

ON SOME ENTIRE MODULAR FORMS OF WEIGHTS 5  
AND 6 FOR THE CONGRUENCE GROUP  $\Gamma_0(4N)$

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ABSTRACT. Two entire modular forms of weight 5 and two of weight 6 for the congruence subgroup  $\Gamma_0(4N)$  are constructed, which will be useful for revealing the arithmetical sense of additional terms in formulas for the number of representations of positive integers by quadratic forms in 10 and 12 variables.

In this paper  $N, a, k, n, r, s, t$  denote positive integers;  $u$  are odd positive integers;  $H, c, g, h, j, m, \alpha, \beta, \gamma, \delta, \xi, \eta$  are integers;  $A, B, C, D$  are complex numbers, and  $z, \tau$  ( $\text{Im } \tau > 0$ ) are complex variables. Further,  $(\frac{h}{u})$  is the generalized Jacobi symbol,  $\binom{n}{t}$  a binomial coefficient,  $\varphi(k)$  Euler's function, and  $e(z) = \exp 2\pi iz$ ,  $\eta(\gamma) = 1$  if  $\gamma \geq 0$  and  $\eta(\gamma) = -1$  if  $\gamma < 0$ .

Let

$$\Gamma = \left\{ \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \mid \alpha\delta - \beta\gamma = 1 \right\},$$
$$\Gamma_0(4N) = \left\{ \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \in \Gamma \mid \gamma \equiv 0 \pmod{4N} \right\}.$$

**Definition.** We shall say that a function  $F$  defined on  $\mathcal{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$  is an entire modular form of weight  $r$  and character  $\chi(\delta)$  for the congruence subgroup  $\Gamma_0(4N)$  if

- 1)  $F$  is regular on  $\mathcal{H}$ ,
- 2) for all substitutions from  $\Gamma_0(4N)$  and all  $\tau \in \mathcal{H}$

$$F\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \chi(\delta)(\gamma\tau + \delta)^r F(\tau),$$

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3) in the neighborhood of the point  $\tau = i\infty$

$$F(\tau) = \sum_{m=0}^{\infty} A_m e(m\tau),$$

4) for all substitutions from  $\Gamma$ , in the neighborhood of each rational point  $\tau = -\frac{\delta}{\gamma}$  ( $\gamma \neq 0$ ,  $(\gamma, \delta) = 1$ )

$$(\gamma\tau + \delta)^r F(\tau) = \sum_{m=0}^{\infty} A'_m e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right).$$

For  $\eta \neq 0$ ,  $\xi$ ,  $g$ ,  $h$ ,  $N$  with  $\xi g + \eta h + \xi \eta N \equiv 0 \pmod{2}$ , put

$$S_{gh} \left( \begin{matrix} \xi \\ \eta \end{matrix} ; c, N \right) = \sum_{\substack{m \bmod N | \eta| \\ m \equiv c \pmod{N}}} (-1)^{h(m-c)/N} e\left(\frac{\xi}{2N\eta} \left(m + \frac{g}{2}\right)^2\right).$$

It is known ([2] p. 323, formula (2.4)–(2.6)) that

$$\begin{aligned} S_{g+2j, h} \left( \begin{matrix} \xi \\ \eta \end{matrix} ; c, N \right) &= S_{gh} \left( \begin{matrix} \xi \\ \eta \end{matrix} ; c + j, N \right), \\ S_{g, h+2j} \left( \begin{matrix} \xi \\ \eta \end{matrix} ; c, N \right) &= S_{gh} \left( \begin{matrix} \xi \\ \eta \end{matrix} ; c, N \right), \\ S_{gh} \left( \begin{matrix} \xi \\ \eta \end{matrix} ; c + N_j, N \right) &= (-1)^{hj} S_{gh} \left( \begin{matrix} \xi \\ \eta \end{matrix} ; c, N \right). \end{aligned} \quad (1)$$

Let

$$\begin{aligned} &\vartheta_{gh}(z|\tau; c, N) = \\ &= \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} e\left(\frac{1}{2N} \left(m + \frac{g}{2}\right)^2 \tau\right) e\left(\left(m + \frac{g}{2}\right)z\right), \end{aligned} \quad (2)$$

hence

$$\begin{aligned} \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) &= (\pi i)^n \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} (2m + g)^n \times \\ &\times e\left(\frac{1}{2N} \left(m + \frac{g}{2}\right)^2 \tau\right) e\left(\left(m + \frac{g}{2}\right)z\right). \end{aligned} \quad (3)$$

Put

$$\begin{aligned} \vartheta_{gh}^{(n)}(\tau; c, N) &= \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) \Big|_{z=0}, \\ \vartheta_{gh}^{(0)}(\tau; c, N) &= \vartheta_{gh}(\tau; c, N) = \vartheta_{gh}(0|\tau; c, N). \end{aligned} \quad (4)$$

It is known ([2], p.318, formula (1.3); p. 321, formula (1.12); p.327, formulas (3.9), (3.5), (3.3), (3.7); p.324, formula (2.16); p.327, formulas (3.10), (3.11)) that

$$\vartheta_{g,h+2j}(z|\tau; c, N) = \vartheta_{gh}(z|\tau; c, N); \quad (5)$$

$$\begin{aligned} \vartheta_{gh}\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}; c, N\right) &= \left(-\frac{i\tau}{N}\right)^{1/2} e\left(\frac{Nz^2}{2\tau}\right) \times \\ &\times \sum_{H \bmod N} e\left(-\frac{1}{N}\left(c + \frac{g}{2}\right)\left(H + \frac{h}{2}\right)\right) \vartheta_{hg}(z|\tau; H, N); \end{aligned} \quad (6)$$

$$\begin{aligned} \vartheta_{gh}\left(\frac{z}{\gamma\tau + \delta} \middle| \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; c, N\right) &= \left(\frac{-i(\gamma\tau + \delta) \operatorname{sgn} \gamma}{N|\gamma|}\right)^{\frac{1}{2}} e\left(\frac{N\gamma z^2}{2(\gamma\tau + \delta)}\right) \times \\ &\times \sum_{H \bmod N} \varphi_{g'gh}(c, H; N) \vartheta_{g'h'}(z|\tau; H, N) \quad (\gamma \neq 0), \end{aligned} \quad (7)$$

where

$$g' = \alpha g + \gamma h + \alpha\gamma N, \quad h' = \beta g + \delta h + \beta\delta N, \quad (8)$$

$$\begin{aligned} \varphi_{g'gh}(c, H; N) &= e\left(-\frac{\beta\delta}{2N}\left(H + \frac{g'}{2}\right)^2\right) e\left(\frac{\beta}{N}\left(c + \frac{g}{2}\right)\left(H + \frac{g'}{2}\right)\right) \times \\ &\times S_{g-\delta g', h+\beta g'}\left(\frac{\alpha}{\gamma}; c - \delta H, N\right); \end{aligned} \quad (9)$$

$$\begin{aligned} \vartheta_{gh}(z|\tau + \beta; c, N) &= e\left(\frac{\beta}{2N}\left(c + \frac{g}{2}\right)^2\right) \vartheta_{g,h+\beta g+\beta N}(z|\tau; c, N), \\ \vartheta_{gh}(-z|\tau - \beta; c, N) &= \\ &= e\left(-\frac{\beta}{2N}\left(c + \frac{g}{2}\right)^2\right) \vartheta_{-g,-h+\beta g-\beta N}(z|\tau; -c, N). \end{aligned} \quad (10)$$

From (5) and (10), according to the notations (4), it follows that

$$\begin{aligned} \vartheta_{g,h+2j}(\tau; c, N) &= \vartheta_{gh}(\tau; c, N), \\ \vartheta_{g,h+2j}^{(n)}(\tau; c, N) &= \vartheta_{gh}^{(n)}(\tau; c, N); \\ \vartheta_{gh}^{(n)}(\tau + \beta; c, N) &= e\left(\frac{\beta}{2N}\left(c + \frac{g}{2}\right)^2\right) \vartheta_{g,h+\beta g+\beta N}^{(n)}(\tau; c, N), \\ \vartheta_{gh}^{(n)}(\tau - \beta; c, N) &= \\ &= (-1)^n e\left(-\frac{\beta}{2N}\left(c + \frac{g}{2}\right)^2\right) \vartheta_{-g,-h+\beta g-\beta N}^{(n)}(\tau; -c, N). \end{aligned} \quad (11)$$

From (2) and (3), according to the notations (4), in particular, it follows

that

$$\begin{aligned}
\vartheta_{gh}(\tau; 0, N) &= \sum_{m=-\infty}^{\infty} (-1)^{hm} e\left(\frac{1}{2N}\left(Nm + \frac{g}{2}\right)^2 \tau\right), \\
\vartheta_{gh}^{(n)}(\tau; 0, N) &= \\
&= (\pi i)^n \sum_{m=-\infty}^{\infty} (-1)^{hm} (2Nm + g)^n e\left(\frac{1}{2N}\left(Nm + \frac{g}{2}\right)^2 \tau\right).
\end{aligned} \tag{13}$$

1.

**Lemma 1.** For  $n \geq 0$

$$\begin{aligned}
\vartheta_{gh}^{(n)}\left(-\frac{1}{\tau}; c, N\right) &= (Ni)^n \left(-\frac{i\tau}{N}\right)^{(2n+1)/2} \times \\
&\times \sum_{H \bmod N} e\left(-\frac{1}{N}\left(c + \frac{g}{2}\right)\left(H + \frac{h}{2}\right)\right) \times \\
&\times \left\{ \vartheta_{hg}^{(n)}(\tau; H, N) + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{hg}^{(n-t)}(\tau; H, N) \right\},
\end{aligned}$$

where

$$\begin{aligned}
A_{tk} \Big|_{z=0} &= \begin{cases} (2k)! \left(\frac{N\pi i}{\tau}\right)^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases} \\
&(t = 1, 2, \dots, n; k = 1, 2, \dots, t).
\end{aligned} \tag{1.1}$$

*Proof.* From (6), by Leibniz's formula, we obtain

$$\begin{aligned}
\frac{\partial^n}{\partial z^n} \vartheta_{gh}\left(\frac{z}{\tau} \Big| -\frac{1}{\tau}; c, N\right) &= \tau^n \left(\frac{-i\tau}{N}\right)^{1/2} \frac{\partial^n}{\partial z^n} \left\{ e\left(\frac{Nz^2}{2\tau}\right) \times \right. \\
&\times \sum_{H \bmod N} e\left(-\frac{1}{N}\left(c + \frac{g}{2}\right)\left(H + \frac{h}{2}\right)\right) \vartheta_{hg}(z|\tau; H, N) \left. \right\} = \\
&= (Ni)^n \left(\frac{-i\tau}{N}\right)^{(2n+1)/2} \times \\
&\times \sum_{H \bmod N} e\left(-\frac{1}{N}\left(c + \frac{g}{2}\right)\left(H + \frac{h}{2}\right)\right) \left\{ e\left(\frac{Nz^2}{2\tau}\right) \frac{\partial^n}{\partial z^n} \vartheta_{hg}(z|\tau; H, N) \right\} + \\
&+ \sum_{t=1}^n \binom{n}{t} \frac{\partial^n}{\partial z^n} e\left(\frac{Nz^2}{2\tau}\right) \frac{\partial^{n-t}}{\partial z^{n-t}} \vartheta_{hg}(z|\tau; H, N) \left. \right\}.
\end{aligned} \tag{1.2}$$

According to formulae (a) and (b) of [1], p. 37<sup>1</sup>

$$\frac{\partial^t}{\partial z^t} e\left(\frac{Nz^2}{2\tau}\right) = A_{t1} e\left(\frac{Nz^2}{2\tau}\right) + \frac{A_{t2}}{2!} e\left(\frac{Nz^2}{2\tau}\right) + \dots + \frac{A_{tt}}{t!} e\left(\frac{Nz^2}{2\tau}\right)$$

$$(t = 1, 2, \dots, n),$$

where

$$A_{tk} = \frac{\partial^t}{\partial z^t} \left(\frac{N\pi iz^2}{\tau}\right)^k - k \left(\frac{N\pi iz^2}{\tau}\right) \frac{\partial^t}{\partial z^t} \left(\frac{N\pi iz^2}{\tau}\right)^{k-1} +$$

$$+ \frac{k(k-1)}{2!} \left(\frac{N\pi iz^2}{\tau}\right)^2 \frac{\partial^t}{\partial z^t} \left(\frac{N\pi iz^2}{\tau}\right)^{k-2} +$$

$$+ \dots + (-1)^{k-1} k \left(\frac{N\pi iz^2}{\tau}\right)^{k-1} \frac{\partial^t}{\partial z^t} \left(\frac{N\pi iz^2}{\tau}\right) \quad (k = 1, 2, \dots, t),$$

hence

$$\frac{\partial^t}{\partial z^t} e\left(\frac{Nz^2}{2\tau}\right) \Big|_{z=0} = \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \quad (t = 1, 2, \dots, n) \quad (1.3)$$

and

$$A_{tk} \Big|_{z=0} = \frac{\partial^t}{\partial z^t} \left(\frac{N\pi iz^2}{\tau}\right)^k \Big|_{z=0} \quad (k = 1, 2, \dots, t). \quad (1.4)$$

Thus, in view of notations (4), the lemma follows from (1.2)–(1.4).  $\square$

**Lemma 2.** *If  $\gamma \neq 0$ , then for  $n \geq 0$*

$$\vartheta_{gh}^{(n)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; c, N\right) = (N|\gamma| i \operatorname{sgn} \gamma)^n \left(-i(\gamma\tau + \delta) \frac{\operatorname{sgn} \gamma}{N|\gamma|}\right)^{(2n+1)/2} \times$$

$$\times \sum_{H \bmod N} \varphi_{g'h'}(c, H; N) \left\{ \vartheta_{g'h'}^{(n)}(\tau; H, N) + \right.$$

$$\left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{g'h'}^{(n-t)}(\tau; H, N) \right\},$$

where  $g'h'$  and  $\varphi_{g'h'}(c, H; N)$  are defined by the formulas (8) and (9),

$$A_{tk} \Big|_{z=0} = \begin{cases} (2k)! \left(\frac{N\gamma\pi i}{\gamma\tau + \delta}\right)^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases}$$

$$(t = 1, 2, \dots, n; k = 1, 2, \dots, t).$$

<sup>1</sup>Page 75 in the Russian version of [1] published in 1933.

*Proof.* From (7), according to Leibnitz's formula, we obtain

$$\begin{aligned}
\frac{\partial^n}{\partial z^n} \vartheta_{gh} \left( \frac{z}{\gamma\tau + \delta} \middle| \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; c, N \right) &= (\gamma\tau + \delta)^n \left( -i(\gamma\tau + \delta) \frac{\operatorname{sgn} \gamma}{N|\gamma|} \right)^{1/2} \times \\
&\times \frac{\partial^n}{\partial z^n} \left\{ e \left( \frac{N\gamma z^2}{2(\gamma\tau + \delta)} \right) \sum_{H \bmod N} \varphi_{g'gh}(c, H; N) \vartheta_{g'h'}(z|\tau; H, N) \right\} = \\
&= (N|\gamma| i \operatorname{sgn} \gamma)^n \left( -i(\gamma\tau + \delta) \frac{\operatorname{sgn} \gamma}{N|\gamma|} \right)^{(2n+1)/2} \times \\
&\times \sum_{H \bmod N} \varphi_{g'gh}(c, H; N) \left\{ e \left( \frac{N\gamma z^2}{2(\gamma\tau + \delta)} \right) \frac{\partial^n}{\partial z^n} \vartheta_{g'h'}(z|\tau; H, N) + \right. \\
&\left. + \sum_{t=1}^n \binom{n}{t} \frac{\partial^t}{\partial z^t} e \left( \frac{N\gamma z^2}{2(\gamma\tau + \delta)} \right) \frac{\partial^{n-t}}{\partial z^{n-t}} \vartheta_{g'h'}(z|\tau; H, N) \right\}. \quad (1.5)
\end{aligned}$$

As in Lemma 1, but with  $e\left(\frac{N\gamma z^2}{2(\gamma\tau + \delta)}\right)$  instead of  $e\left(\frac{Nz^2}{2\tau}\right)$ , we have

$$\frac{\partial^t}{\partial z^t} e \left( \frac{N\gamma z^2}{2(\gamma\tau + \delta)} \right) \Big|_{z=0} = \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \quad (t = 1, 2, \dots, n) \quad (1.6)$$

and

$$A_{tk} \Big|_{z=0} = \frac{\partial^t}{\partial z^t} \left( \frac{N\gamma\pi iz}{\gamma\tau + \delta} \right)^{2k} \Big|_{z=0} \quad (k = 1, 2, \dots, t). \quad (1.7)$$

Thus, according to the notations (4), the lemma follows from (1.5)-(1.7).  $\square$

**Lemma 3.** *If  $g$  is even, then for  $n \geq 0$  and all substitutions from  $\Gamma_0(4N)$  we have*

$$\begin{aligned}
&\vartheta_{gh}^{(n)} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N \right) = \\
&= (\operatorname{sgn} \delta)^n i^{(2n+1)\eta(\gamma)(\operatorname{sgn} \delta - 1)/2} i^{(1-|\delta|)/2} \left( \frac{2\beta N \operatorname{sgn} \delta}{|\delta|} \right) \times \\
&\times (\gamma\tau + \delta)^{(2n+1)/2} e \left( -\frac{\alpha\gamma\delta^2 h^2}{16N} \right) e \left( \frac{\beta\delta g^2}{4} \frac{\delta^{2\varphi(2N)-2}}{4N} \right) \vartheta_{\alpha g, h}^{(n)}(\tau; 0, 2N).
\end{aligned}$$

*Proof.* 1) Let  $\gamma \neq 0$ . In [4] (p.18, formula (5.1)) it is shown that

$$\begin{aligned}
&S_{g0} \left( \frac{\beta}{\delta}; 0, 2N \right) = \\
&= e \left( \frac{\beta\delta g^2}{4} \frac{\delta^{2\varphi(2N)-2}}{4N} \right) i^{(1-|\delta|)/2} \left( \frac{2\beta N \operatorname{sgn} \delta}{|\delta|} \right) |\delta|^{1/2}. \quad (1.8)
\end{aligned}$$

Replacing  $\alpha, \beta, \gamma, \delta, \tau, c, N$  by  $\beta, -\alpha, \delta, -\gamma, \tau', 0, 2N$  in Lemma 2, we obtain

$$\begin{aligned} \vartheta_{gh}^{(n)}\left(\frac{\beta\tau' - \alpha}{\delta\tau' - \gamma}; 0, 2N\right) &= (2N|\delta| i \operatorname{sgn} \delta)^n \left(-i(\delta\tau' - \gamma) \frac{\operatorname{sgn} \delta}{2N|\delta|}\right)^{(2n+1)/2} \times \\ &\times \sum_{H \bmod 2N} \varphi_{h'gh}(0, H; 2N) \left\{ \vartheta_{h', \alpha g}^{(n)}(\tau'; H, 2N) + \right. \\ &\left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{h', \alpha g}^{(n-t)}(\tau'; H, 2N) \right\}, \end{aligned} \quad (1.9)$$

where by (8),(9),(11) and (1)

$$h' = \beta g + \delta h + 2\beta\delta N, \quad (1.10)$$

$$\begin{aligned} &\varphi_{h'gh}(0, H; 2N) = \\ &= e\left(-\frac{\alpha\gamma h'^2}{16N}\right) e\left(-\frac{\alpha g}{4N}\left(H + \frac{h'}{2}\right)\right) S_{g0}\left(\frac{\beta}{\delta}; 0, 2N\right), \end{aligned} \quad (1.11)$$

$$A_{tk} \Big|_{z=0} = \begin{cases} (2k)! \left(\frac{2N\delta\pi i}{\delta\tau' - \gamma}\right)^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases} \quad (k = 1, 2, \dots, t). \quad (1.12)$$

Writing  $-\frac{1}{\tau}$  instead of  $\tau'$  in (1.9) and (1.12), according to (1.11), we obtain

$$\begin{aligned} \vartheta_{gh}^{(n)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N\right) &= (2Ni)^n (|\delta| \operatorname{sgn} \delta)^n \left(\frac{-i(-\frac{\delta}{\tau} - \gamma) \operatorname{sgn} \delta}{2N|\delta|}\right)^{(2n+1)/2} \times \\ &\times e\left(-\frac{\alpha\gamma h'^2}{16N}\right) e\left(\frac{\beta\delta g^2}{4} \frac{\delta^{2\varphi(2N)-2}}{4N}\right) i^{(1-|\delta|)/2} \left(\frac{2\beta N \operatorname{sgn} \delta}{|\delta|}\right) |\delta|^{1/2} \times \\ &\times \sum_{H \bmod 2N} e\left(-\frac{\alpha g}{4N}\left(H + \frac{h'}{2}\right)\right) \left\{ \vartheta_{h', \alpha g}^{(n)}\left(-\frac{1}{\tau}; H, 2N\right) + \right. \\ &\left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{h', \alpha g}^{(n-t)}\left(-\frac{1}{\tau}; H, 2N\right) \right\}, \end{aligned} \quad (1.13)$$

where

$$A_{tk} \Big|_{z=0} = \begin{cases} (2k)! \left(\frac{2N\delta\pi i\tau}{\gamma\tau + \delta}\right)^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases} \quad (k = 1, 2, \dots, t).$$

Writing  $\alpha g$ ,  $h'$ ,  $-\frac{1}{\tau}$ ,  $0$ ,  $2N$  instead of  $g$ ,  $h$ ,  $\tau$ ,  $c$ ,  $N$  in Lemma 1, we obtain

$$\begin{aligned} \vartheta_{\alpha g, h'}^{(n)}(\tau; 0, 2N) &= (2Ni)^n \left(\frac{i}{2N\tau}\right)^{(2n+1)/2} \sum_{H \bmod 2N} e\left(-\frac{\alpha g}{4N}\left(H + \frac{h'}{2}\right)\right) \times \\ &\quad \times \left\{ \vartheta_{h', \alpha g}^{(n)}\left(-\frac{1}{\tau}; H, 2N\right) + \right. \\ &\quad \left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{h', \alpha g}^{(n-t)}\left(-\frac{1}{\tau}; H, 2N\right) \right\}, \end{aligned} \quad (1.14)$$

where

$$A_{tk} \Big|_{z=0} = \begin{cases} (2k)!(-2N\pi i\tau)^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases} \quad (k = 1, 2, \dots, t).$$

From (1.14) it follows that

$$\begin{aligned} &\sum_{H \bmod 2N} e\left(-\frac{\alpha g}{4N}\left(H + \frac{h'}{2}\right)\right) \left\{ \vartheta_{h', \alpha g}^{(n)}\left(-\frac{1}{\tau}; H, 2N\right) + \right. \\ &\quad \left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{h', \alpha g}^{(n-t)}\left(-\frac{1}{\tau}; H, 2N\right) \right\} = \\ &= (2\pi i)^{-n} (-2Ni\tau)^{(2n+1)/2} \vartheta_{\alpha g, h'}^{(n)}(\tau; 0, 2N). \end{aligned} \quad (1.15)$$

From (1.13) and (1.15) we obtain

$$\begin{aligned} \vartheta_{gh}^{(n)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N\right) &= (-2Ni\tau)^{(2n+1)/2} (|\delta| \operatorname{sgn} \delta)^n \times \\ &\quad \times \left(\frac{-i(-\frac{\delta}{\tau} - \gamma) \operatorname{sgn} \delta}{2N|\delta|}\right)^{(2n+1)/2} i^{(1-|\delta|)/2} e\left(-\frac{\alpha\gamma h'^2}{16N}\right) \times \\ &\quad \times e\left(\frac{\beta\delta g^2}{4} \frac{\delta^{2\varphi(2N)-2}}{4N}\right) \left(\frac{2\beta N \operatorname{sgn} \delta}{|\delta|}\right) |\delta|^{1/2} \vartheta_{\alpha g, h'}^{(n)}(\tau; 0, 2N). \end{aligned} \quad (1.16)$$

In [3] (p.85, formula (6.16) with  $2N$  instead of  $N$ ) it is shown that

$$\begin{aligned} &\left(\frac{-i(-\frac{\delta}{\tau} - \gamma) \operatorname{sgn} \delta}{2N|\delta|}\right)^{1/2} (-2Ni\tau)^{1/2} = \\ &= i^{\operatorname{sgn} \gamma (\operatorname{sgn} \delta - 1)/2} \left(\frac{\gamma\tau + \delta}{|\delta|}\right)^{1/2}. \end{aligned} \quad (1.17)$$



From (1.16) and (1.17) it follows that

$$\begin{aligned} & \vartheta_{gh}^{(n)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N\right) = \\ & = (|\delta| \operatorname{sgn} \delta)^n \left(i \operatorname{sgn} \gamma (\operatorname{sgn} \delta - 1) \frac{\gamma\tau + \delta}{|\delta|}\right)^{(2n+1)/2} i^{(1-|\delta|)/2} \times \\ & \times e\left(-\frac{\alpha\gamma h'^2}{16N}\right) e\left(\frac{\beta\delta g^2}{4} \frac{\delta^2 \varphi(2N)-2}{4N}\right) \left(\frac{2\beta N \operatorname{sgn} \delta}{|\delta|}\right) |\delta|^{1/2} \vartheta_{\alpha g, h'}^{(n)}(\tau; 0, 2N). \end{aligned}$$

Thus, in view of (1.10) and (1.11), the lemma is proved for  $\gamma \neq 0$ .

2) Let  $\gamma = 0$ . Then  $\alpha = \delta = 1$  or  $\alpha = \delta = -1$  in (12). Putting  $c = 0$  in (12) and writing  $2N$  instead of  $N$ , by (11), we obtain

$$\begin{aligned} \vartheta_{gh}^{(n)}(\tau + \beta; 0, 2N) &= e\left(\frac{\beta g^2}{16N}\right) \vartheta_{gh}^{(n)}(\tau; 0, 2N), \\ \vartheta_{gh}^{(n)}(\tau - \beta; 0, 2N) &= (-1)^n e\left(-\frac{\beta g^2}{16N}\right) \vartheta_{-g, h}^{(n)}(\tau; 0, 2N), \end{aligned}$$

i.e., the lemma is also proved for  $\gamma = 0$ .  $\square$

**Lemma 4.** *If  $\gamma \neq 0$ , then for  $n \geq 0$*

$$\begin{aligned} & (\gamma\tau + \delta)^{(2n+1)/2} \vartheta_{gh}^{(n)}(\tau; 0, 2N) = \\ & = e((2n+1) \operatorname{sgn} \gamma / 8) (2N|\gamma|)^{-1/2} (-i \operatorname{sgn} \gamma)^n \times \\ & \times \sum_{H \bmod 2N} \varphi_{g'gh}(0, H; 2N) \left\{ \vartheta_{g'h'}^{(n)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H, 2N\right) + \right. \\ & \left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{g'h'}^{(n-t)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H, 2N\right) \right\}, \end{aligned}$$

where

$$A_{tk} \Big|_{z=0} = \begin{cases} (2k)! (-2N\gamma\pi i (\gamma\tau + \delta))^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases} \quad (t = 1, 2, \dots; k = 1, 2, \dots, t). \quad (1.18)$$

*Remark.* From (1.18) it follows that

$$\sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} = \begin{cases} \frac{A_{t, t/2}}{(t/2)!} \Big|_{z=0} & \text{if } 2|t \\ 0 & \text{if } 2 \nmid t. \end{cases} \quad (1.19)$$

*Proof.* Replacing  $\alpha, \beta, \gamma, \delta, \tau, c, N$  by  $\delta, -\beta, -\gamma, \alpha, \tau', 0, 2N$  in Lemma 2, we obtain

$$\begin{aligned} & \vartheta_{gh}^{(n)}\left(\frac{\delta\tau' - \beta}{-\gamma\tau' + \alpha}; 0, 2N\right) = \\ & = (-2N|\gamma| i \operatorname{sgn} \gamma)^n (i(-\gamma\tau' + \alpha) \frac{\operatorname{sgn} \gamma}{2N|\gamma|})^{(2n+1)/2} \times \\ & \times \sum_{H \bmod 2N} \varphi_{g'gh}(0, H; 2N) \left\{ \vartheta_{g'h'}^{(n)}(\tau'; H, 2N) + \right. \\ & \left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{g'h'}^{(n-t)}(\tau'; H, 2N) \right\}, \end{aligned} \quad (1.20)$$

where

$$g' = \delta g - \gamma h - 2\gamma\delta N, \quad h' = -\beta g + \alpha h - 2\alpha\beta N, \quad (1.21)$$

$$\begin{aligned} \varphi_{g'gh}(0, H; 2N) & = e\left(-\frac{\alpha\beta}{4N}\left(H + \frac{g'}{2}\right)^2\right) e\left(-\frac{\beta g}{4N}\left(H + \frac{g'}{2}\right)\right) \times \\ & \times S_{g-\alpha g', h-\beta g'}\left(\frac{\delta}{-\gamma}; -\alpha H, 2N\right), \end{aligned} \quad (1.22)$$

$$A_{tk} \Big|_{z=0} = \begin{cases} (2k)! \left(\frac{2N\gamma\pi i}{\gamma\tau' - \alpha}\right)^k & \text{if } t = 2k, \\ 0 & \text{if } t \neq 2k. \end{cases} \quad (1.23)$$

In [3, p.87, formula (6.23)] (with  $2N$  instead of  $N$ ) it is shown that

$$\left(\frac{i \operatorname{sgn} \gamma}{2N|\gamma|(\gamma\tau + \delta)}\right)^{1/2} = \frac{e(\operatorname{sgn} \gamma/8)}{(2N|\gamma|)^{1/2}(\gamma\tau + \delta)^{1/2}}. \quad (1.24)$$

Taking  $\frac{\alpha\tau + \beta}{\gamma\tau + \delta}$  instead of  $\tau'$  in (1.20) and (1.23) and using (1.24), we complete the proof of the lemma.  $\square$

## 2.

**Lemma 5.** *For a given  $N$  let*

$$\begin{aligned} \Psi_1(\tau) & = \Psi_1(\tau; g_1, g_2, h_1, h_2, c_1, c_2, N_1, N_2) = \\ & = \frac{1}{N_1} \vartheta_{g_1 h_1}'''(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}'(\tau; c_2, 2N_2) - \\ & - \frac{1}{N_2} \vartheta_{g_2 h_2}'''(\tau; c_2, 2N_2) \vartheta_{g_1 h_1}'(\tau; c_1, 2N_1) \end{aligned} \quad (2.1)$$

and

$$\Psi_2(\tau) = \Psi_2(\tau; g_1, g_2, h_1, h_2, c_1, c_2, N_1, N_2) =$$

$$\begin{aligned}
&= \frac{1}{N_1^2} \vartheta_{g_1 h_1}^{(4)}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) + \\
&+ \frac{1}{N_2^2} \vartheta_{g_2 h_2}^{(4)}(\tau; c_2, 2N_2) \vartheta_{g_1 h_1}(\tau; c_1, 2N_1) - \\
&- \frac{6}{N_1 N_2} \vartheta_{g_1 h_1}''(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}''(\tau; c_2, 2N_2), \tag{2.2}
\end{aligned}$$

where

$$2|g_1, 2|g_2, N_1|N, N_2|N, 4|N\left(\frac{h_1}{N_1} + \frac{h_2}{N_2}\right). \tag{2.3}$$

For all substitutions from  $\Gamma$  in the neighborhood of each rational point  $\tau = -\frac{\delta}{\gamma}$  ( $\gamma \neq 0, (\gamma, \delta) = 1$ ), we then have

$$\begin{aligned}
&(\gamma\tau + \delta)^5 \Psi_j(\tau; g_1, g_2, h_1, h_2, 0, 0, N_1, N_2) = \\
&= \sum_{n=0}^{\infty} C_n^{(j)} e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \quad (j = 1, 2). \tag{2.4}
\end{aligned}$$

*Proof.* I. From Lemma 4 for  $n = 3$  (with  $g_1, h_1, N_1, g'_1, h'_1, H_1$  instead of  $g, h, N, g', h', H$ ) and  $n = 1$  (with  $g_2, h_2, N_2, g'_2, h'_2, H_2$  instead of  $g, h, N, g', h', H$ ), according to (1.19), it follows that

$$\begin{aligned}
&\frac{1}{N_1} (\gamma\tau + \delta)^5 \vartheta_{g'_1 h'_1}'''(\tau; 0, 2N_1) \vartheta'_{g_2 h_2}(\tau; 0, 2N_2) = \\
&= \frac{1}{N_1} e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\
&\times \sum_{H_1 \bmod 2N_1} \varphi_{g'_1 h'_1}(0, H_1; 2N_1) \left\{ \vartheta_{g'_1 h'_1}''' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) + \right. \\
&\quad \left. + \frac{3 \cdot 2}{2!} A_{21} \Big|_{z=0} \cdot \vartheta'_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\} \times \\
&\times \sum_{H_2 \bmod 2N_2} \varphi_{g'_2 h'_2}(0, H_2; 2N_2) \vartheta'_{g_2 h_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) = \\
&= e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\
&\times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 h'_1}(0, H_1; 2N_1) \varphi_{g'_2 h'_2}(0, H_2; 2N_2) \times \\
&\times \left\{ \frac{1}{N_1} \vartheta_{g'_1 h'_1}''' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta'_{g_2 h_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - \right. \\
&\quad \left. - 12\gamma\pi i (\gamma\tau + \delta) \vartheta'_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \times \right.
\end{aligned}$$

$$\times \vartheta'_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \}. \quad (2.5)$$

Replacing  $N_1, g_1, h_1, H_1, g'_1, h'_1$  by  $N_2, g_2, h_2, H_2, g'_2, h'_2$  in (2.5) and vice versa, we obtain

$$\begin{aligned} & \frac{1}{N_2} (\gamma\tau + \delta)^5 \vartheta'''_{g'_2 h'_2}(\tau; 0, 2N_2) \vartheta'_{g'_1 h'_1}(\tau; 0, 2N_1) = \\ & = e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\ & \times \sum_{\substack{H_2 \bmod 2N_2 \\ H_1 \bmod 2N_1}} \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \varphi_{g'_1 g_1 h_1}(\tau; 0, H_1; 2N_1) \times \\ & \times \left\{ \frac{1}{N_2} \vartheta'''_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \vartheta'_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) - \right. \\ & \quad - 12\gamma\pi i (\gamma\tau + \delta) \vartheta'_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \times \\ & \quad \left. \times \vartheta'_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\}. \quad (2.6) \end{aligned}$$

Subtracting (2.6) from (2.5), according to (2.1), we obtain

$$\begin{aligned} & (\gamma\tau + \delta)^5 \Psi_1(\tau; g_1, g_2, h_1, h_2, 0, 0, N_1, N_2) = \\ & = e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\ & \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \times \\ & \quad \times \Psi_1 \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_1, g'_2, h'_1, h'_2, H_1, H_2, N_1, N_2 \right). \quad (2.7) \end{aligned}$$

In (2.7) let  $\gamma$  be even. Then, by (1.21) and (2.3),  $g'_1$  and  $g'_2$  are also even. Therefore, according to (3) and the notations (4), we have

$$\begin{aligned} & \vartheta'''_{g'_r h'_r} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_r, 2N_r \right) \vartheta'_{g'_s h'_s} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_s, 2N_s \right) = \\ & = \sum_{n_1=0}^{\infty} B_{n_1} e\left(\frac{Nn_1/N_r}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \sum_{n_2=0}^{\infty} B_{n_2} e\left(\frac{Nn_2/N_s}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \\ & = \sum_{n=0}^{\infty} B_n e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \\ & \text{for } r = 1, s = 2 \text{ and } r = 2, s = 1. \quad (2.8) \end{aligned}$$

Hence, for even  $\gamma$ , (2.4) follows from (2.1), (2.7), and (2.8) if  $j = 1$ .

Now let  $\gamma$  in (2.7) be odd. If  $h_1$  and  $h_2$  are both even, then by (1.21) and (2.3),  $g'_1$  and  $g'_2$  are also even, and we obtain the same result. But if  $h_r$  is odd, then by (1.21)  $g'_r$  will also be odd, and in (3) we have

$$\left(m + \frac{g'_r}{2}\right)^2 = \left(m + \frac{1}{2}(g'_r - 1)\right)^2 + \left(m + \frac{1}{2}(g'_r - 1)\right) + \frac{1}{4},$$

hence

$$\begin{aligned} \vartheta'''_{g'_r h'_r} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_r, 2N_r \right) &= -\pi^3 i e \left( \frac{1}{16N_r} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ &\times \sum_{m \equiv H_r \pmod{2N_r}} (-1)^{h'_r(m-H_r)/2N_r} (2m + g'_r)^3 \times \\ &\times e \left\{ \frac{1}{4N_r} \left( \left(m + \frac{1}{2}(g'_r - 1)\right)^2 + \left(m + \frac{1}{2}(g'_r - 1)\right) \right) \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right\} = \\ &= e \left( \frac{h_r}{16N_r} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \sum_{n_1=0}^{\infty} B'_{n_1} e \left( \frac{n_1}{4N_r} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right), \end{aligned}$$

since (5) implies that  $h_r = 1$ . Analogously,

$$\vartheta'_{g'_s h'_s} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_s, 2N_s \right) = e \left( \frac{h_s}{16N_s} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \sum_{n_2=0}^{\infty} B'_{n_2} e \left( \frac{n_2}{4N_s} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right),$$

which implies that  $h_s = 1$  if  $h_s$  is odd and  $h_s = 0$  if  $h_s$  is even. Thus, if among  $h_1$  and  $h_2$  at least one is odd, then we have, for  $r = 1, s = 2$  and  $r = 2, s = 1$ ,

$$\begin{aligned} &\vartheta'''_{g'_r h'_r} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_r, 2N_r \right) \vartheta'_{g'_s h'_s} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_s, 2N_s \right) = \\ &= e \left( \frac{N/4(h_r/N_r + h_s/N_s)}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \sum_{n_1=0}^{\infty} B'_{n_1} e \left( \frac{n_1}{4N_r} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ &\times \sum_{n_2=0}^{\infty} B'_{n_2} e \left( \frac{n_2}{4N_s} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = \sum_{n=0}^{\infty} B'_n e \left( \frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right), \end{aligned} \quad (2.9)$$

since, by (2.3),  $\frac{N}{4} \left( \frac{h_r}{N_r} + \frac{h_s}{N_s} \right)$  is an integer. Hence for odd  $\gamma$  (2.4) follows from (2.1), (2.7), and (2.9) if  $j = 1$ .

II. From Lemma 4 for  $n = 4$  and  $n = 0$ , as in I, it follows that

$$\begin{aligned} &\frac{1}{N_2^2} (\gamma\tau + \delta)^5 \vartheta_{g_1 h_1}^{(4)}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) = \\ &= \frac{1}{N_1^2} e \left( \frac{5}{4} \operatorname{sgn} \gamma \right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \end{aligned}$$

$$\begin{aligned}
& \times \sum_{H_1 \bmod 2N_1} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \left\{ \vartheta_{g'_1 h'_1}^{(4)} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) + \right. \\
& \quad + \frac{4 \cdot 3}{2!} A_{21} \Big|_{z=0} \cdot \vartheta_{g'_1 h'_1}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) + \\
& \quad \left. + \frac{A_{42}}{2!} \Big|_{z=0} \cdot \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\} \times \\
& \times \sum_{H_2 \bmod 2N_2} \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \vartheta_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) = \\
& = e \left( \frac{5}{4} \operatorname{sgn} \gamma \right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\
& \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \times \\
& \quad \times \left\{ \frac{1}{N_1^2} \vartheta_{g'_1 h'_1}^{(4)} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \times \right. \\
& \times \vartheta_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - \frac{24\gamma\pi i}{N_1} (\gamma\tau + \delta) \vartheta_{g'_1 h'_1}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \times \\
& \vartheta_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - 48\gamma^2 \pi^2 (\gamma\tau + \delta)^2 \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \times \\
& \quad \left. \times \vartheta_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \right\}. \tag{2.10}
\end{aligned}$$

Replacing  $N_1, g_1, h_1, H_1, g'_1, h'_1$  by  $N_2, g_2, H_2, g'_2, h'_2$  in (2.10) and vice versa, we obtain

$$\begin{aligned}
& \frac{1}{N_1^2} (\gamma\tau + \delta)^5 \vartheta_{g_2 h_2}^{(4)}(\tau; 0, 2N_2) \vartheta_{g_1 h_1}(\tau; 0, 2N_1) = \\
& = e \left( \frac{5}{4} \operatorname{sgn} \gamma \right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\
& \times \sum_{\substack{H_2 \bmod 2N_2 \\ H_1 \bmod 2N_1}} \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \times \\
& \quad \times \left\{ \frac{1}{N_2^2} \vartheta_{g'_2 h'_2}^{(4)} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \times \right. \\
& \times \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) - \frac{24\gamma\pi i}{N_2} (\gamma\tau + \delta) \vartheta_{g'_2 h'_2}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \times \\
& \times \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) - 48\gamma^2 \pi^2 (\gamma\tau + \delta)^2 \vartheta_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \times
\end{aligned}$$

$$\times \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \}. \quad (2.11)$$

Analogously, from Lemma 4 for  $n = 2$  it follows that

$$\begin{aligned} & \frac{6}{N_1 N_2} (\gamma\tau + \delta)^5 \vartheta''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta''_{g_2 h_2}(\tau; 0, 2N_2) = \\ & = e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\ & \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \times \\ & \times \left\{ \frac{6}{N_1 N_2} \vartheta''_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta''_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - \right. \\ & - \frac{24\gamma\pi i}{N_2} (\gamma\tau + \delta) \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - \\ & - \frac{24\gamma\pi i}{N_1} (\gamma\tau + \delta) \vartheta''_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta''_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - \\ & \quad - 96\gamma^2 \pi^2 (\gamma\tau + \delta)^2 \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \times \\ & \quad \left. \times \vartheta_{g'_2 h'_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \right\}. \quad (2.12) \end{aligned}$$

Subtracting (2.12) from the sum of (2.10) and (2.11), according to (2.2), we obtain

$$\begin{aligned} & (\gamma\tau + \delta)^5 \Psi_2(\tau; g_1, g_2, h_1, h_2, 0, 0, N_1, N_2) = \\ & = e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \times \\ & \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \times \\ & \quad \times \Psi_2 \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_1, g'_2, h'_1, h'_2, H_1, H_2, N_1, N_2 \right). \quad (2.13) \end{aligned}$$

Further, reasoning as in I, from (2.2) and (2.13) we obtain (2.4) if  $j = 2$ .  $\square$

**Theorem 1.** *For a given  $N$  the functions  $\Psi_1(\tau)$  and  $\Psi_2(\tau)$  with  $c_1 = c_2 = 0$  are entire modular forms of weight 10 and character  $\chi(\delta) = \operatorname{sgn} \delta \left( \frac{-\Delta}{|\delta|} \right)$  ( $\Delta$  is the determinant of an arbitrary positive quadratic form in 5 variables)*

for the congruence group  $\Gamma_0(4N)$  if the following conditions hold:

$$1) \quad 2|g_1, 2|g_2, N_1|N, N_2|N, \quad (2.14)$$

$$2) \quad 4|N\left(\frac{h_1^2}{N_1} + \frac{h_2^2}{N_2}\right), 4|\frac{g_1^2}{4N_1} + \frac{g_2^2}{4N_2}, \quad (2.15)$$

$$3) \quad \text{for all } \alpha \text{ and } \delta \text{ with } \alpha \equiv \delta \equiv 1 \pmod{4N}$$

$$\begin{aligned} & \left(\frac{N_1 N_2}{|\delta|}\right) \Psi_j(\tau; \alpha g_1, \alpha g_2, h_1, h_2, 0, 0, N_1, N_2) = \\ & = \left(\frac{\Delta}{|\delta|}\right) \Psi_j(\tau; g_1, g_2, h_1, h_2, 0, 0, N_1, N_2) \quad (j = 1, 2). \end{aligned} \quad (2.16)$$

*Proof.*

I. It is wellknown that the thetaseries (2)-(3) are regular on  $\mathcal{H}$ , hence the functions  $\Psi_1(\tau)$  and  $\Psi_2(\tau)$  satisfy condition 1 of the definition.

II. From (2.15), since  $\delta^2 \equiv 1 \pmod{4}$ , it follows that

$$4|N\delta^2\left(\frac{h_1^2}{N_1} + \frac{h_2^2}{N_2}\right), 4|\frac{g_1^2}{4N_1}\delta^{2\varphi(2N_1)-2} + \frac{g_2^2}{4N_2}\delta^{2\varphi(2N_2)-2}. \quad (2.17)$$

By Lemma 3 for  $n = 3$  and  $n = 1$  (with  $g_r, h_r, N_r$  and  $g_s, h_s, N_s$  instead of  $g, h, N$ ), according to (2.14) and (2.17), for each substitution from  $\Gamma_0(4N)$ , we have

$$\begin{aligned} & \vartheta'''_{g_r h_r}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r\right) \vartheta'_{g_s h_s}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_s\right) = i^{\eta(\gamma)(sgn\delta-1)} i^{1-|\delta|} \times \\ & \times (\gamma\tau + \delta)^5 \left(\frac{N_r N_s}{|\delta|}\right) \vartheta'''_{\alpha g_r, h_r}(\tau; 0, 2N_r) \vartheta'_{\alpha g_s, h_s}(\tau; 0, 2N_s) = \\ & = \text{sgn } \delta \left(\frac{-N_r N_s}{|\delta|}\right) (\gamma\tau + \delta)^5 \times \\ & \times \vartheta'''_{\alpha g_r, h_r}(\tau; 0, 2N_r) \vartheta'_{\alpha g_s, h_s}(\tau; 0, 2N_s) \end{aligned} \quad (2.18)$$

for  $r = 1, s = 2$  and  $r = 2, s = 1$ .

Analogously, by Lemma 3, we have:

1) for  $n = 4$  and  $n = 0$

$$\begin{aligned} & \vartheta^{(4)}_{g_r h_r}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r\right) \vartheta_{g_s h_s}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_s\right) = i^{\eta(\gamma)(sgn\delta-1)} i^{1-|\delta|} \times \\ & \times (\gamma\tau + \delta)^5 \left(\frac{N_r N_s}{|\delta|}\right) \vartheta^{(4)}_{\alpha g_r, h_r}(\tau; 0, 2N_r) \vartheta_{\alpha g_s, h_s}(\tau; 0, 2N_s) = \\ & = \text{sgn } \delta \left(\frac{-N_r N_s}{|\delta|}\right) (\gamma\tau + \delta)^5 \times \\ & \times \vartheta^{(4)}_{\alpha g_r, h_r}(\tau; 0, 2N_r) \vartheta_{\alpha g_s, h_s}(\tau; 0, 2N_s) \end{aligned} \quad (2.19)$$

if  $r = 1, s = 2$  and  $r = 2, s = 1$ ;



2) for  $n = 2$

$$\begin{aligned} & \vartheta''_{g_1 h_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_1 \right) \vartheta''_{g_2 h_2} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_2 \right) = \\ & = \operatorname{sgn} \delta \left( \frac{-N_1 N_2}{|\delta|} \right) (\gamma\tau + \delta)^5 \times \\ & \times \vartheta''_{\alpha g_1, h_1}(\tau; 0, 2N_1) \vartheta''_{\alpha g_2, h_2}(\tau; 0, 2N_2). \end{aligned} \quad (2.20)$$

Hence, according to (2.16), for all  $\alpha$  and  $\delta$  with  $\alpha\delta \equiv 1 \pmod{4N}$ , we have

$$\begin{aligned} & \Psi_j \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, g_2, h_1, h_2, 0, 0, N_1, N_2 \right) = \\ & = \operatorname{sgn} \delta \left( \frac{-N_1 N_2}{|\delta|} \right) (\gamma\tau + \delta)^5 \Psi_j(\tau; \alpha g_1, \alpha g_2, h_1, h_2, 0, 0, N_1, N_2) = \\ & = \operatorname{sgn} \delta \left( \frac{-\Delta}{|\delta|} \right) (\gamma\tau + \delta)^5 \Psi_j(\tau; g_1, g_2, h_1, h_2, 0, 0, N_1, N_2) \end{aligned}$$

for  $j = 1$ , by (2.1) and (2.16), and for  $j = 2$ , by (2.2), (2.19) and (2.20). Thus the functions  $\Psi_1(\tau)$  and  $\Psi_2(\tau)$  with  $c_1 = c_2 = 0$  satisfy condition 2 of the definition.

III. From (13) it follows that, for  $r = 1, s = 2$  and  $r = 2, s = 1$ ,

$$\begin{aligned} & \vartheta''_{g_r h_r}(\tau; 0, 2N_r) \vartheta'_{g_s h_s}(\tau; 0, 2N_s) = \pi^4 \sum_{m_r, m_s = -\infty}^{\infty} (-1)^{h_r m_r + h_s m_s} \times \\ & \times (4N_r m_r + g_r)^3 (4N_s m_s + g_s) e(\Lambda\tau), \\ & \vartheta_{g_r h_r}^{(4)}(\tau; 0, 2N_r) \vartheta_{g_s h_s}(\tau; 0, 2N_s) = \pi^4 \sum_{m_r, m_s = -\infty}^{\infty} (-1)^{h_r m_r + h_s m_s} \times \\ & \times (4N_r m_r + g_r)^4 e(\Lambda\tau) \end{aligned} \quad (2.21)$$

and also

$$\begin{aligned} & \vartheta''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta''_{g_2 h_2}(\tau; 0, 2N_2) = \pi^4 \sum_{m_1, m_2 = -\infty}^{\infty} (-1)^{h_1 m_1 + h_2 m_2} \times \\ & \times (4N_1 m_1 + g_1)^2 (4N_2 m_2 + g_2)^2 e(\Lambda\tau), \end{aligned}$$

where

$$\Lambda = \sum_{k=1}^2 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 = \sum_{k=1}^2 (N_k m_k^2 + m_k g_k/2) + \frac{1}{4} \sum_{k=1}^2 g_k^2 / 4N_k,$$

by (2.14) and (2.15), is an integer. Thus, the functions  $\Psi_1(\tau)$  and  $\Psi_2(\tau)$  with  $c_1 = c_2 = 0$  satisfy condition 3 of the definition.

IV. By Lemma 5 the functions  $\Psi_1(\tau)$  and  $\Psi_2(\tau)$  with  $c_1 = c_2 = 0$  also satisfy condition 4 of the definition.  $\square$

3.

**Lemma 6.** *For a given  $N$  let*

$$\begin{aligned} \Phi_1(\tau) &= \Phi_1(\tau; g_1, \dots, g_4, h_1, \dots, h_4, c_1, \dots, c_4, N_1, \dots, N_4) = \\ &= \frac{1}{N_1} \vartheta'''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta'_{g_2 h_2}(\tau; c_2, 2N_2) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; c_k, 2N_k) - \\ &\quad - \frac{1}{N_2} \vartheta'''_{g_2 h_2}(\tau; c_2, 2N_2) \vartheta'_{g_1 h_1}(\tau; c_1, 2N_1) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; c_k, 2N_k) \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \Phi_2(\tau) &= \Phi_2(\tau; g_1, \dots, g_4, h_1, \dots, h_4, c_1, \dots, c_4, N_1, \dots, N_4) = \\ &= \frac{1}{N_1^2} \vartheta_{g_1 h_1}^{(4)}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; c_k, 2N_k) + \\ &\quad + \frac{1}{N_2^2} \vartheta_{g_2 h_2}^{(4)}(\tau; c_2, 2N_2) \vartheta_{g_1 h_1}(\tau; c_1, 2N_1) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; c_k, 2N_k) - \\ &\quad - \frac{6}{N_1 N_2} \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; c_k, 2N_k), \end{aligned} \quad (3.2)$$

where

$$2|g_k, N_k|N \ (k = 1, 2, 3, 4), \quad 4|N \sum_{k=1}^4 \frac{h_k}{N_k}. \quad (3.3)$$

For all substitutions from  $\Gamma$  in the neighborhood of each rational point  $\tau = -\frac{\delta}{\gamma}$  ( $\gamma \neq 0$ ,  $(\gamma, \delta) = 1$ ), we then have

$$\begin{aligned} (\gamma\tau + \delta)^6 \Phi_j(\tau; g_1, \dots, g_4, h_1, \dots, h_4, 0, \dots, 0, N_1, \dots, N_4) &= \\ &= \sum_{n=0}^{\infty} D_n^{(j)} e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \quad (j = 1, 2). \end{aligned} \quad (3.4)$$

*Proof.* From Lemma 4 for  $n = 3$ ,  $n = 1$ , and  $n = 0$ , as in the proof of Lemma 5, it follows that

$$\frac{1}{N_1} (\gamma\tau + \delta)^6 \vartheta'''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta'_{g_2 h_2}(\tau; 0, 2N_2) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) =$$

$$\begin{aligned}
&= e\left(\frac{3}{2} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times \\
&\quad \times \left\{ \left(\frac{1}{N_1} \vartheta'''_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) - \right. \right. \\
&\quad \left. \left. -12\gamma\pi i(\gamma\tau + \delta) \vartheta'_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right)\right) \times \right. \\
&\quad \left. \times \vartheta'_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right) \prod_{k=3}^4 \vartheta_{g'_k h'_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k\right) \right\} \quad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{N_2} (\gamma\tau + \delta)^6 \vartheta'''_{g_2 h_2}(\tau; 0, 2N_2) \vartheta'_{g_1 h_1}(\tau; 0, 2N_1) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) = \\
&= e\left(\frac{3}{2} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times \\
&\quad \times \left\{ \left(\frac{1}{N_2} \vartheta'''_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right) - \right. \right. \\
&\quad \left. \left. -12\gamma\pi i(\gamma\tau + \delta) \vartheta'_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right)\right) \times \right. \\
&\quad \left. \times \vartheta'_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) \prod_{k=3}^4 \vartheta_{g'_k h'_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k\right) \right\}. \quad (3.6)
\end{aligned}$$

As in the proof of Lemma 5, from Lemma 4 it follows that

1) for  $n = 4$  and  $n = 0$

$$\begin{aligned}
&\frac{1}{N_1^2} (\gamma\tau + \delta)^6 \vartheta^{(4)}_{g_1 h_1}(\tau; 0, 2N_1) \prod_{k=2}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) = \\
&= e\left(\frac{3}{2} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times \\
&\quad \times \left\{ \left(\frac{1}{N_1^2} \vartheta^{(4)}_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) - \right. \right. \\
&\quad \left. \left. -\frac{24\gamma\pi i}{N_1} (\gamma\tau + \delta) \vartheta''_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) - \right. \right. \\
&\quad \left. \left. -48\gamma^2 \pi^2 (\gamma\tau + \delta)^2 \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right)\right) \times \right.
\end{aligned}$$

$$\times \prod_{k=2}^4 \vartheta_{g'_k h'_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k \right) \} \quad (3.7)$$

and

$$\begin{aligned} & \frac{1}{N_2^2} (\gamma\tau + \delta)^6 \vartheta_{g_2 h_2}^{(4)}(\tau; 0, 2N_2) \vartheta_{g_1 h_1}(\tau; 0, 2N_1) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) = \\ & = e \left( \frac{3}{2} \operatorname{sgn} \gamma \right) \left( 4\gamma^2 \left( \prod_{k=1}^4 N_k \right)^{1/2} \right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k h'_k}(0, H_k; 2N_k) \times \\ & \quad \times \left\{ \left( \frac{1}{N_2^2} \vartheta_{g'_2 h'_2}^{(4)} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - \right. \right. \\ & \quad \left. \left. - \frac{24\gamma\pi i}{N_2} (\gamma\tau + \delta) \vartheta_{g'_2 h'_2}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \right) \times \right. \\ & \quad \times \vartheta_{g'_1 h'_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \prod_{k=3}^4 \vartheta_{g'_k h'_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k \right) - \\ & \quad \left. - 48\gamma^2 \pi^2 (\gamma\tau + \delta)^2 \prod_{k=1}^4 \vartheta_{g'_k h'_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k \right) \right\}, \quad (3.8) \end{aligned}$$

2) for  $n = 2$  and  $n = 0$

$$\begin{aligned} & \frac{6}{N_1 N_2} (\gamma\tau + \delta)^6 \vartheta_{g'_1 h'_1}''(\tau; 0, 2N_1) \vartheta_{g_2 h_2}''(\tau; 0, 2N_2) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) = \\ & = e \left( \frac{3}{2} \operatorname{sgn} \gamma \right) \left( 4\gamma^2 \left( \prod_{k=1}^4 N_k \right)^{1/2} \right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \varphi_{g'_k h'_k}(0, H_k; 2N_k) \times \\ & \quad \times \left\{ \left( \frac{6}{N_1 N_2} \vartheta_{g'_1 h'_1}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) - \right. \right. \\ & \quad \left. \left. - \frac{24\gamma\pi i}{N_2} (\gamma\tau + \delta) \vartheta_{g'_1 h'_1}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right) \times \right. \\ & \quad \times \vartheta_{g'_2 h'_2}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \prod_{k=3}^4 \vartheta_{g'_k h'_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k \right) - \\ & \quad \left. - \left( \frac{24\gamma\pi i}{N_1} (\gamma\tau + \delta) \vartheta_{g_1 h_1}'' \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) + \right. \right. \\ & \quad \left. \left. + 96\gamma^2 \pi^2 (\gamma\tau + \delta)^2 \vartheta_{g_1 h_1} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right) \times \right. \end{aligned}$$

$$\times \prod_{k=2}^4 \vartheta_{g'_k h'_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k \right) \}. \quad (3.9)$$

Subtracting (3.6) from (3.5), and (3.9) from the sum of (3.7) and (3.8), according to (3.1) and (3.2) respectively, we obtain

$$\begin{aligned} & (\gamma\tau + \delta)^6 \Phi_j(\tau; g_1, \dots, g_4, h_1, \dots, h_4, 0, \dots, N_1, \dots, N_4) = \\ & = e\left(\frac{3}{2} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \varphi_{g'_k h'_k}(0, H_k; 2N_k) \times \\ & \times \Phi_j\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, \dots, g_4, h_1, \dots, h_4, H_1, \dots, H_4, N_1, \dots, N_4\right) \\ & \quad (j = 1, 2). \end{aligned} \quad (3.10)$$

Further, reasoning as in Lemma 5, from (3.10) we obtain (3.4).  $\square$

**Theorem 2.** For a given  $N$  the functions  $\Phi_1(\tau)$  and  $\Phi_2(\tau)$  with  $c_1 = c_2 = c_3 = c_4 = 0$  are entire modular forms of weight 12 and character  $\chi(\delta) = \left(\frac{\Delta}{|\delta|}\right)$  ( $\Delta$  is the determinant of an arbitrary positive quadratic form in 6 variables) for the congruence group  $\Gamma_0(4N)$  if the following conditions hold:

$$1) \quad 2|g_k, N_k|N \quad (k = 1, 2, 3, 4), \quad (3.11)$$

$$2) \quad 4|N \sum_{k=1}^4 \left(\frac{h_k^2}{N_k}\right), 4\left|\sum_{k=1}^4 \frac{g_k^2}{4N_k}\right|, \quad (3.12)$$

$$3) \quad \text{for all } \alpha \text{ and } \delta \text{ with } \alpha\delta \equiv 1 \pmod{4N}$$

$$\begin{aligned} & \left(\frac{\prod_{k=1}^4 N_k}{|\delta|}\right) \Phi_j(\tau; \alpha g_1, \dots, \alpha g_4, h_1, \dots, h_4, 0, \dots, 0, N_1, \dots, N_4) = \\ & = \left(\frac{\Delta}{|\delta|}\right) \Phi_j(\tau; g_1, \dots, g_4, h_1, \dots, h_4, 0, \dots, 0, N_1, \dots, N_4) \\ & \quad (j = 1, 2). \end{aligned} \quad (3.13)$$

*Proof.*

I. As in the case of Theorem 1, the functions  $\Phi_1(\tau)$  and  $\Phi_2(\tau)$  with  $c_1 = c_2 = c_3 = c_4 = 0$  satisfy condition 1 and, by Lemma 6, also condition 4 of the definition.

II. From (3.12), since  $\delta^2 \equiv 1 \pmod{4}$ , it follows that

$$4|N\delta^2 \sum_{k=1}^4 \frac{h_k^2}{N_k}, 4\left|\sum_{k=1}^4 \frac{g_k^2}{4N_k} \delta^{2\varphi(2N_k)-2}\right|. \quad (3.14)$$

By Lemma 3, for  $n = 3$ ,  $n = 1$  and  $n = 0$ , according to (3.11) and (3.14), for each substitution from  $\Gamma_0(4N)$ , we have

$$\begin{aligned}
& \vartheta'''_{g_r h_r} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r \right) \vartheta'_{g_s h_s} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_s \right) \times \\
& \quad \times \prod_{k=3}^4 \vartheta_{g_k h_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right) = \\
& = (\gamma\tau + \delta)^6 \left( \frac{\prod_{k=1}^4 N_k}{|\delta|} \right) \vartheta'''_{\alpha g_r, h_r}(\tau; 0, 2N_r) \vartheta'_{\alpha g_s, h_s}(\tau; 0, 2N_s) \times \\
& \quad \times \prod_{k=3}^4 \vartheta_{\alpha g_k, h_k}(\tau; 0, 2N_k) \tag{3.15}
\end{aligned}$$

for  $r = 1, s = 2$  and  $r = 2, s = 1$ .

Analogously, by Lemma 3, we have:

1) for  $n = 4$  and  $n = 0$

$$\begin{aligned}
& \vartheta^{(4)}_{g_r h_r} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r \right) \vartheta_{g_s h_s} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_s \right) \times \\
& \quad \times \prod_{k=3}^4 \vartheta_{g_k h_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right) = \\
& = (\gamma\tau + \delta)^6 \left( \frac{\prod_{k=1}^4 N_k}{|\delta|} \right) \vartheta^{(4)}_{\alpha g_r, h_r}(\tau; 0, 2N_r) \vartheta_{\alpha g_s, h_s}(\tau; 0, 2N_s) \times \\
& \quad \times \prod_{k=3}^4 \vartheta_{\alpha g_k, h_k}(\tau; 0, 2N_k) \tag{3.16}
\end{aligned}$$

for  $r = 1, s = 2$  and  $r = 2, s = 1$ ,

2) for  $n = 2$  and  $n = 0$

$$\begin{aligned}
& \prod_{k=1}^2 \vartheta''_{g_k h_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right) \prod_{k=3}^4 \vartheta_{g_k h_k} \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right) = \\
& = (\gamma\tau + \delta)^6 \left( \frac{\prod_{k=1}^4 N_k}{|\delta|} \right) \prod_{k=1}^2 \vartheta''_{\alpha g_k, h_k}(\tau; 0, 2N_k) \prod_{k=3}^4 \vartheta_{\alpha g_k, h_k}(\tau; 0, 2N_k).
\end{aligned}$$

Hence, according to (3.13), for all  $\alpha$  and  $\delta$  with  $\alpha\delta \equiv 1 \pmod{4N}$ , we have

$$\begin{aligned}
& \Phi_j \left( \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, \dots, g_4, h_1, \dots, h_4, 0, \dots, N_1, \dots, N_4 \right) = \\
& = \left( \frac{\prod_{k=1}^4 N_k}{|\delta|} \right) (\gamma\tau + \delta)^6 \times
\end{aligned}$$

$$\begin{aligned} & \times \Phi_j(\tau; \alpha g_1, \dots, \alpha g_4, h_1, \dots, h_4, 0, \dots, 0, N_1, \dots, N_4) = \\ & = \left( \frac{\Delta}{|\delta|} \right) (\gamma\tau + \delta)^6 \Phi_j(\tau; g_1, \dots, g_4, h_1, \dots, h_4, 0, \dots, N_1, \dots, N_4) \end{aligned}$$

for  $j = 1$ , by (3.1) and (3.15), and for  $j = 2$ , by (3.2), (3.16), and (3.17). Thus the functions  $\Phi_1(\tau)$  and  $\Phi_2(\tau)$  satisfy condition 2 of the definition.

III. From (13) it follows that, for  $r = 1, s = 2$  and  $r = 2, s = 1$ ,

$$\begin{aligned} & \vartheta'''_{g_r h_r}(\tau; 0, 2N_r) \vartheta'_{g_s h_s}(\tau; 0, 2N_s) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) = \\ & = \pi^4 \sum_{m_r, m_s, m_3, m_4 = -\infty}^{\infty} (-1)^{m_r + m_s + m_3 + m_4} (4N_r m_r + g_r)^3 (4N_s m_s + g_s) e(\Lambda\tau), \\ & \vartheta^{(4)}_{g_r h_r}(\tau; 0, 2N_r) \vartheta_{g_s h_s}(\tau; 0, 2N_s) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) = \\ & = \pi^4 \sum_{m_r, m_s, m_3, m_4 = -\infty}^{\infty} (-1)^{m_r + m_s + m_3 + m_4} (4N_r m_r + g_r)^4 e(\Lambda\tau), \end{aligned}$$

and also

$$\begin{aligned} & \vartheta''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta''_{g_2 h_2}(\tau; 0, 2N_2) \prod_{k=3}^4 \vartheta_{g_k h_k}(\tau; 0, 2N_k) = \\ & = \pi^4 \sum_{m_1, \dots, m_4 = -\infty}^{\infty} (-1)^{\sum_{k=1}^4 h_k m_k} (4N_1 m_1 + g_1)^2 (4N_2 m_2 + g_2)^2 e(\Lambda\tau), \end{aligned}$$

where

$$\Lambda = \sum_{k=1}^4 \frac{1}{4N_k} \left( 2N_k m_k + \frac{g_k}{2} \right)^2 = \sum_{k=1}^4 \left( N_k m_k^2 + \frac{1}{2} m_k g_k \right) + \frac{1}{4} \sum_{k=1}^4 \frac{g_k^2}{4N_k},$$

by (3.11) and (3.12), is an integer. Thus the functions  $\Phi_1(\tau)$  and  $\Phi_2(\tau)$  with  $c_1 = c_2 = c_3 = c_4 = 0$  satisfy condition 3 of the definition.  $\square$

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