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WEIGHTED ESTIMATES FOR THE HILBERT TRANSFORM OF ODD FUNCTIONS

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Abstract. The aim of the present paper is to characterize the classes of weights which ensure the validity of one-weighted strong, weak or extra-weak type estimates in Orlicz classes for the integral operator

$$H_0 f(x) = \frac{2}{\pi}\int_0^\infty \frac{yf(y)}{x^2-y^2}\,dy, \quad x\in(0,\infty).$$

1. Introduction. The Hilbert transform is given for any function f satisfying

$$\int_{-\infty}^{\infty} |f(x)|\,(1+|x|)^{-1}\,dx < \infty$$

by the Cauchy principal value integral

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{\mathbf{R}\setminus(x-\varepsilon,x+\varepsilon)} \frac{f(y)}{x-y}\,dy.$$

If f is an odd function, then Hf is even, and $Hf(x) = H_o f(|x|)$, where

$$H_o f(x) = \frac{2}{\pi}\int_0^\infty \frac{yf(y)}{x^2-y^2}\,dy, \qquad x\in(0,\infty).$$

The Hilbert transform is closely related to the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{I\ni x}\frac{1}{|I|}\int_I |f(y)|\,dy.$$

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If ϱ is a weight (measurable and nonnegative function) and $1 \le p < \infty$, strong type inequalities

$$\int_\Omega |Tf(x)|^p \varrho(x)\,dx \le C \int_\Omega |f(x)|^p \varrho(x)\,dx, \tag{1.1}$$

as well as weak type inequalities

$$\varrho(\{|Tf| > \lambda\}) \le C\,\lambda^{-p} \int_\Omega |f(x)|^p \varrho(x)\,dx, \tag{1.2}$$

have been widely studied by many authors. The pioneering result of Muckenhoupt [13] stated that (1.1) holds with $\Omega = \mathbf{R}$, $T = M$ and $p > 1$ if and only if $\varrho \in A_p$, that is,

$$\sup_I \varrho_I \cdot ((\varrho^{1-p'})_I)^{p-1} \le C,$$

and (1.2) holds with $\Omega = \mathbf{R}, T = M$ and $p \ge 1$ if and only if $\varrho \in A_p$, where $\varrho \in A_1$ means $\varrho_I \le C \operatorname{ess\,inf}_I \varrho$. Hunt, Muckenhoupt and Wheeden [10] proved the same result for $\Omega = \mathbf{R}$ and $T = H$. The class of good weights for (1.1) or (1.2) with $\Omega = (0,\infty)$ and $T = H_o$ appears to be strictly larger than A_p. This result is due to Andersen who showed that (1.1) with $\Omega = (0,\infty)$, $p > 1$, and $T = H_o$ holds if and only if $\varrho \in A_p^o$, that is,

$$\varrho(a,b)\Big(\int_a^b \varrho^{1-p'}(x)x^{p'}\,dx\Big)^{p-1} \le C(b^2 - a^2)^p, \qquad (a,b) \subset \mathbf{R}, \tag{1.3}$$

and (1.2) with $\Omega = (0,\infty)$, $p \ge 1$, and $T = H_o$ holds if and only if $\varrho \in A_p^o$, where $\varrho \in A_1^o$ means

$$\frac{\varrho(a,b)}{b^2 - a^2} \le C \operatorname{ess\,inf}_{(a,b)} \frac{\varrho(x)}{x}.$$

Our aim is to study analogous inequalities where the power function t^p is replaced by a general convex function $\Phi(t)$. More precisely, we shall study the inequalities

$$\int_\Omega \Phi(|Tf(x)|)\,\varrho(x)\,dx \le C \int_\Omega \Phi(C|f(x)|)\,\varrho(x)\,dx, \tag{1.4}$$

$$\varrho(\{|Tf| > \lambda\}) \cdot \Phi(\lambda) \le C \int_\Omega \Phi(C|f(x)|)\,\varrho(x)\,dx, \tag{1.5}$$

and

$$\varrho(\{|Tf| > \lambda\}) \le C \int\limits_{\Omega} \Phi(C\lambda^{-1}|f(x)|)\, \varrho(x)\, dx. \tag{1.6}$$

We call (1.4) strong type inequality, (1.5) weak type inequality, and (1.6) extra–weak type inequality. While (1.4) is an analogue of (1.1), (1.5) and (1.6) are two different analogues of (1.2). It is always true that (1.4) $\Rightarrow$ (1.5) $\Rightarrow$ (1.6), and none of these implications is reversible in general. The interest in these types of inequalities stems from their use in various problems of Fourier analysis. For example, extra–weak type inequalities have interesting interpolation applications (see [2]).

We assume throughout that Φ is a convex nondecreasing function on $(0,\infty)$, $\Phi(0) = 0$. In fact, it is not hard to prove that for all the above operators the inequalities (1.4) or (1.5) always imply at least quasiconvexity of Φ. For more detaile, see [9].

Weak and extra–weak type inequalities together were apparently firstly studied in [14] for $T = M$ and $\Omega = \mathbf{R}^n$. In [9] the following results were obtained (for definitions see Section 2 below):

Theorem A. *The inequality* (1.5) *holds with* $T = H$ *and* $\Omega = \mathbf{R}$ *if and only if* $\Phi \in \Delta_2$ *and* $\varrho \in A_\Phi$.

Theorem B. *Let* $\Phi \in \Delta_2^0$. *Then* (1.6) *holds with* $T = H$ *and* $\Omega = \mathbf{R}$ *if and only if* $\varrho \in E_\Phi$.

The main aim of the present paper is to characterize the classes of weights for which the inequalities (1.4–6) hold with $\Omega = (0,\infty)$ and $T = H_o$ (Theorems 3–5 in Section 4). Moreover, we get a characterization for the strong type inequality (1.4) with $\Omega = \mathbf{R}$ and $T = H$. This is given in Section 3 (Theorem 2) as well as the similar assertion for $T = M$ (Theorem 1). However, in the case $T = M$ we do not obtain a full characterization but are left with a small but significant gap between the necessary and the sufficient condition.

It should be mentioned that Andersen obtained in [1] L^p–results also for the operator

$$H_e f(x) = \int\limits_0^\infty \frac{x f(y)}{x^2 - y^2}\, dy, \tag{1.7}$$

the Hilbert transform for even functions. However, our methods do not provide analogous results for H_e with t^p replaced by $\Phi(t)$.

Let us finally mention that the result of Andersen was generalized to the case of multiple Hilbert transform in [17]. For other related results see also [15], [16]. Some of the results of this paper were included in the comprehensive monograph [18].

2. Preliminaries. Let Φ be a convex nondecreasing function on $[0,\infty)$, $\Phi(0)=0$, which does not vanish identically on $[0,\infty)$, but it is allowed that $\Phi\equiv 0$ on $[0,a]$ and/or $\Phi\equiv\infty$ on (a,∞) for some $a>0$ provided that $\Phi(a-)$ is finite. The *complementary function to* Φ, $\tilde{\Phi}(t)=\sup_{s>0}(st-\Phi(s))$, has the same properties as Φ (for example, convexity of $\tilde{\Phi}$ follows easily from the subadditivity of supremum). Moreover, the Young inequality

$$st\le\Phi(s)+\tilde{\Phi}(t) \tag{2.1}$$

holds for all s,t positive. Both Φ and $\tilde{\Phi}$ are invertible on $(0,\infty)$ and it follows immediately from (2.1) that

$$\Phi^{-1}(t)\cdot\tilde{\Phi}^{-1}(t)\le 2t,\qquad t>0. \tag{2.2}$$

We say that Φ satisfies *the* Δ_2 *condition,* ($\Phi\in\Delta_2$), if $\Phi(2t)\le C\Phi(t)$. If this estimate holds merely near 0 (near ∞), we write $\Phi\in\Delta_2^0$ ($\Phi\in\Delta_2^\infty$). We recall that $\Phi\in\Delta_2$ is equivalent to $2\Phi^{-1}(t)\le\Phi^{-1}(Ct)$.

The functions

$$R_\Phi(t)=\Phi(t)/t,\quad S_\Phi(t)=\tilde{\Phi}(t)/t$$

will play a crucial role in the sequel. Clearly, R_Φ and S_Φ are nondecreasing on $[0,\infty)$. It is known [14], [9] that

$$\Phi(S_\Phi(t))\le C\tilde{\Phi}(t),\qquad t\ge 0, \tag{2.3}$$

and, by convexity,

$$\Phi(\lambda S_\Phi(t))\le C\lambda\tilde{\Phi}(t),\qquad t\ge 0,\lambda\in(0,1). \tag{2.4}$$

We say that Φ is *of bounded type near zero (near infinity)*, and write $\Phi\in B_0$ ($\Phi\in B_\infty$) if $R_\Phi(t)\ge a>0$ (or $R_\Phi(t)\le a<\infty$) for all $t>0$. This classification was introduced in [9]. It was proved in [9] that

$$\begin{aligned}
R_\Phi(t)\ge a,\qquad t>0\qquad&\Leftrightarrow\qquad\tilde{\Phi}(t)\equiv 0,\qquad t\in[0,a],\\
R_\Phi(t)\le a,\qquad t>0\qquad&\Leftrightarrow\qquad\tilde{\Phi}(t)\equiv\infty,\qquad t\in(a,\infty),\\
\Phi(t)\equiv 0,\qquad t\in[0,a]\qquad&\Leftrightarrow\qquad S_\Phi(t)\ge a,\qquad t>0,\\
\Phi(t)\equiv\infty,\qquad t\in(a,\infty)\qquad&\Leftrightarrow\qquad S_\Phi(t)\le a,\qquad t>0.
\end{aligned}$$

The functions R_Φ and S_Φ need not be injective. However, thanks to convexity of Φ, they can be constant on intervals only in a few special cases (this is the main difference between R_Φ and Φ'), namely, if R_Φ is equal to a constant on an interval (a,b), then it must be $a=0$ (b may be ∞). On the rest of its domain R_Φ is strictly increasing and thus invertible. Of course, the same holds for S_Φ.

It follows easily from (2.3) that

$$R_\Phi(t) \le S_\Phi^{-1}(Ct) \tag{2.5}$$

holds for admissible t (that is, for t such that Ct belongs to the range of the invertible part of S_Φ). We shall also make use of the (converse) estimate

$$S_\Phi^{-1}(t) \le 2R_\Phi(2t), \qquad \text{admissible } t. \tag{2.6}$$

To prove (2.6) substitute in (2.2) $t \to \tilde{\Phi}(t)$ to get $\tilde{\Phi}(t) \le \Phi(2S_\Phi(t))$. The complementary version of the last inequality reads as $\Phi(t) \le \tilde{\Phi}(2R_\Phi(t))$, which yields $t \le 2S_\Phi(2R_\Phi(t))$. Putting now $t \to 2t$ and assuming that $2t$ is admissible we get (2.6).

Let us introduce the notion of index of a nondecreasing function.

Putting $h(\lambda) = \sup_{t>0} \Phi(\lambda t)/\Phi(t)$, $\lambda \ge 0$, we define the *lower index of* Φ as $i(\Phi) = \lim\limits_{\lambda\to 0+} \log h(\lambda)/\log\lambda$ and the *upper index of* Φ as $I(\Phi) = \lim\limits_{\lambda\to\infty} \log h(\lambda)/\log\lambda$.

It follows easily from the definitions that for every $\varepsilon > 0$ there exists $C_\varepsilon \ge 1$ such that

$$\Phi(\lambda t) \le C_\varepsilon \max\left\{\lambda^{i(\Phi)-\varepsilon},\, \lambda^{I(\Phi)+\varepsilon}\right\} \Phi(t), \quad t \ge 0, \quad \lambda \ge 0, \tag{2.7}$$

and

$$\min\left\{\mu^{i(\Phi)-\varepsilon},\, \mu^{I(\Phi)+\varepsilon}\right\} \Phi(t) \le C_\varepsilon \Phi(\mu t), \quad t \ge 0, \quad \mu \ge 0. \tag{2.8}$$

Let us recall that $\Phi \in \Delta_2$ is equivalent to $I(\Phi) < \infty$, and $\tilde{\Phi} \in \Delta_2$ is equivalent to $i(\Phi) > 1$.

We define the *weighted modular* by $m_\varrho(f,\Phi) = \int_{-\infty}^{\infty} \Phi(|f(x)|)\varrho(x)\,dx$; then the *weighted Orlicz space* $L_{\Phi,\varrho}$ is the set of all functions f for which $m_\varrho(f/\lambda,\Phi)$ is finite for some $\lambda > 0$. This space can be equipped with *the Orlicz norm*

$$|||f|||_{\Phi,\varrho} = \sup\left\{ \int_{-\infty}^{\infty} fg\varrho,\ m_\varrho(g,\tilde{\Phi}) \le 1 \right\},$$

and also with *the Luxemburg norm*

$$\|f\|_{\Phi,\varrho} = \inf\{\lambda > 0, m_\varrho(f/\lambda,\Phi) \le 1\}.$$

The norms are equivalent, and the unit ball in $L_{\Phi,\varrho}$ with respect to the Luxemburg norm coincides with the set of all functions f such that $m_\varrho(f,\Phi) \le 1$. The Hölder inequality

$$\int_{-\infty}^{\infty} fg\varrho\,dx \le \|f\|_{\Phi,\varrho} \cdot |||g|||_{\tilde{\Phi},\varrho}$$

holds, and is saturated in the sense that

$$\|f\|_{\Phi,\varrho} = \sup\left\{ \int_{-\infty}^{\infty} fg\varrho\,dx, |||g|||_{\tilde{\Phi},\varrho} \le 1 \right\}$$

and

$$|||f|||_{\Phi,\varrho} = \sup\left\{ \int_{-\infty}^{\infty} fg\varrho\,dx, \|g\|_{\tilde{\Phi},\varrho} \le 1 \right\}.$$

The norm topology is stronger than the modular one, whence the modular inequality $\int \Phi(Tf)\varrho \le C\int \Phi(C|f|)\varrho$ implies its norm counterpart $\|Tf\|_{\Phi,\varrho} \le C\|f\|_{\Phi,\varrho}$, where T is any positive homogeneous operator.

As usual, given measurable functions h, g and a measurable set E, $|E|$ means $\int_E dx$, $h(E)$ means $\int_E h$, h_E means $|E|^{-1}h(E)$, and $h(\{g > \lambda\})$ means $\int_{\{x\in\mathbf{R}, g(x)>\lambda\}} h(t)dt$.

The letter I will always denote an interval in $\mathbf{R}$, and if $I = (a,b)$, we put $I' = (b, 2b-a)$, $I^* = (a, 2b-a)$, and let αI, $\alpha > 0$, be the interval concentric with I and α times as long.

If $\varrho(2I) \le C\varrho(I)$ for all I, we say that ϱ is a *doubling weight*.

We say that $\varrho \in A_\Phi$ if either $\Phi \notin B_0 \cup B_\infty$ and there exist C, ε such that

$$\sup_{\alpha>0} \sup_I \alpha\varrho_I R_\Phi\Big(\frac{\varepsilon}{|I|}\int_I S_\Phi\Big(\frac{1}{\alpha\varrho(x)}\Big)\,dx\Big) \le C,$$

or $\Phi \in B_0 \cup B_\infty$ and $\varrho \in A_1$.

We say that $\varrho \in E_\Phi$ if there exist $C, \varepsilon > 0$ such that

$$\sup_I \frac{1}{|I|}\int_I S_\Phi\Big(\varepsilon\frac{\varrho_I}{\varrho(x)}\Big)\,dx \le C.$$

3. Strong type inequalities for the maximal operator and the Hilbert transform. We start by considering the strong type inequality for the operator M. As known [6], the nonweighted inequality

$$\int \Phi(Mf) \le C\int \Phi(|f|)$$

holds if and only if $\tilde{\Phi} \in \Delta_2$. Kerman and Torchinsky [11] proved that under the assumption that both Φ and $\tilde{\Phi}$ satisfy the Δ_2 condition the weighted inequality

$$\int \Phi(Mf)\varrho \le C \int \Phi(|f|)\,\varrho$$

is equivalent to the condition

$$\sup_{\alpha,I} \left(\frac{\alpha}{|I|} \int_I \varrho(x)\,dx \right) \phi \left(\frac{1}{|I|} \int_I \phi^{-1} \left(\frac{1}{\alpha\varrho(x)} \right) dx \right) \le C,$$

where $\phi = \Phi'$.

As we shall see, the assumption $\Phi \in \Delta_2$ can be removed. On the other hand, the assumption $\tilde{\Phi} \in \Delta_2$, at least near infinity, is necessary.

Theorem 1. *Assume that ϱ and Φ are such that $\varrho \in A_\Phi$ and $\tilde{\Phi} \in \Delta_2$. Then there exists C such that for every f the inequality*

$$\int_{-\infty}^{\infty} \Phi(Mf(x))\varrho(x)\,dx \le C \int_{-\infty}^{\infty} \Phi(|f(x)|)\varrho(x)\,dx \tag{3.1}$$

holds.

Conversely, if (3.1) *holds with C independent of f, then $\varrho \in A_\Phi$ and $\tilde{\Phi} \in \Delta_2^\infty$.*

We shall need the following two observations:

Lemma 1. *If*

$$|||Mf|||_{\Phi,\varrho} \le C\,|||f|||_{\Phi,\varrho}, \tag{3.2}$$

then $\tilde{\Phi} \in \Delta_2^\infty$.

Lemma 2. *If $\tilde{\Phi} \in \Delta_2$ and $\varrho \in A_\Phi$, then there exists a function Φ_0 such that $\varrho \in A_{\Phi_0}$ and $i(\Phi_0) < i(\Phi)$.*

Proof of Theorem 1. *Necessity.* As mentioned above, the modular inequality (3.1) implies (3.2). Necessity of $\Phi \in \Delta_2^\infty$ thus follows from Lemma 1. As proved in [14], $\varrho \in A_\Phi$ is necessary even for the weak type inequality, the more so for (3.1).

Sufficiency. Let Φ_0 be the function from Lemma 2. Put $p = i(\Phi_0)$ and $F_p(t) = \Phi(t^{1/p})$. Then $i(F_p) = \frac{1}{p} i(\Phi) > 1$, whence the weighted maximal operator $M_\varrho f = \sup \varrho(I)^{-1} \int_I |f|\varrho$ is bounded on $L_{F_p,\varrho}$ [6]. Moreover, $\varrho \in A_{\Phi_0}$ implies $\varrho \in A_p$, and $(Mf)^p \le C M_\varrho(f^p)$ [11]. Thus,

$$\int \Phi(Mf)\varrho = \int F_p\left((Mf)^p\right)\varrho \le \int F_p\left(M_\varrho\left(C|f|^p\right)\right)\varrho \le$$
$$\le C\int F_p\left(C|f|^p\right)\varrho = C\int \Phi(C|f|)\varrho. \quad \blacksquare$$

Proof of Lemma 1. Fix a $K>0$ such that the set $E=\{\,K^{-1}\le\varrho(x)\le K\}$ has positive measure. Let x be a density point of E; with no loss of generality let $x=0$. Fix a_0 such that $|E\cap[0,a)|\ge\frac{3}{4}a$ for all $a\le a_0$. Then, for such a,

$$|E\cap(4^{-1}a,a)|\ge\frac{1}{2}a. \tag{3.3}$$

Indeed, we have

$$|E\cap(4^{-1}a,a)|=|E\cap(0,a)|-|E\cap(0,4^{-1}a)|\ge\frac{3}{4}a-\frac{1}{4}a.$$

From this we obtain the following observation to be used below: Since $\frac{1}{x}$ is a decreasing function, we have for every $a\in(0,a_0)$

$$\int\limits_{E\cap(4^{-1}a,a)}\frac{dx}{x}\ge\int\limits_{2^{-1}a}^{a}\frac{dx}{x}=\log 2. \tag{3.4}$$

Moreover, for every $a\in(0,a_0)$,

$$|E\cap(0,a)|\le a=4^m|(0,4^{-m}a)|\le\ 3^{-1}4^{m+1}|E\cap(0,4^{-m}a)|,$$

and so, by the definition of E,

$$\varrho(E\cap(0,a))\le K|E\cap(0,a)|\le K3^{-1}4^{m+1}|E\cap(0,4^{-m}a)|\le$$
$$\le K^2 3^{-1}4^{m+1}\varrho(E\cap(0,4^{-m}a)). \tag{3.5}$$

For $m\in\mathbf{N}$ and a fixed $b\in(0,a_0)$ put $f_m(x)=\chi_{E\cap(0,4^{-m}b)}(x)$. Then, by (3.5),

$$|||f_m|||_{\Phi,\varrho}=\varrho(E\cap(0,4^{-m}b))\cdot\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E\cap(0,4^{-m}b))}\right)\le$$
$$\le K|E\cap(0,4^{-m}b)|\cdot\tilde{\Phi}^{-1}\big(\frac{K^2 3^{-1}4^{m+1}}{\varrho(E\cap(0,b))}\big). \tag{3.6}$$

Moreover, for $x \in (4^{-m}b, b)$, $Mf_m(x) \geq x^{-1}|E \cap (0, 4^{-m}b)|$. Therefore, setting

$$g(x) = \tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) \cdot \chi_{E\cap(0,b)}(x),$$

we get $\int \tilde{\Phi}(g)\varrho = 1$, and thus

$$\begin{aligned}
&|||Mf_m|||_{\Phi,\varrho} \geq \int\limits_{-\infty}^{\infty} Mf_m(x)g(x)\varrho(x)\,dx \geq \\
&\geq |E \cap (0, 4^{-m}b)|\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) \int\limits_{E\cap(4^{-m}b,b)} \frac{\varrho(x)}{x}\,dx \geq \\
&\geq |E \cap (0, 4^{-m}b)|\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) K^{-1} \sum_{n=1}^{m} \int\limits_{E\cap(4^{-n}b,4^{-n+1}b)} \frac{dx}{x} \geq \\
&\geq \text{by (3.4)} \geq \\
&\geq |E \cap (0, 4^{-m}b)|\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) K^{-1} m \log 2. \qquad (3.7)
\end{aligned}$$

Combining (3.2), (3.6) and (3.7), we arrive at

$$\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) \cdot m \leq \frac{CK^2}{\log 2} \cdot \tilde{\Phi}^{-1}\left(\frac{K^2 3^{-1} 4^{m+1}}{\varrho(E \cap (0,b))}\right).$$

Choose $m = 2CK^2/\log 2$. Since m does not depend on b, the last inequality can be rewritten as

$$2\tilde{\Phi}^{-1}(t) \leq \tilde{\Phi}^{-1}(C_0 t), \quad t \geq t_0,$$

with $C_0 = 3^{-1}4^{m+1}K^2$ and $t_0 = (\varrho(E \cap (0, a_0)))^{-1}$. In other words, $\tilde{\Phi} \in \Delta_2^{\infty}$. ■

Proof of Lemma 2. Fix $\alpha > 0$ and I and define $v = S_\Phi(1/\alpha\varrho)$. We claim that $v \in A_\infty$, that is, there exist α and β independent of I such that the set

$$E_\beta = \{x \in I;\ v(x) > \beta v_I\}$$

satisfies $|E_\beta| > \alpha|I|$.

We have to distinguish several cases. First assume that $i(\Phi) = \infty$. Since $\varrho \in A_\Phi$ always implies $\varrho \in A_\infty$ [9], and $\varrho \in A_\infty$ always implies $\varrho \in A_p$ for certain $p < \infty$, in this case the assertion of the lemma is easily satisfied.

Suppose $i(\Phi) < \infty$. Since $\tilde{\Phi} \in \Delta_2$, Φ cannot be of any bounded type. However, this is not true for $\tilde{\Phi}$, so it can be either

(i) $S_\Phi(0,\infty) = (0,\infty)$;

(ii) $S_\Phi(0,\infty) = (0,a)$;

(iii) $S_\Phi(0,\infty) = (a,\infty)$; or
(iv) $S_\Phi(0,\infty) = [a,\infty)$;
with some positive a. Note that in case (ii) S_Φ is invertible on $(0,a)$, and in cases (iii) and (iv) S_Φ is invertible on (a,∞). Choose $\gamma \in (1, i(\Phi))$ arbitrarily. Then, by (2.7),

$$\Phi(\lambda t) \le C_\gamma \cdot \lambda^\gamma \cdot \Phi(t), \qquad t \ge 0, \quad \lambda \in (0,1). \tag{3.8}$$

Let ε be the constant from A_Φ. Choose $\beta \le \varepsilon/2$ in order that

$$C_\varrho \cdot 2^\gamma \cdot C_\gamma \cdot \left(\frac{\beta}{\varepsilon}\right)^{\gamma-1} \le \frac{1}{2}, \tag{3.9}$$

where C_ϱ is an A_Φ constant for the weight ϱ. Given fixed I, suppose that βv_I is admissible for S_Φ^{-1}. We may then conclude from A_Φ that

$$\frac{C_\varrho}{R_\Phi(\varepsilon v_I)} \ge \frac{1}{|I|}\int_I \frac{dx}{S_\Phi^{-1}(v(x))} \ge \frac{|I\backslash E_\beta|}{|I|}\frac{1}{S_\Phi^{-1}(\beta v_I)}. \tag{3.10}$$

Hence, by (3.10), (2.6), (3.8) and (3.9),

$$\begin{aligned} \frac{|I\backslash E_\beta|}{|I|} &\le C_\varrho \frac{S_\Phi^{-1}(\beta v_I)}{R_\Phi(\varepsilon v_I)} \le 2C_\varrho \frac{R_\Phi(2\beta v_I)}{R_\Phi(\varepsilon v_I)} = \frac{C_\varrho \varepsilon}{\beta}\frac{\Phi(\frac{2\beta}{\varepsilon}\varepsilon v_I)}{\Phi(\varepsilon v_I)} \le \\ &\le C_\varrho 2^\gamma C_\gamma \left(\frac{\beta}{\varepsilon}\right)^{\gamma-1} \le \frac{1}{2}, \end{aligned} \tag{3.11}$$

or $|E_\beta| > \frac{1}{2}|I|$.

Now suppose that βv_I is not admissible for S_Φ^{-1}. This is possible only in case (iii) or (iv) if $\beta v_I \le a$. But then, of course, $E_\beta = I$, and the desired estimate is trivial. Therefore, $v \in A_\infty$.

Now, as known [5], v satisfies the reverse Hölder inequality, that is, there are positive C and δ such that

$$\left(\frac{1}{|I|}\int_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)} \le \frac{C}{|I|}\int_I v(x)\,dx \tag{3.12}$$

for all I.

We define the function Φ_0 by means of its complementary function: put

$$S_{\Phi_0} = S_\Phi^{1+\delta}, \quad \text{that is,} \quad \tilde\Phi_0(t) = t\cdot[S_\Phi(t)]^{1+\delta}.$$

Then, obviously,

$$I(\tilde\Phi_0) = I(\tilde\Phi) + \delta(I(\tilde\Phi) - 1).$$

The case $I(\tilde{\Phi}) = 1$ (that is, $i(\Phi) = \infty$) was already excluded at the beginning. On the other hand, $I(\tilde{\Phi})$ cannot be ∞, since $\tilde{\Phi} \in \Delta_2$. Consequently, $I(\tilde{\Phi}_0) > I(\tilde{\Phi})$, which is of course equivalent to $i(\Phi_0) < i(\Phi)$.

It remains to prove $\varrho \in A_{\Phi_0}$. We start by rewriting (3.12) as

$$\frac{\varepsilon}{2C}\left(\frac{1}{|I|}\int\limits_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)} \le \frac{\varepsilon}{2|I|}\int\limits_I v(x)\,dx. \tag{3.13}$$

Suppose first that everything is admissible for S_Φ^{-1}. Then, as S_Φ^{-1} is nondecreasing,

$$S_\Phi^{-1}\left(\frac{\varepsilon}{2C}\left(\frac{1}{|I|}\int\limits_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)}\right) \le S_\Phi^{-1}\left(\frac{\varepsilon}{2|I|}\int\limits_I v(x)\,dx\right). \tag{3.14}$$

Note that $S_{\Phi_0}^{-1}(t) = S_\Phi^{-1}(t^{1/(1+\delta)})$ for admissible t. So, (3.14) gives

$$S_{\Phi_0}^{-1}\left(\left(\frac{\varepsilon}{2C}\right)^{1+\delta}\frac{1}{|I|}\int\limits_I v^{1+\delta}(x)\,dx\right) \le \le S_\Phi^{-1}\left(\frac{\varepsilon}{2|I|}\int\limits_I v(x)\,dx\right), \tag{3.15}$$

which by means of (2.5) and (2.6) yields

$$R_{\Phi_0}\left(\left(\frac{\varepsilon}{2C}\right)^{1+\delta}\frac{1}{|I|}\int\limits_I S_{\Phi_0}\left(\frac{1}{\alpha\varrho(x)}\right)\,dx\right) \le \le 2R_\Phi\left(\frac{\varepsilon}{|I|}\int\limits_I S_\Phi\left(\frac{1}{\alpha\varrho(x)}\right)\,dx\right). \tag{3.16}$$

Now assume that it was not possible to apply S_Φ^{-1} in (3.13). This can happen only in case (iii) or (iv) and

$$\frac{\varepsilon}{2C}\left(\frac{1}{|I|}\int\limits_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)} < a. \tag{3.17}$$

Note that in the cases (iii) or (iv) it is for all t $S_\Phi(t) \geq a$, that is, $S_{\Phi_0}(t) \geq a^{1+\delta}$, which is equivalent to $R_{\Phi_0}(t) = 0$ for $t \leq a^{1+\delta}$. Thus, in this case (3.16) holds trivially. It is clear that from A_Φ and (3.16) A_{Φ_0} already follows. The proof is finished. ■

The method of the proof of Lemma 2 is the same as that in [11], the only slight change being that we have replaced derivatives of Φ and $\tilde{\Phi}$ by R_Φ and S_Φ. Actually, our proof shows that the condition $\tilde{\Phi} \in \Delta_2$ is not required as an assumption, and allows us to insert it as part of the statement of the strong maximal theorem.

Let us turn our attention to the Hilbert transform.

It follows easily from the Kerman – Torchinsky theorem that if $\Phi \in \Delta_2$ and $\tilde{\Phi} \in \Delta_2$, then $\varrho \in A_\Phi$ is necessary and sufficient for

$$\int \Phi(|Hf|)\varrho \leq C \int \Phi(C|f|)\varrho. \tag{3.18}$$

Indeed, for sufficiency we use Coifman's inequality [4]

$$\int \Phi(|Hf|)\varrho \leq C \int \Phi(Mf)\varrho$$

which is valid provided that $\Phi \in \Delta_2$ and $\varrho \in A_\infty$. However, $\Phi \in \Delta_2$ is an assumption, and $\varrho \in A_\infty$ follows from $\varrho \in A_\Phi$ [9].

It may be of some interest that both $\Phi \in \Delta_2$ and $\tilde{\Phi} \in \Delta_2$ are also necessary for (3.18). We have the following characterization of the strong type inequality for the Hilbert transform.

Theorem 2. *The inequality*

$$\int_{-\infty}^{\infty} \Phi(|Hf(x)|)\varrho(x)\,dx \leq C \int_{-\infty}^{\infty} \Phi(|f(x)|)\varrho(x)\,dx \tag{3.19}$$

holds if and only if $\Phi \in \Delta_2$, $\tilde{\Phi} \in \Delta_2$, *and* $\varrho \in A_\Phi$.

We shall make use of the following assertion.

Lemma 3. *Let us define the operator*

$$G_\varrho f(x) = \frac{1}{\varrho(x)} \cdot H(f\varrho)(x).$$

Then the following statements are equivalent.

(i) *There is* C *such that*

$$\int_{-\infty}^{\infty} \Phi(|Hf(x)|)\varrho(x)\,dx \leq C \int_{-\infty}^{\infty} \Phi(C|f(x)|)\varrho(x)\,dx;$$

(ii) *There is C such that*

$$\int_{-\infty}^{\infty} \tilde{\Phi}(|G_\varrho f(x)|)\varrho(x)\,dx \le C \int_{-\infty}^{\infty} \tilde{\Phi}(C|f(x)|)\varrho(x)\,dx. \tag{3.20}$$

Proof of Theorem 2. The "if part" has already been established. It thus remains to show $\Phi \in \Delta_2$ and $\tilde{\Phi} \in \Delta_2$. It was proved in [9] that even the weak type inequality with Hilbert transform implies $\Phi \in \Delta_2$ and $\varrho \in A_\Phi$.

It remains to prove $\tilde{\Phi} \in \Delta_2$. By Lemma 3, (3.19) is equivalent to (3.20). Of course, (3.20) implies the weak type inequality

$$\varrho(\{|G_\varrho f| > \lambda\}) \cdot \Phi(\lambda) \le C \int_{-\infty}^{\infty} \Phi(Cf)\varrho. \tag{3.21}$$

Take K positive such that the set $E = \{K^{-1} \le \varrho(x) \le K\}$ has positive measure, and for any $\lambda > 0$ define $f = \frac{\lambda}{2C}\chi_{E_0}$, where E_0 is any bounded subset of E. Inserting f into (3.21) we get

$$\tilde{\Phi}(\lambda) \le C \frac{\varrho(E_0)}{\varrho(\{|G_\varrho f| > 2C\})} \cdot \tilde{\Phi}(\frac{\lambda}{2}),$$

in other words, $\tilde{\Phi} \in \Delta_2$. The idea is due to A. Gogatishvili [8]. ■

To prove Lemma 3 we employ the following result of D. Gallardo which was communicated to the author personally [7]. We give a sketch of the proof since as far as we know the author has not published it. When this manuscript was written, we learned that the same assertion was proved in a preprint by Bloom and Kerman ([3], Proposition 2.5).

Lemma 4. *Let T be a positively homogeneous operator. Then the modular estimate*

$$\int \Phi(|Tf(x)|)\varrho(x)\,dx \le C \int \Phi(C|f(x)|)\varrho(x)\,dx$$

is equivalent to the existence of C such that for all ε and f the norm inequality $\|Tf\|_{\Phi,\varepsilon\varrho} \le C\|f\|_{\Phi,\varepsilon\varrho}$ holds.

Proof of Lemma 4. Let the norm inequality be satisfied with C independent of ε and f. Then, by the definition of the Luxemburg norm, $\int \Phi((C\|f\|_{\Phi,\varepsilon\varrho})^{-1}|Tf(x)|)\varepsilon\varrho(x)\,dx \le 1$. Fix f, a function with finite modular, and put $\varepsilon = (\int \Phi(C|f|)\varrho)^{-1} > 0$. Then $\|Cf\|_{\Phi,\varepsilon\varrho} = 1$, and so

$$\int \Phi(|Tf|)\varepsilon\varrho = \int \Phi\left(\frac{C|Tf(x)|}{C\|Cf\|_{\Phi,\varepsilon\varrho}}\right)\varepsilon\varrho \le 1.$$

Inserting ε, we are done. The converse implication is obvious. ■

Proof of Lemma 3. By Lemma 4, (i) is equivalent to

$$\|Hf\|_{\Phi,\varepsilon\varrho} \le C\|f\|_{\Phi,\varepsilon\varrho}, \quad \text{all} \quad \varepsilon.$$

That is,

$$\begin{aligned} C &\ge \sup_{\|f\|_{\Phi,\varepsilon\varrho}\le 1} \|Hf\|_{\Phi,\varepsilon\varrho} = \\ &= \sup_{\|f\|_{\Phi,\varepsilon\varrho}\le 1} \sup_{\|g\|_{\tilde\Phi,\varepsilon\varrho}\le 1} \int_{-\infty}^{\infty} |Hf(x)| g(x)\varepsilon\varrho(x)\,dx = \\ &= \sup_{\|g\|_{\tilde\Phi,\varepsilon\varrho}\le 1} \sup_{\|f\|_{\Phi,\varepsilon\varrho}\le 1} \int_{-\infty}^{\infty} |f(x)| \cdot \frac{1}{\varrho(x)} |H(g\varrho)(x)|\varepsilon\varrho(x)\,dx = \\ &= \sup_{\|g\|_{\tilde\Phi,\varepsilon\varrho}\le 1} \|G_\varrho g\|_{\tilde\Phi,\varepsilon\varrho}, \end{aligned}$$

which is, again by Lemma 4, equivalent to (ii). ■

4. The Hilbert transform for odd functions. In this section we shall make use of the measure ν defined on $(0,\infty)$ by $d\nu(x) = x\,dx$.

We say that $\varrho \in A^o_\Phi$ if either $\Phi \notin B_0 \cup B_\infty$ and there exist positive C, ε such that

$$\sup_{\alpha, I} \left(\frac{\alpha}{\nu(I)} \int_I \frac{\varrho(x)}{x}\,d\nu \right) R_\Phi \left(\frac{\varepsilon}{\nu(I)} \int_I S_\Phi \left(\varepsilon \frac{x}{\alpha\varrho(x)} \right) d\nu \right) \le C$$

or $\Phi \in B_0 \cup B_\infty$ and $\varrho \in A^o_1$, that is,

$$\frac{\varrho(I)}{\nu(I)} \le C \operatorname{ess\,inf}_I \frac{\varrho(x)}{x}.$$

We say that $\varrho \in E^o_\Phi$ if there exist positive C, ε such that

$$\sup_I \frac{1}{\nu(I)} \int_I S_\Phi \left(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)} \right) d\nu \le C.$$

Remark. Obviously, $\varrho \in A^o_1$ implies $\varrho \in A^o_\Phi$ for any Φ. Further, putting $\alpha = \frac{\nu(I)}{\varrho(I)}$ we obtain that $\varrho \in A^o_\Phi$ implies $\varrho \in E^o_\Phi$ for any Φ.

We shall prove the following theorems.

Theorem 3. *The strong type inequality*

$$\int_0^\infty \Phi(|H_o f(x)|)\varrho(x)\,dx \le C\int_0^\infty \Phi(C|f(x)|)\varrho(x)\,dx \tag{4.1}$$

holds with C independent of f if and only if $\Phi \in \Delta_2$, $\tilde{\Phi} \in \Delta_2$, and $\varrho \in A^o_\Phi$.

Theorem 4. *The weak type inequality*

$$\varrho(\{|H_o f| > \lambda\}) \cdot \Phi(\lambda) \le C\int_0^\infty \Phi(C\,|f(x)|)\,\varrho(x)\,dx \tag{4.2}$$

holds with C independent of f and λ if and only if $\Phi \in \Delta_2$, and $\varrho \in A^o_\Phi$.

Theorem 5. *Let $\Phi \in \Delta^0_2$. Then the extra–weak type inequality*

$$\varrho(\{|H_o f| > \lambda\}) \le C\int_0^\infty \Phi(\frac{C\,|f(x)|}{\lambda})\,\varrho(x)\,dx \tag{4.3}$$

holds with C independent of f and λ if and only if $\varrho \in E^o_\Phi$.

The following auxiliary assertion is a modification of Lemma 1 from [1].

Lemma 5. *Define*

$$\sigma(x) = \frac{\varrho(\sqrt{|x|})}{2\sqrt{|x|}}, \qquad x \ne 0.$$

Then $\varrho \in A^o_\Phi$ if and only if $\sigma \in A_\Phi$, and $\varrho \in E^o_\Phi$ if and only if $\sigma \in E_\Phi$.

Proof of Lemma 5. Let $I = (a,b)$, $a > 0$, and put $J = (\sqrt{a}, \sqrt{b})$. Easily, $|I| = 2\nu(J)$ and $\sigma(I) = \varrho(J)$. Therefore,

$$\alpha\sigma_I R_\Phi\left(\frac{\gamma}{|I|}\int_I S_\Phi\left(\frac{1}{\alpha\sigma(x)}\right)dx\right) =$$
$$= \frac{\alpha\varrho(J)}{2\nu(J)} R_\Phi\left(\frac{\gamma}{\nu(J)}\int_J S_\Phi\left(\frac{2y}{\alpha\varrho(y)}\right)d\nu(y)\right),$$

and analogously

$$\int_I \tilde{\Phi}\left(\varepsilon\frac{\sigma_I}{\sigma(x)}\right)\frac{\sigma(x)}{\sigma(I)}dx = \int_J \tilde{\Phi}\left(\varepsilon\frac{y\varrho(J)}{\varrho(y)\nu(J)}\right)\frac{\varrho(y)}{\varrho(J)}dy.$$

Similar argument holds for $b < 0$, and, in the remaining case, we split the interval into two. ■

Proof of Theorem 4. *Necessity.* First we claim that if (4.3) holds, then ϱ is a doubling weight.

Let $I = (a,b)$, $0 < a < b < \infty$. Assume that $\operatorname{supp} f \subset I$. Since $\nu(I^*) = 2b(b-a) > (b^2 - a^2)$,

$$H_o f(x) \geq \frac{1}{\pi\nu(I^*)} \int\limits_I f(y)\, d\nu(y), \qquad x \in I'. \tag{4.4}$$

Now, easily $4\nu(I) > \nu(I^*)$, so inserting $f = \chi_I$ in (4.4) yields $H_o\chi_I(x) > (4\pi)^{-1}$ for all $x \in I'$. This together with (4.3) leads to $\varrho(I') \leq C\varrho(I)$. By symmetry,

$$C^{-1}\varrho(I) \leq \varrho(I') \leq C\varrho(I), \tag{4.5}$$

and the doubling condition follows.

Now we shall show, using again the idea from [8], that $\Phi \in \Delta_2$. Given λ, set $f = (2C)^{-1}\lambda\chi_I$, where C is from (4.2) and I is an appropriate interval. It then follows from (4.2) that

$$\Phi(\lambda) \leq C\frac{\varrho(I)}{\varrho(\{|H_o\chi_I| > 2C\})}\Phi(\lambda/2), \lambda > 0,$$

that is, $\Phi \in \Delta_2$.

It remains to show that $\varrho \in A_\Phi^o$. Given $\alpha > 0$, and $I = (a,b)$, put

$$f = C^{-1}S_\Phi\left(\frac{\gamma}{\alpha}\frac{x}{\varrho(x)}\right)\chi_I(x),$$

where C is from (4.2), and

$$\lambda = \frac{1}{2\pi\nu(I^*)}\int\limits_I f\, d\nu.$$

Then by (4.4), (2.1), (4.2) and (2.3)

$$\varrho(I')\cdot\Phi(\lambda) \leq C\int\limits_I \Phi S_\Phi\left(\frac{\gamma}{\alpha}\frac{x}{\varrho(x)}\right)\varrho(x)\, dx \leq C\gamma\alpha^{-1}\int\limits_I f\, d\nu,$$

or, by (4.5),

$$\frac{\alpha\varrho(I)}{\nu(I)}\Phi\left(\frac{1}{8\pi C\nu(I)}\int_I S_\Phi\left(\frac{\gamma}{\alpha}\frac{x}{\varrho(x)}\right)d\nu(x)\right)\leq$$
$$\leq\frac{C\gamma}{\nu(I)}\int_I S_\Phi\left(\frac{\gamma}{\alpha}\frac{x}{\varrho(x)}\right)d\nu(x)\ . \tag{4.6}$$

Denote $A=\int_I f\,d\nu$. Obviously, $A>0$. Assume that $A=\infty$. Then $\int_I \tilde{\Phi}(\frac{\gamma}{\alpha}\frac{x}{\varrho(x)})\varrho(x)dx=\infty$, and there must exist a function $g\in L_{\Phi,\varrho}(I)$ such that

$$\infty=\int_I\frac{\gamma}{\alpha}\frac{x}{\varrho(x)}g(x)\,\varrho(x)\,dx=\frac{\gamma}{\alpha}\int_I g\,d\nu.$$

This and (4.4) would give $H_o(\varepsilon g)(x)=\infty$ for all $x\in I'$ and $\varepsilon>0$ and, by (4.2),

$$\varrho(I')\Phi(\lambda)\leq C\int_I\Phi(\varepsilon Cg(x))\varrho(x)\,dx,\qquad \lambda,\varepsilon>0.$$

Since $g\in L_{\Phi,\varrho}$, there must be ε such that the last integral is finite, and so it follows that $\varrho(I')=0$. However, since ϱ is doubling and nontrivial, this is impossible. Hence $0<A<\infty$ and we can divide both sides of (4.6) by $\nu(I)^{-1}A$ to get $\varrho\in A^o_\Phi$. ■

Sufficiency. By Lemma 5, $\varrho\in A^o_\Phi$ implies $\sigma\in A_\Phi$. We thus have from Theorem A

$$\sigma(\{Hg>\lambda\})\cdot\Phi(\lambda)\leq C\int_{-\infty}^{\infty}\Phi(C|g(x)|)\sigma(x)\,dx.$$

For given f on $(0,\infty)$ put $g(x)=f(\sqrt{x})$ for $x>0$, and 0 elsewhere. Then $Hg(x)=(H_of)(\sqrt{x})$ [1], and therefore

$$\varrho(\{x>0,|H_of(x)|>\lambda\})\leq\sigma(\{x\in\mathbf{R},|Hg(x)|>\lambda\})\leq$$
$$\leq\frac{C}{\Phi(\lambda)}\int_0^\infty\Phi(Cf(y))\varrho(y)\,dy.\quad ■$$

Proof of Theorem 3. *Necessity.* By Theorem 4, $\varrho\in A_\Phi$ and $\Phi\in\Delta_2$ are necessary even for the weak type inequality. It remains to prove $\tilde{\Phi}\in\Delta_2$. In the same way as in Lemma 3 and Lemma 4 we can prove that (4.1) implies

$$\varrho(\{\frac{1}{\varrho(x)}|H_e(f\varrho)(x)| > \lambda\})\tilde{\Phi}(\lambda) \le C\int_0^\infty \tilde{\Phi}(C|f(x)|)\varrho(x)\,dx.$$

For the definition of H_e see (1.7). Putting $f = \frac{\lambda}{2C}\chi_{E_0}$ similarly as in the proof of Lemma 3 we get $\tilde{\Phi} \in \Delta_2$.

Sufficiency. By Lemma 5 and Lemma 2, $\varrho \in A_\Phi^o$ implies $\sigma \in A_{\Phi_0}$ with $i(\Phi_0) < i(\Phi)$. By Theorem A, $\sigma \in A_{\Phi_0}$ and $\Phi_0 \in \Delta_2$ imply the weak type inequality

$$\sigma(\{x \in \mathbf{R}; Hg(x) > \lambda\})\Phi_0(\lambda) \le C\int_{-\infty}^\infty \Phi_0(Cg(x))\sigma(x)\,dx$$

for every g. Given f on $(0,\infty)$, we put $g = f(\sqrt{x})\cdot\chi_{\{x>0\}}$, $x \in \mathbf{R}$. A change of variables then gives

$$\varrho(\{x > 0; H_o f > \lambda\})\Phi_0(\lambda) \le C\int_0^\infty \Phi_0(Cf(x))\varrho(x)\,dx,$$

which yields the assertion by the usual interpolation argument. ■

Proof of Theorem 5. *Necessity.* First assume that $\Phi \notin B_\infty$. Note that then S_Φ is finite on $(0,\infty)$. Fix $k \in \mathbf{N}$ and an interval I, put $I_k = \{x \in I, x \le k\varrho(x)\}$, and define

$$h(x) = h_k(x) = S_\Phi\left(\varepsilon\frac{x}{\varrho(x)}\frac{\varrho(I)}{\nu(I)}\right)\chi_{I_k}(x)$$

with ε to be specified later. Put

$$\beta_I = \frac{1}{\nu(I)}\int_I h\,d\nu.$$

Now, assume that K is the biggest of the constants C from (2.4), (4.3), and (4.5). We then have from (4.5) that $\varrho(I) \le K\varrho(\{|H_0 h| \ge (4\pi)^{-1}\beta_I\})$. Therefore, by (4.3) with $f = h$ and $\lambda < (4\pi)^{-1}\beta_I$,

$$\int_{I_k}\tilde{\Phi}\big(\varepsilon\frac{x}{\varrho(x)}\frac{\varrho(I)}{\nu(I)}\big)\varrho(x)\,dx = \varepsilon\frac{\varrho(I)}{\nu(I)}\int_{I_k} S_\Phi\big(\varepsilon\frac{x}{\varrho(x)}\frac{\varrho(I)}{\nu(I)}\big)\,d\nu =$$
$$= \varepsilon\varrho(I)\beta_I \le 4\pi K\varepsilon\varrho(I) + \delta_I,$$

where $\delta_I = 0$ if $\beta_I \le 4\pi K$, and

$$\delta_I = K^2 \varepsilon \beta_I \int\limits_{I_k} \Phi\big(\frac{4\pi K}{\beta_I} h(x)\big) \varrho(x)\, dx, \qquad \text{if } \beta_I > 4\pi K.$$

In any case, using (2.4) with $\lambda = 4\pi K/\beta_I$ we get

$$\int\limits_{I_k} \tilde{\Phi}\big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\big) \varrho(x)\, dx \le$$
$$\le 4\pi K \varepsilon \varrho(I) + 4\pi K^3 \varepsilon \int\limits_{I_k} \tilde{\Phi}\big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\big) \varrho(x)\, dx.$$

Now, since S_Φ is finite, we have

$$\int\limits_{I_k} \tilde{\Phi}\big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\big) \varrho(x)\, dx = \varepsilon \frac{\varrho(I)}{\nu(I)} \int\limits_{I_k} S_\Phi\big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\big)\, d\nu \le$$
$$\le \varepsilon \varrho(I) S_\Phi\big(\varepsilon k \frac{\varrho(I)}{\nu(I)}\big) < \infty,$$

and hence we can put $\varepsilon < (4\pi K^3)^{-1}$ and subtract to get

$$\int\limits_{I_k} \tilde{\Phi}\big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\big) \varrho(x)\, dx \le \frac{4\pi K \varepsilon}{1 - 4\pi K^3 \varepsilon} \varrho(I),$$

which yields $\varrho \in E_\Phi^o$ sonce the constant on the right does not depend on k.

If $\Phi \in B_\infty$, then $\Phi(t) \le Ct$ for all t and therefore, inserting $f = \chi_E$ and $\lambda = \nu(E)/(2\pi\nu(I^*))$ into (4.3) we obtain

$$\varrho(I) \le C\varrho(E)\Phi\big(C\frac{\nu(I^*)}{\nu(E)}\big) \le C\varrho(E)\frac{\nu(I^*)}{\nu(E)},$$

that is, $\varrho \in A_1^o$. Therefore, in this case $\varrho \in A_\Phi^o$ for any Φ (see Remark above).

Sufficiency. By Lemma 5, $\varrho \in E_\Phi^o$ implies $\sigma \in E_\Phi$, whence, using Theorem B, we have

$$\sigma(\{Hg > \lambda\}) \le C \int\limits_{-\infty}^{\infty} \Phi\big(\frac{C|g(x)|}{\lambda}\big) \sigma(x)\, dx.$$

The same argument as in the proof of Theorem 4 now leads to the assertion. ■

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REFERENCES

1. K.F. Andersen, Weighted norm inequalities for Hilbert transforms and conjugate functions of even and odd functions. *Proc. Am. Math. Soc.* **56**, (1976), No.4, 99–107.

2. R.J. Bagby, Weak bounds for the maximal function in weighted Orlicz spaces. *Stud. Math.* **95**(1990), 195–204.

3. S. Bloom and R. Kerman, Weighted L_Φ integral inequalities for operators of Hardy type. *Preprint.*

4. R.R. Coifman, Distribution function inequalities for singular integrals. *Proc. Natl. Acad. Sci. USA* **69**(1972), 2838–2839.

5. R.R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals. *Stud. Math.* **51**(1974), 241–250.

6. D. Gallardo, Orlicz spaces for which the Hardy–Littlewood maximal operator is bounded. *Publ. Math.* **32**(1988), 261–266.

7. D. Gallardo, *Personal communication.*

8. A. Gogatishvili, Riesz transforms and maximal functions in $\Phi(L)$ classes. *Bull. Acad. Sci. Georg. SSR* **137**(1990), No.3, 489–492

9. A. Gogatishvili and L. Pick, Weighted inequalities of weak and extra–weak type for the maximal operator and the Hilbert transform. *To appear in Czech. Math. J.*

10. R.A. Hunt, B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176**(1973), 227–251.

11. R.A. Kerman and A. Torchinsky, Integral inequalities with weights for the Hardy maximal function. *Stud. Math.* **71**(1982), 277-284.

12. W.A.J. Luxemburg, Banach function spaces. *Thesis, Delft*, 1955.

13. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165**(1972), 207–227.

14. L. Pick, Two-weight weak type maximal inequalities in Orlicz classes. *Stud. Math.* **100**(1991), No.1, 207-218.

15. V.M. Kokilashvili, Singular integral operators in weighted spaces. *Colloq. Math. Soc. János Bolyai.* **35.** *Functions, Series, Operators* (*Budapest, Hungary,* 1980), 707-714, *North-Holland, Amsterdam-New York,* 1982.

16. —–, Singular operators in weighted Orlicz spaces. (Russian) *Trudy Tbilis. Mat. Inst. Razmadze* **89**(1988), 42-50.

17. —–, Weighted inequalities for some integral transforms. (Russian) *Trudy Tbilis. Mat. Inst. Razmadze* **76**(1985), 101-106.

18. V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces. *World Scientific, Singapore*, 1991.

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