho)) \supset d_k$ and $f(D^+(e^{i\theta_2}, \rho)) \supset d'_j$ for $k > N$ and $j > N$, then we find that $d_k \cap d'_j = \varnothing$ holds for sufficiently large values of the indices k and j. From this it immediately follows that the curve γ cannot have points of self-intersection and if $\gamma(s') = \gamma(s'')$, $(s' \neq s'')$ and $\gamma(s)$ is differentiable at s' and s'', then $\gamma'(s') = -\gamma'(s'')$.

Lemma 4. *Let $\gamma[s's'']$ be a closed Jordan arc of the curve γ, $0 \le s' < s'' < S$, and for some $s_1 \in (s', s'')$ there exist a value s_2 such that $\gamma(s_1) = \gamma(s_2)$. Then $s_2 \in (s', s'')$.*

Assume that the opposite assumption is true. Let, for definiteness, $s_2 > s''$. Denote $\widetilde{\gamma} = \gamma[s'', s_2]\cdot\gamma[s_2, S]\cdot\gamma[0, s']$. Since γ has no points of self-intersection, then either $\gamma[s', s'']$ separates $W(\widetilde{\gamma})$ from the point at infinity or $\widetilde{\gamma}$ separates $W(\gamma[s', s''])$ from the said point. Let us consider the first case. Then $B \subset$

$W(\gamma[s',s''])$ and for any point $z_0 \in B$ we have

$$\varkappa\big(z_0,\gamma[s',s'']\big)=\frac{1}{2\pi}\int\limits_{\gamma[s',s'']} d\,\mathrm{Arg}(t-z_0)=1.$$

Hence

$$\varkappa(z_0,\widetilde{\gamma})=\frac{1}{2\pi}\int\limits_{\widetilde{\gamma}} d\,\mathrm{Arg}(t-z_0)=0,$$

i.e., $B\not\subset W(\widetilde{\gamma})$ and therefore $B\subset W_\infty(\widetilde{\gamma})$, from which we obtain $B\subset W(\gamma[s',s''])\cap B_\infty(\widetilde{\gamma})$. Let us represent γ as $\gamma=\gamma[0,s']\cdot\gamma[s',s_1]\cdot\gamma[s_1,s'']\cdot\gamma[s'',s_2]\cdot\gamma[s_2,S]$ and consider two closed curves $\gamma_1=\gamma[s',s_1]\cdot\gamma[s_2,S]\cdot\gamma[0,s']$ and $\gamma_2=\gamma[s_1,s'']\cdot\gamma[s'',s_2]$. From the equality

$$\varkappa(z_0,\gamma)=\varkappa(z_0,\gamma_1)+\varkappa(z_0,\gamma_2)=1$$

we conclude that one of the values $\varkappa(z_0,\gamma_j)$, $j=1,2$, is equal to zero, while the second to unity. Let $\varkappa(z_0,\gamma_2)=0$. This means that $W(\gamma_2)\cap B=\varnothing$. But since $W(\widetilde{\gamma})\cap B=\varnothing$, it follows that $\gamma[\widetilde{s}'',\widetilde{s}_2]$, where $[\widetilde{s}'',\widetilde{s}_2]\subset(s'',s_2)$ is separated from B, which contradicts the initial assumption that each point of the curve γ is a boundary point.

Arguments for the case with $\widetilde{\gamma}$ separating $W(\gamma[s',s''])$ from the point at infinity do not differ from those used above.

Two values s' and s'' are called twin if in any neighborhoods $V(s')$ and $V(s'')$ there are different values s_1 and s_2 such that $\gamma(s_1)=\gamma(s_2)$. Let us show that if $s'\neq s''$, then s' and s'' are twin if and only if $\gamma(s')=\gamma(s'')$.

Indeed, if $\gamma(s')=\gamma(s'')$, then the values s' and s'' themselves can be taken as s_1 and s_2. Assume now that $\gamma(s')\neq\gamma(s'')$. Then since $\gamma(s)$ is continuous, there are neighborhoods $V(s')$ and $V(s'')$ such that $|\gamma(s_1)-\gamma(s_2)|\geq d>0$ for any $s_1\in V(s')$ and $s_2\in V(s'')$. An example of the self-twin value is the value s_0 characterized by the fact that the arcs $\gamma[s_0-\delta,s_0]$ and $\gamma[s_0,s_0+\delta]$ coincide up to orientation.

Denote by $\mathfrak{M}(\gamma)$ the set of all segments $I=[s',s'']$ $(s'\leq s'')$ whose end-points are twin values. The set $\mathfrak{M}(\gamma)$ is partially ordered with respect to the inclusion. Let $r=\{I_\alpha\}$, $\alpha\in\mathcal{A}$, be a maximal chain (a maximal linearly ordered subset of the set $\mathfrak{M}(\gamma)$)) [5]. As above, the continuity of $\gamma(s)$ readily implies that $\underline{I}=\bigcap\limits_{\alpha\in\mathcal{A}} I_\alpha\in r$ and $\overline{I}_r=\bigcup\limits_{\alpha\in\mathcal{A}} I_\alpha\in r$, i.e., any maximal chain contains both the first and the last element. We will show that $\overline{I}=[\overline{s}\,'_r,\overline{s}\,''_r]$ is the last element of a maximal chain if and only if $\gamma(\overline{s}\,')=\gamma(\overline{s}\,'')\in\partial B_\infty(\gamma)$. Let

$$s^-=\sup s,\quad s\leq\overline{s}\,'_r,\quad \gamma(s)\in\partial B_\infty(\gamma),$$
$$s^+=\inf s,\quad s\geq\overline{s}\,''_r,\quad \gamma(s)\in\partial B_\infty(\gamma).$$

Let us show that $\gamma(s^-)\in\partial B_\infty(\gamma)$ and $\gamma(s^+)\in\partial B_\infty(\gamma)$. Indeed, let $\gamma(s_k)\in\partial B_\infty(\gamma)$, $k=1,2,\dots$, $\lim\limits_{k\to\infty}s_k=s^-$ and $z_m^{(k)}\in B_\infty(\gamma)$, $\lim\limits_{m\to\infty}z_m^{(k)}=\gamma(s_k)$, $k=1,2,\dots$. Then it is clear that $\lim\limits_{m\to\infty}z_m^{(m)}=\gamma(s^-)$, i.e., $\gamma(s^-)\in\partial B_\infty(\gamma)$.

Analogously, $\gamma(s^+) \in \partial B_\infty(\gamma)$. From the definition of s^- and s^+ it follows that $\gamma(s) \notin \partial B_\infty(\gamma)$ when $s \in (s^-, s^+)$ and therefore the assumption $\gamma(s^-) \neq \gamma(s^+)$ would imply the existence of s_0, $s^- < s_0 < s^+$ such that $\gamma(s_0) \in \partial B_\infty(\gamma)$. Thus $\gamma(s^-) = \gamma(s^+) \in \partial B_\infty(\gamma)$, but $[s^-, s^+] \supseteq \overline{I}_r$ and therefore $[s^-, s^+] = \overline{I}_r$.

Conversely, let $[s', s'']$ be some element of the maximal chain and $\gamma(s') = \gamma(s'') \in \partial B_\infty(\gamma)$. Assuming that the last element of the chain is another element $[\widetilde{s}', \widetilde{s}''] \supset [s', s'']$, by virtue of what has been proved above we would have $\widetilde{s}^- \le s'$ and $\widetilde{s}^+ \ge s''$, where $\widetilde{s}^-$ and $\widetilde{s}^+$ are defined for $\widetilde{s}'$ and $\widetilde{s}''$. But then $\gamma(s') = \gamma(s'') \notin \partial B_\infty(\gamma)$. The statement is proved.

Let $\underline{I}_r = [\underline{s}'_r, \underline{s}''_r]$ be the first element of a maximal chain. There are two possible cases:

I. $\underline{s}'_r < s''_r$;

II. $\underline{s}'_r = \underline{s}''_r$, i.e., $[\underline{s}'_r, \underline{s}''_r]$ degenerates into a point.

From Lemma 4 and the definition of $\underline{I}_r$ it follows that in case I the curve $\gamma[\underline{s}'_r, \underline{s}''_r]$ is a Jordan curve.

Denote by B_r the domain bounded by the curve $\gamma[\underline{s}'_r, \underline{s}''_r]$ and not containing the point at infinity. Lemma 4 implies that two segments belonging to $\mathfrak{M}(\gamma)$ either have no interior points or one of them is wholly contained within the other. Therefore the number of maximal chains is at most countable and $B_{r_1} \cap B_{r_2} = \varnothing$. We will prove that the number of maximal chains of type I is finite, which is equivalent to proving that the number of domains B_r is finite. Choose a point a_r on each curve $\gamma[\underline{s}'_r, \underline{s}''_r]$. If the set of chosen points is infinite, then it should have at least one limit point which is a boundary point by virtue of the fact that the set of boundary points is closed. Denote it by $a_0 = \gamma(s_0)$. From the set $\{a_r\}$ choose a sequence $\{a_k\}_{k=1}^\infty$ tending to a_0 and assume that γ_k are those Jordan curves from the set $\{\gamma[\underline{s}'_r, \underline{s}''_r]\}$ on which these points lie. Since $\lim\limits_{k\to\infty} \operatorname{diam}\langle\gamma_k\rangle = 0$, any neighborhood of a_0 will contain an infinite number of γ_k. Since the curve γ is piecewise-smooth, it can be assumed without loss of generality that γ_k are smooth and therefore by the property $\gamma'(\underline{s}'_k) = -\gamma'(\underline{s}''_k)$ we have $|\arg\gamma'(\underline{s}'_k) - \arg\gamma'(\underline{s}''_k)| = \pi$, $k = 1, 2, \ldots$. But for sufficiently small $\delta > 0$ we have either $|\arg\gamma'(s) - \arg\gamma'(s_0)| < \varepsilon$ when $|s - s_0| < \delta$ when $\gamma'(s)$ is continuous at the point s_0 or $|\arg\gamma'(s) - \arg\gamma'(s_0 - 0)| < \varepsilon$ when $s_0 - \delta \le s \le s_0$ and $|\arg\gamma'(s) - \arg\gamma'(s_0 - 0) - \pi\nu_0| < \varepsilon$ when $s_0 \le s < s_0 + \delta$ at a corner of opening $\pi\nu_0$. In both cases the interval of length π cannot be covered by the above-said sets. Hence the set of chains of type I is finite.

Let us consider case II. Again, neglecting the finite number of points of discontinuity of $\gamma'(s)$, it can be assumed that $\gamma(s)$ has a derivative at the ends of each interval $I = [s'_\alpha, s''_\alpha]$ contained in a chain r of type II and in that case the equality $\gamma'(s'_\alpha) = -\gamma'(s''_\alpha)$ is fulfilled again. Therefore $\lim\limits_{s\to s_0-0}\gamma'(s) = -\lim\limits_{s\to s_0+0}\gamma'(s)$, where $s_0 = \bigcap\limits_{\alpha\in\mathcal{A}} I_\alpha$, i.e., the opening of the angle at the point $\gamma(s_0)$ is equal to 2π. Since, by assumption that γ has a finite number of corners, the number of chains of type II is also finite.

4. Let us fix some maximal chain r' and consider $I = \cup I_\alpha$, where $I_\alpha \in \mathfrak{M}(\gamma)\backslash r'$. Since any union of the form $\cup I_\alpha$, where I_α are contained in the same chain, is an element of this chain (i.e., is a segment) and the number of maximal chains is finite, we conclude that the set I is a finite union of segments. Let $\widetilde{I}=\bigcup_{k=1}^{m}[s'_k,s''_k]$, where $\overline{s}\,'_{r'}\le s'_1<s''_1<s'_2<s''_2<\cdots<s'_m<s''_m\le\overline{s}\,''_{r'}$. Choose in $B_{r'}$ a point $z_{r'}$ and connect it with the point $\gamma(s_{r'})\in\gamma[\underline{s}\,'_{r'},\underline{s}\,''_{r'}]$ by means of the simple arc $l_{r'}$ passing through $B_{r'}$. Let $s'_j<s_{r'}$, $j=\overline{1,k}$, $s''_j<s_{r'}$, $j=\overline{1,k}$, and $s'_{k+1}>s_{r'}$. Consider the curve

$$C_{r'}=L_{r'}\cdot\gamma[\overline{s}\,'_{r'},s'_1]\cdot\gamma[s''_1,s'_2]\cdot\cdots\cdot\gamma[s''_k,s_{r'}]\cdot l_{r'}, \tag{10}$$

where $L_{r'}$ is a simple curve passing through $B_\infty(\gamma)$ and connecting the point at infinity with $\gamma(\overline{s}\,'_{r'})$. The curve $C_{r'}$ is simple by construction. Fix in $\overline{\mathbb{C}}\backslash\langle C_{r'}\rangle$ a one-valued branch of the function $P(r'):w=\sqrt{z-z_{r'}}$. The function $P(r')$ conformally maps B onto some domain $B(r')$ whose boundary contains simple arcs $P(r')\circ\gamma[\overline{s}\,'_{r'},s'_1], P(r')\circ\gamma[s''_1,s'_2],\ldots,P(r')\circ\gamma[s''_m,\overline{s}\,''_{r'}]$. Since $P(r')$ is analytically continuable across the both sides of the cut $C_{r'}$, the images of the corners of the curve γ are the corners of the boundary of the domain $B(r')$ of the same openings, while new corners do not appear. Since all twin values corresponding to the end-points of segments, contained in the chain r', cease being twin, the number of maximal chains in $\mathfrak{M}(P_{r'}\circ\gamma)$ is less by one than in $\mathfrak{M}(\gamma)$, while the points $P(r')(\gamma(s'_k))$ and $P(r')(\gamma(s''_k))$, $k=\overline{1,m}$, turn out to lie on $\partial B_\infty(P_{r'}\circ\gamma)$ and thus become the last elements of the respective chains.

If now the procedure described above is applied to $B(r')$, then, without violating the piecewise-smoothness of the boundary, we again decrease the number of maximal chains by one.

Continuing this process, after a finite number of steps we come to the domain B_0 bounded by the piecewise-smooth curve γ_0 with the same number of corners and the same angle openings as those of the initial curve γ. But if the set $\mathfrak{M}(\gamma)$ contains maximal chains of type II, the new curve γ_0 will keep them and it will not be a Jordan curve.

Lemma 5. *Let z_* be an accessible from $B_\infty(\gamma_*)$ corner of opening $\nu_*\pi$, $\nu_*<1$, on ∂B_*. Then there exists a holomorphic and univalent function $w=\Phi_*(z)$ in $\overline{B}_*$ such that the mapping $\zeta=(w-\Phi_*(z_*))^{\frac{1}{\nu_*}}$ is univalent in $\overline{\Phi_*(B_*)}$.*

We will construct the function Φ_* with more specific properties. Namely, $\Phi_*(\gamma_*)=0$ and the direction of the bisectrix of the angle E_* with vertex at the point $\Phi_*(z_*)$ will coincide with the direction of the positive real semi-axis.

Choose a point $a\in D(z_*,\delta)\cap C\overline{B}$, where $D(z_*,\delta)=\{z;\,|z-z_*|<\delta\}$, and connect it by means of the curve l_*, having no common points with $\overline{B}_*$, with the point at infinity. This can be done since z_* is accessible from $B_\infty(\gamma_*)$. Cut the plane along l_* and map the obtained domain conformally onto the plane cut along the negative real semi-axis. Normalize the mapping $w=F(z)$ by the condition $F(z_*)=u_0$, $\arg F'(z_*)=\eta$, where $u_0>0$ and η is chosen so that the direction of the bisectrix of the angle E_* with vertex at z_* be mapped on the direction of the positive real semi-axis.

Take $\varepsilon > 0$ such that $\frac{\nu_*}{2} + \varepsilon < \frac{1}{2}$ and choose $\delta > 0$ so that $w(s) = F(z(s)) \in E_{\frac{\nu_*}{2}+\varepsilon}(u_0)$ for $s \in (s_* - \delta, s_* + \delta)$, where $z_* = \gamma_*(s_*)$. Next, choose q and $0 < q < u_0$ so that $(D(q,\rho) \backslash E_{\frac{\nu_*}{2}+\varepsilon}(u_0)) \cap F(B_*) = \varnothing$. Denote by w_1 some point of the intersection of the circumference $C(q,\rho) = \{w : |w - q| = \rho\}$ with the boundary $E_{\frac{\nu_*}{2}+\varepsilon}(u_0)$ and let $\pi\beta = |\arg w_1|$. It is obvious that $0 < \beta < 1$. Consider the translation $T_q : \zeta = w - q$. For the mapping $P_\beta \circ T_q$ the circle $D'(q,\rho)$, cut along the radius directed towards the negative real semi-axis, is mapped into $E_\beta(0)$, while the angle $E_{\frac{\nu_*}{2}+\varepsilon}(u_0)$ is mapped by Lemma 3 onto the domain contained in $E_{\frac{\nu_*}{2}+\varepsilon}((u_0 - q)^\beta)$. Thus the domain $E_\beta(0) \backslash E_{\frac{\nu_*}{2}+\varepsilon}((u_0 - q)^\beta)$ contains no points of the domain $P_\beta(T_q(F(B_*)))$. Make another translation $\widetilde{T}_{(u_0-q)^\beta} : \widetilde{\zeta} = \zeta - (u_0 - q)^\beta$. Then $\widetilde{T}_{(u_0-q)^\beta}\big(P_\beta(T_q(F(B_*)))\big) \subset E_{\frac{\nu_*}{2}+\varepsilon}(0)$. But $E_{\frac{\nu_*}{2}+\varepsilon}(0)$ is a subdomain of the domain where the function $P_{\frac{1}{\nu_*}}$ is univalent. Each of the mappings $\widetilde{T}_{(u_0-q)^\beta}$, P_β, T_q and F is univalent on the closure of those domains on which they are defined and therefore the mapping $\Phi_* = \widetilde{T}_{(u_0-q)^\beta} \circ P_\beta \circ T_q \circ F$ satisfies required conditions.

Note that the statement of the lemma also holds for $\nu_* \geq 1$ since in the case $\nu_* = 1$ the point z_* is not a corner, while for $\nu_* > 1$ we can take as Φ_* the identical mapping. However, for the symmetrical notation of the expressions arising below we will use the common symbols in all cases. In this context, for $\nu_* > 1$ we will take as $\Phi_*(z)$ the entire linear function which maps a corner on the origin and the direction of the interior angle bisectrix on the direction of the positive real semi-axis. Our next task is to map the domain B_0 onto the domain bounded by a Jordan curve without corners.

Let $z_1 = \gamma_0(s_*)$ be the first element of the maximal chain $r_1 \in \mathfrak{M}(\gamma_0)$ of type II. The natural parameter on the curve γ_0 is again denoted by s and it is assumed that s'_j and s''_j are the same notations for r_1 as in the case of a chain of type I. An auxiliary curve C_{r_1} has the same form as (10) but with the only difference that the curve l_{r_1} is absent and the last cofactor in the expression for C_{r_1} is $\gamma_0[s''_k, s_*]$, where $s''_k < s_*$, and $s'_{k+1} > s_*$. Let us make a mapping $P_{\frac{1}{2}} \circ \Phi_1$, where Φ_1 is the holomorphic function from Lemma 5 (the entire linear function in the considered case). By Lemma 2 the point $P_{\frac{1}{2}} \circ \Phi_1(z_1)$ is not a corner of the curve $\gamma_1 = P_{\frac{1}{2}} \circ \Phi_1 \circ \gamma_0$. Moreover, like in the case of a chain of type I, the number of maximal chains in $\mathfrak{M}(\gamma_1)$ is less by one than the number of chains in $\mathfrak{M}(\gamma_0)$. All corners of the curve γ_1, except the point $P_{\frac{1}{2}} \circ \Phi_1(z_1)$, have the same opening as their preimages.

Continue this process until after performing a finite number of steps the obtained curve γ_{n_0} becomes a Jordan curve. Number the remaining corners in an arbitrary manner starting from $n_0 + 1$. The following notation will be used below: Φ_j $(j \geq 2)$ will denote the function from Lemma 5 for the domain

$$\big(P_{\frac{1}{\nu_{j-1}}} \circ \Phi_{j-1} \circ \cdots \circ P_{\frac{1}{\nu_1}} \circ \Phi_1\big)(B_0)$$

and the point $\big(P_{\frac{1}{\nu_{j-1}}} \circ \Phi_{j-1} \circ \cdots \circ P_{\frac{1}{\nu_1}} \circ \Phi_1\big)(z_j)$.

Recall that $\nu_1 = \nu_2 = \cdots = \nu_{n_0} = 2$ and $z_1, z_2, \ldots, z_{n_0}$ are the points corresponding to the first elements of type II in $\mathfrak{M}(\gamma_0)$.

Denote $\widetilde{\omega} = P_{\frac{1}{\nu_k}} \circ \Phi_n \circ \cdots \circ P_{\frac{1}{\nu_1}} \circ \Phi_1$. From Lemma 2 and the above constructions it follows that the curve $\gamma_n = \widetilde{\omega} \circ \gamma_0$ is a Jordan smooth curve.

5. Fix k, $1 \le k \le n$, and write $\widetilde{\omega}$ in the form

$$\widetilde{\omega} = X_k \circ P_{\frac{1}{\nu_k}} \circ \widetilde{X}_k,$$

where

$$\widetilde{X}_k = \Phi_k \circ P_{\frac{1}{\nu_k}} \circ \cdots \circ P_{\frac{1}{\nu_1}} \circ \Phi_1,$$
$$X_k = \Phi_{\frac{1}{\nu_n}} \circ \Phi_n \circ P_{\frac{1}{\nu_{n-1}}} \circ \cdots \circ \Phi_{k+1}.$$

Since the univalent function $\overline{B}_0$ in $\widetilde{X}_k$ is holomorphic at the point z_k and $\widetilde{X}_k(z_k) = 0$, for z sufficiently close to z_k we have

$$\widetilde{X}_k(z) = \sum_{m=1}^{\infty} \widetilde{a}_m (z - z_k)^m, \quad \widetilde{a}_1 \neq 0.$$

Hence we obtain

$$\widetilde{X}(z) = (z - z_k) R_k(z),$$

where $R_k(z) \neq 0$ for $|z - z_k| < \delta_k$. Therefore

$$\begin{aligned} \frac{d}{dz}\big(P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z))\big) &= \frac{1}{\nu_k}(z - z_k)^{\frac{1}{\nu_k}-1}(R_k(z))^{\frac{1}{\nu_k}-1}\big[R_k(z) + (z - z_k)R_k'(z)\big] \\ &= (z - z_k)^{\frac{1}{\nu_k}-1} g_k(z), \end{aligned}$$

where $g_k(z) = \frac{1}{\nu_k}(R_k(z))^{\frac{1}{\nu_k}-1}[R_k(z) + (z - z_k)R_k'(z)]$ is a non-vanishing holomorphic function in $D(z_k, \delta_k)$.

Since $X_k(w)$ is holomorphic in the neighborhood $w = 0$, for $w = P_{\frac{1}{\nu_k}}(\widetilde{X}(z))$, where $z \in D^+(z_k, \delta_k)$, we have

$$\begin{aligned} \frac{d\widetilde{\omega}(z)}{dz} = \frac{dX_k(w)}{dw} \cdot \frac{dw}{dz} &= \frac{dX_k\big(P_{\frac{1}{\nu_k}}(\widetilde{X}(z))\big)}{dw} \cdot \frac{d}{dz}\big(P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z))\big) \\ &= \frac{dX_k\big(P_{\frac{1}{\nu_k}}(\widetilde{X}(z))\big)}{dw} \cdot (z - z_k)^{\frac{1}{\nu_k}-1} \cdot g_k(z). \end{aligned}$$

Denoting

$$\frac{dX_k\big(P_{\frac{1}{\nu_k}}(\widetilde{X}(z))\big)}{dw} \cdot g_k(z) = \widetilde{\omega}_k(z),$$

we obtain the local representation

$$\widetilde{\omega}\,'(z) = (z - z_k)^{\frac{1}{\nu_k}-1} \cdot \widetilde{\omega}_k(z), \tag{11}$$

where $\widetilde{\omega}_k(z)$ is holomorphic in $\overline{D}^{\,+}(z_k, \delta_k) \backslash \{z_k\}$, continuous in $\overline{D}(z_k, \delta_k)$ because $P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z))$ is continuous and non-vanishing because $X_k(w)$ is univalent in the neighborhood of zero.

Consider the function

$$\widetilde{\omega}_0(z) = \widetilde{\omega}\,'(z) \prod_{k=1}^{n} (z - z_k)^{1-\frac{1}{\nu_k}}. \tag{12}$$

By virtue of (11) the function $\widetilde{\omega}_0(z)$ is holomorphic in $\overline{B}_0 \setminus \bigcup_{k=1}^{n} \{z_k\}$, continuous in $\overline{B}_0$ and non-vanishing. From (12) we obtain the representation

$$\widetilde{\omega}\,' = \widetilde{\omega}_0\,(z) \prod_{k=1}^{n} (z - z_k)^{\frac{1}{\nu_k}-1}.$$

To obtain a similar representation of the function $\widetilde{\Omega}\,'(\zeta) = (\widetilde{\omega}^{\,-1}(\zeta))'$, where $\zeta \in B_n = \widetilde{\omega}(B_0)$, we write $\widetilde{\Omega}$ in the form

$$\widetilde{\Omega} = \widetilde{X}_k^{\,-1} \circ P_{\nu_k} \circ X_k^{-1},$$

where

$$X_k^{-1} = \Phi_{k+1}^{-1} \circ P_{\nu_{k+1}} \circ \Phi_{k+2}^{-1} \circ \cdots \circ P_{\nu_n},$$
$$\widetilde{X}_k^{\,-1} = \Phi_1^{-1} \circ P_{\nu_1} \circ \Phi_2^{-1} \circ \cdots \circ \Phi_k^{-1}$$

and investigate the behavior of its derivative in the neighborhood of the point $\zeta_k = \widetilde{\omega}(z_k)$. Repeating the previous arguments for the function $\widetilde{\Omega}$, we obtain the representation

$$\widetilde{\Omega}\,'(\zeta) = \widetilde{\Omega}_0(\zeta) \prod_{k=1}^{n} (\zeta - \zeta_k)^{\nu_k - 1}, \tag{13}$$

where the function $\widetilde{\Omega}_0(z)$ is holomorphic in $\overline{B}_n \setminus \bigcup_{k=1}^{n} \{\zeta_k\}$, continuous and non-vanishing in $\overline{B}_n$.

So far we have been investigating the behavior of the derivative of the function mapping B_0 onto $\mathbb{D}$. But B_0 is obtained from the domain B by means of the conformal mapping violating neither the piecewise-smoothness of the boundary nor the openings of corners and therefore the reasoning used above for $\widetilde{\omega}$ and $\widetilde{\Omega}$ can be applied both to the function mapping B onto $\mathbb{D}$ and to the inverse function.

6. Before we proceed to investigating the nature of the continuity of the considered functions it is appropriate to make the following remark: if the domain B (or B_0) is bounded by a non-Jordan curve, then an inequality of form (2) cannot be satisfied globally all over the boundary. Again, since the mapping $B \to B_0$, as mentioned above, is analytically continuable across the boundary, it inherits all the local boundary properties of the function $\widetilde{\omega}$ which we are interested in.

Let $0 < \nu_k < 2$. Then, as is known ([4], §6 and Appendix 1), the function $t^{\frac{1}{\nu_k}}$ satisfies on the curve $t = \widetilde{X}_k \circ \gamma_0(s)$ the condition $H(\mu)$, where $\mu = \min(1, \frac{1}{\nu_k})$.

Therefore for $z_1, z_2 \in \langle\gamma\rangle \cap D(z_k, \delta_k)$ we have

$$\left|\frac{dX_k\big(P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_1))\big)}{dw} - \frac{dX_k\big(P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_2))\big)}{dw}\right| \le M_5\big|P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_1) - P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_2))\big|$$
$$\le M_6\big|\widetilde{X}_k(z_1) - \widetilde{X}_k(z_2)\big|^{\mu} \le M_7|z_1 - z_2|^{\mu}. \tag{14}$$

It obviously follows that on $\partial D(z_k, \delta_k) \cap \overline{B}_0$ the function $\frac{dX_k\left(P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z))\right)}{dw}$ satisfies the condition $H(1)$ and thus it satisfies the condition $H(\mu)$ all over $\overline{D}^+(z_k, \delta_k) = \overline{D(z_k, \delta_k) \cap B_0}$ ([4], § 15 and Appendix II).

Let us consider the case $\nu_k = 2$ (cusp). Let L_1 and L_2 be the arcs of the curve $\widetilde{X}_k \circ \gamma$ which are adjacent to the corner $\widetilde{z}_k = 0$, and let in a sufficiently small neighborhood of zero these arcs be represented in the form

$$L_1 : \widetilde{y} = \chi_1(\widetilde{x}), \quad L_2 : \widetilde{y} = -\chi_2(\widetilde{x}),$$

where $\chi_j(\widetilde{x}) = |\widetilde{x}|^{p_j}\varphi_j(\widetilde{x})$, $0 < k \le \varphi_j(\widetilde{x}) \le K$, $j = 1, 2$; $0 \le -\widetilde{x} \le \varepsilon$, $1 < p_j < \infty$. In that case the point z_k is called a cusp of finite order. It is obvious that the numbers p_j, $j = 1, 2$, are invariant with respect to diffeomorphisms of the domain enclosing $\overline{B}_0$. Let $\widetilde{z}_j = \widetilde{x}_j + i\widetilde{y}_j \in L_j$, $j = 1, 2$. Then

$$|\widetilde{z}_j| = \sqrt{\widetilde{x}_j^{\,2} + \chi_j^2(\widetilde{x}_j)} = \sqrt{\left[\frac{\chi_j(\widetilde{x}_j)}{\varphi_j(\widetilde{x}_j)}\right]^{\frac{2}{pj}} + \chi_j^2(\widetilde{x}_j)}$$
$$= \big|\chi_j(\widetilde{x}_j)\big|^{\frac{1}{pj}}\sqrt{\big[\varphi_j(\widetilde{x}_j)\big]^{-\frac{2}{pj}} + \big[\chi_j(\widetilde{x}_j)\big]^{2(1-\frac{1}{pj})}}$$
$$\le M_8\big|\chi_j(\widetilde{x}_j)\big|^{\frac{1}{pj}} \le M_8\big|\chi_j(x_j)\big|^{\frac{1}{p}}, \tag{15}$$

where $p = \max(p_1, p_2)$.

On the other hand, we have

$$|\widetilde{z}_1 - \widetilde{z}_2| \ge |\chi_1(\widetilde{x}_1)| + |\chi_2(\widetilde{x}_2)|. \tag{16}$$

From (15) and (16) we obtain

$$|\widetilde{z}_1 - \widetilde{z}_2| \ge M_9\big(|\widetilde{z}_1|^p + |\widetilde{z}_2|^p\big). \tag{17}$$

Now using the inequality

$$a^p + b^p \ge 2^{1-p}(a+b)^p, \tag{18}$$

where $a \ge 0$, $b \ge 0$ and $p > 1$ ([6], Section 3.5), from (17) we get

$$|\widetilde{z}_1 - \widetilde{z}_2| \ge M_{10}\big(|\widetilde{z}_1| + |\widetilde{z}_2|\big)^p.$$

Let $w_j = P_{\frac{1}{2}}(\widetilde{z}_j)$, $j = 1, 2$. Then

$$|\widetilde{z}_1 - \widetilde{z}_2| \ge M_{10}\big(|w_1|^2 + |w_2|^2\big)^p$$

and again using inequality (18) we obtain

$$|\widetilde{z}_1 - \widetilde{z}_2| \ge M_{11}\big(|w_1| + |w_2|\big)^{2p} \ge M_{11}|w_1 - w_2|^{2p},$$

i.e.,

$$|w_1 - w_2| \le M_{12}|z_1 - z_2|^{\frac{1}{2p}}. \tag{19}$$

Inequality (19) means that for a cusp of finite order the function $w = \widetilde{z}^{\,\frac{1}{2}}$ satisfies, on $\widetilde{X}_k \circ \gamma$ in a neighborhood of zero, the condition $H(\frac{1}{2p})$ in the so-called strong form ([4], Appendix II). Hence, repeating the arguments we used for (14), we conclude that $\widetilde{\omega}_0(z)$ also satisfies, on $\langle\gamma\rangle \cap D(z_k, \delta_k))$, the condition $H(\frac{1}{2p})$ in the strong form and therefore satisfies this condition in $\overline{D}^{\,+}(z_k, \delta_k)$.

Since $\widetilde{\omega}_0$ is holomorphic in $\overline{B}_0 \setminus \bigcup\limits_{k=1}^{n} \{z_k\}$, the latter fact immediately implies that if γ_0 is a Jordan curve and all cusps are of finite order, then $\widetilde{\omega}_0$ satisfies the Hölder condition all over $\overline{B}_o$.

The proof that $\overline{\Omega}_0$ is a Hölder continuous function in $\overline{B}_n$ is simpler since $\gamma_n = \partial B_n$ is smooth and an estimate of form (14) for the function

$$\frac{dX_k^{-1}\big(P_{\nu_k}(X_k^{-1}(\zeta))\big)}{d\widetilde{z}},$$

where $\widetilde{z} = P_{\nu_k}(X_k^{-1}(\zeta))$, $\zeta \in \overline{B}_n$, implies that it is a Hölder continuous function in $\overline{B}_n$.

7. Let us perform the last mapping $\Phi : B_n \to \mathbb{D}$. Note that if z_k is a corner of opening $\pi\nu_k$, $\nu_k < 2$, or a cusp of finite order, then for $z_1, z_2 \in \overline{D}^{\,+}(z_k, \delta_k)$ we have

$$\begin{aligned} \big|\widetilde{\omega}(z_1) - \widetilde{\omega}(z_2)\big| &= \Big|X_k\big(P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_1))\big) - X_k\big(P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_2))\big)\Big| \\ \le M_{13}\big|P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_1)) - P_{\frac{1}{\nu_k}}(\widetilde{X}_k(z_2))\big| &\le M_{14}\big|\widetilde{X}_k(z_1) - \widetilde{X}_k(z_2)\big|^{\mu} \\ &\le M_{15}|z_1 - z_2|^{\mu}, \quad \mu > 0. \end{aligned}$$

Let γ be a piecewise-Lyapunov curve. Then, by the corollary of Lemma 2, ∂B_n is a Lyapunov curve and $\Phi'_\zeta(\zeta)$ satisfies in $\overline{B}_n$ the Hölder condition and is different from zero [7]. Denote $\omega = \Phi \circ \widetilde{\omega}$. Then

$$\frac{d\omega}{dz} = \frac{d\Phi(\zeta)}{d\zeta} \cdot \frac{d\widetilde{\omega}(z)}{dz}.$$

If z_k is a corner with $\nu_k < 2$ or a cusp of finite order, then for $z_1, z_2 \in \overline{D}^{\,+}(z_k, \delta_k)$ we have

$$\big|\Phi'_\zeta(\widetilde{\omega}(z_1)) - \Phi'_\zeta(\widetilde{\omega}(z_2))\big| \le M_{16}\big|\widetilde{\omega}(z_1) - \widetilde{\omega}(z_2)\big|^{\mu_1} \le M_{17}|z_1 - z_2|^{\mu_2}, \quad \mu_2 > 0.$$

Therefore in the representation

$$\omega'(z) = \Phi'_\zeta(\widetilde{\omega}(z))\,\widetilde{\omega}_k(z)(z - z_k)^{\frac{1}{\nu_k} - 1}$$

given by equality (11) the function $\Phi'_\zeta(\widetilde{\omega}(z))\,\widetilde{\omega}_k(z)$ satisfies in $\overline{D}^{\,+}(z_k, \delta_k)$ the Hölder condition and is different from zero. This fact allows us to conclude that in the representation

$$\omega'(z) = \omega_0(z) \prod_{k=1}^{n} (z - z_k)^{\frac{1}{\nu_k} - 1}$$

the holomorphic function $\omega_0(z)$ in B_0 is continuous in $\overline{B}_0$ and satisfies the Hölder condition on each smooth simple arc of the curve γ_0. If however γ_0 is a Jordan curve and all cusps are of finite order, then ω_0 is a Höder continuous function in $\overline{B}_0$.

Let us denote $\Phi^{-1} = \Psi$ and consider the behavior of the function $\Omega = \widetilde{\Omega} \circ \Psi$ in the neighborhood of the point $\tau_k = \Phi(\widetilde{\omega}(z_k)) = \omega(z_k)$. Since ∂B_n is a Lyapunov curve, the function $\Psi'(\tau)$ satisfies in $\overline{\mathbb{D}}$ the Hölder condition and is different from zero [7]. Using (13), we obtain

$$\Omega'(\tau) = \Psi'(\tau) \cdot \widetilde{\Omega}'_\zeta(\Psi(\tau)) = \Psi'(\tau) \cdot \widetilde{\Omega}_0(\Psi(\tau)) \cdot \prod_{k=1}^{n} \big(\Psi(\tau) - \Psi(\tau_k)\big)^{\nu_k - 1}$$

$$= \Psi'(\tau) \cdot \widetilde{\Omega}_0(\Psi(\tau)) \prod_{k=1}^{n} \Big(\frac{\Psi(\tau) - \Psi(\tau_k)}{\tau - \tau_k}\Big)^{\nu_k - 1} \prod_{k=1}^{n} (\tau - \tau_k)^{\nu_k - 1}.$$

Since $\widetilde{\Omega}_0(\zeta)$ and $\Psi(\tau)$ are Hölder continuous functions, the composition $\widetilde{\Omega}_0 \circ \Psi$ is a Hölder continuous function in $\overline{\mathbb{D}}$. Consider the continuous function

$$u(\tau, \tau_k) = \operatorname{Arg} \frac{\Psi(\tau) - \Psi(\tau_k)}{\tau - \tau_k} \tag{20}$$

in $\mathbb{D}$. By Lemma 1 this function satisfies the Hölder condition on each of the arcs $l_k^- = \{e^{i\theta},\ \theta_k - \varepsilon \le \theta \le \theta_k\}$ and $l_k^+ = \{e^{i\theta},\ \theta_k \le \theta \le \theta_k + \varepsilon\}$, where $\tau_k = e^{i\theta_k}$. Therefore $u(\tau, \tau_k)$ satisfies the Hölder condition on $l^- \cup l^+$. On the remaining part of the unit circumference the Hölder continuity of the function $u(\tau, \tau_k)$ is obvious. But in that case the function

$$W(\tau, \tau_k) = \ln \frac{\Psi(\tau) - \Psi(\tau_k)}{\tau - \tau_k}$$

satisfies the Hölder condition in $\overline{\mathbb{D}}$ [4]. Therefore

$$\left| \frac{\Psi(\tau') - \Psi(\tau_k)}{\tau' - \tau_k} - \frac{\Psi(\tau'') - \Psi(\tau_k)}{\tau'' - \tau_k} \right| = \big| \exp W(\tau', \tau_k) - \exp W(\tau'', \tau_k) \big|$$

$$= \left| \sum_{n=1}^{\infty} \frac{W^n(\tau', \tau_k) - W^n(\tau'', \tau_k)}{n!} \right| \le \big| W(\tau', \tau_k) - W(\tau'', \tau_k) \big| \cdot \sum_{n=1}^{\infty} \frac{n M_{18}^n}{n!}$$

$$\le M_{19} |\tau' - \tau''|^\alpha, \quad 0 < \alpha \le 1.$$

In conclusion let us consider the case with the piecewise-smooth curve γ. In that case ∂B_n is a smooth curve and therefore $\Psi'(\tau) \in H_p(\mathbb{D})$ for all $p > 0$([7], Ch. IX). The function $\widetilde{\Omega}_0(\Psi(\tau))$ is bounded in $\mathbb{D}$ since $\widetilde{\Omega}_0(\zeta)$ is continuous in $\overline{B}_n$. The continuity of the function $u(\tau, \tau_k)$ in $\overline{\mathbb{D}}$ implies that

$$\left[\frac{\Psi(\tau) - \Psi(\tau_k)}{\tau - \tau_k} \right]^{\pm 1} \in H_p$$

for all $p > 0$ ([7], Ch. IX), from which it follows that

$$\Psi'(\tau) \widetilde{\Omega}_0(\Psi(\tau)) \prod_{k=1}^{n} \Big[\frac{\Psi(\tau) - \Psi(\tau_k)}{\tau - \tau_k} \Big]^{\nu_k - 1} \in H_p$$

for all $p > 0$.

Finally, the identity

$$\left|\omega(\Omega(\tau))\sqrt[p]{\Omega'(\tau)}\right|^p = |\tau|^p\,|\Omega'(\tau)|, \quad \tau \in \mathbb{D},$$

immediately implies that $\omega(z)$ belongs to $\bigcap\limits_{p>0} E_p(B_0)$.

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