General Mathematics Vol. 9, No. 1–2 (2001), 49–52

Note on a class of delta operators

Emil C. Popa

Abstract

Let Q be a delta operator with basic set (p_n), and

$$A_n(x) = x(x + nb)^{n-1},\ n = 1, 2, ...$$

the Abel polynomials.

For each natural number $n, n \geq 1$ let us consider the delta operator defined on the algebra of polynomials, by

$$\alpha_{n,Q} = Q(Q + nbI)^{n-1},\ b \neq 0. \tag{1}$$

The purpose of this paper is to give a representation theorem for basic set (q_m) relative to $\alpha_{n,D}$, and we define a linear operator $t_{n,D}$ whence we obtain the $R_{n,a}$ and $\alpha_{n,D}$ operators (see [1], [2]).

2000 Mathematical Subject Classification: 05A40

1

Theorem 1 *If* $m = min(k, n)$, *we have*

$$(\alpha_{n,Q} p_k)(x) = \frac{1}{n} \sum_{j=1}^{m} \frac{(-n)_j(-k)_j}{(j-1)!} (nb)^{n-1} p_{k-j}(x) \tag{2}$$

where $(a)_s = a(a + 1) \cdot ... \cdot (a + s - 1)$.

Proof.

$$A_n(x)=\sum_{j=0}^{n-1}\left(\begin{array}{c} n-1 \\ j \end{array}\right)(nb)^{n-j-1}x^{j+1}=$$

$$=\sum_{j=1}^{n}\left(\begin{array}{c} n-1 \\ j-1 \end{array}\right)(nb)^{n-j}x^{j}.$$

Hence

$$\alpha_{n,Q}=\sum_{j=1}^{n}\left(\begin{array}{c} n-1 \\ j-1 \end{array}\right)(nb)^{n-j}Q^{j},$$

whence

$$(\alpha_{n,Q}p_k)(x)=\sum_{j=1}^{m}\left(\begin{array}{c} n-1 \\ j-1 \end{array}\right)\frac{k!}{(k-j)!}(nb)^{n-j}p_{k-j}(x)$$

with $m=min(k,n)$.

Theorem 2 *If (q_m) is a basic sequence for the delta operator $\alpha_{n,D}$, then for $n>0$*

$$(3)\qquad q_m=\frac{1}{(nb)^{mn-m}\Gamma(mn-m)}\int_0^{\infty}e^{-t}t^{mn-m-1}x\left(x-\frac{1}{nb}t\right)^{m-1}dt.$$

Proof. We have

$$\alpha_{n,D}=D(D+nbI)^{n-1}$$

and

$$q_m=x(D+nbI)^{-mn+m}e_{m-1},$$

$$q_m(x)=x\frac{I}{(D+nbI)^{mn-m}}x^{m-1}.$$

Hence

$$q_m(x)=\frac{1}{(nb)^{mn-m}}\cdot\frac{1}{\Gamma(mn-m)}\int_0^{\infty}e^{-t}t^{mn-m-1}x\left(x-\frac{1}{nb}t\right)^{m-1}dt.$$

2

Ler R be a delta operator with two Sheffer sequences (r_n) and $(\widetilde{r}_n)$.

We note

$$T = e^{ax} D^m e^{-ax} = (D - aI)^m$$

and

$$t_n(x) = S(TE^b)^n \widetilde{r}_{n-1}(x)$$

where S is the linear operator defined by

$$S\widetilde{r}_n = r_{n+1}, \ n = 0, 1, 2, ...$$

Theorem 3 *If m is natural number, $a, b \in \mathbb{R}$, $a \neq 0$ and R'_S, $P'_S \in \prod_t^*$ where*

$$P^{-1} = TE^b$$

then

$$t_n(x) = t_n(a, b, m; \widetilde{r}_n; x) = Se^{ax} D^{nm} e^{-ax} \widetilde{r}_{n-1}(x + nb)$$

is a set of Sheffer polynomials.

Proof. It is used the theorem of [3].

For $m = 0$, we find

$$t_n(a, b, 0; \widetilde{r}_n; x) = S\widetilde{r}_{n-1}(x + nb)$$

and for $a = m = 1, \ b = 0$,

$$t_n(1, 0, 1; \widetilde{r}_n; x) = S(D - I)^n \widetilde{r}_{n-1}(x).$$

Now for $\widetilde{r}_n = r_n = x^n$ we have $S = X, \ (XP)(x) = xp(x)$, and

$$t_n(a, b, 0; x^n; x) = x(x + nb)^{n-1},$$

$$t_n(1, 0, 1; x^n, x) = x(D - I)^n x^{n-1}.$$

Hence from

$$t_n(x) = t_n(a, b, m; \widetilde{r}_n; x) = S(D - aI)^{mn} \widetilde{r}_{n-1}(x + nb)$$

we obtain the linear operator

$$t_{n,D} = t_n(D)$$

whence we find $R_{n,\alpha}$ and $\alpha_{n,D}$ operators, (see [1], [2]).

References

[1] E.C.Popa, *A certain class of linear operators*, ICAOR, Cluj-Napoca, 1996, Revue D'Analyse *Numérique* et de *Théorie* de l'Approximation, Tome 26, No. 1-2, 1997.

[2] E.C.Popa, *Linear operators of Abel type*, Bull.Math. de la Soc. des Sci. Math. de Roum., Vol.40, 1997.

[3] E.C.Popa, *Note on Sheffer polynomials*, Octogon, Mathematical Magazine, Vol.5, Nr.2, oct. 1997, 56-57.

[4] E.C.Popa, *Note on a delta operators*, General Mathematics, Vol.8, Nr.1-2, oct. 2000, 79-82.

[5] S.Roman, G-C.Rota, *The umbral calculus*, Adv. in Math. 27, 1978, 96-188.

[6] G-C.Rota, D.Kahaner, A.Odlyzko, *Finite operator calculus*, J.Math.Anal.Appl. 42, 1973, 685-760.

Department of Mathematics
Faculty of Sciences
"Lucian Blaga" University of Sibiu
2400 Sibiu, Romania
E-mail: *emil.popa@ulbsibiu.ro*