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A sufficient condition for univalence[1]

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Abstract

In this paper we obtain sufficient conditions for univalence, which generalize some well known univalence criteria for analytic functions in the unit disk.

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1 Introduction

We denote by $U_r = \{\ z \in C :\ \ |z| < r\}$ the disk of z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0,\infty)$. Let A be the class of functions f analytic in U such that $f(0) = 0,\ f'(0) = 1$.

Theorem 1.1. (see [2]) *Let $f \in A$. If for all $z \in U$*

$$|\{f;z\}| \le \frac{2}{(1-|z|^2)^2} \tag{1}$$

where

$$\{f;z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{2}$$

then the function f is univalent in U.

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Theorem 1.2. (see [1]) *Let $f \in A$. If for all $z \in U$*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1\ , \tag{3}$$

then the function f is univalent in U.

Theorem 1.3. (see [3]) *Let $f \in A$. If for all $z \in U$*

$$\left|\frac{z^2f'(z)}{f^2(z)}-1\right| < 1 \tag{4}$$

then the function f is univalent in U.

2 Preliminaries

Our considerations are based on the theory of Löwner chains; we first recall the basic result of this theory, from Pommerenke.

Theorem 2.1. (see [4]) *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots,\ a_1(t) \neq 0$ be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and locally uniformly with respect to U_r.For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \text{for all } z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition Re $p(z,t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

3 Main results

Theorem 3.1. *Let β be a real number, $\beta > 1/2$ and $f \in A$. If there exist the analytic functions g and h in U, $g(z) = 1+b_1z+\dots,\ h(z) = c_0+c_1z+\dots$, such that the inequalities*

$$\left|\frac{f'(z)}{g(z)}-\beta\right| < \beta \tag{5}$$

and

$$\left|\left(\frac{f'(z)}{g(z)}-\beta\right)|z|^{2\beta}+(1-|z|^{2\beta})\left(\frac{2zf'(z)h(z)}{g(z)}+\frac{zg'(z)}{g(z)}+1-\beta\right)\right.$$
$$\left.+\frac{(1-|z|^{2\beta})^2}{|z|^{2\beta}}\left(\frac{z^2f'(z)h^2(z)}{g(z)}+\frac{z^2g'(z)h(z)}{g(z)}-z^2h'(z)\right)\right|\le\beta \tag{6}$$

are true for all $z\in U$, then the function f is univalent in U.

Proof. The functions f, g, h being analytic in U, it is easy to see that there is a real number $r_1\in(0,1]$ such that the function

$$L(z,t)=f(e^{-t}z)+\frac{(e^{2\beta t}-1)\cdot e^{-t}z\cdot g(e^{-t}z)}{1+(e^{2\beta t}-1)\cdot e^{-t}z\cdot h(e^{-t}z)} \tag{7}$$

is analytic in U_{r_1}, for all $t\in I$. If $L(z,t)=a_1(t)z+a_2(t)z^2+\dots$ is the power series expansion of $L(z,t)$ in the neighborhood U_{r_1}, it can be checked that we have $a_1(t)=e^{(2\beta-1)t}$ and therefore $a_1(t)\neq 0$ for all $t\in I$. From $\beta>1/2$, it follows that $\lim_{t\to\infty}|a_1(t)|=\infty$.

Since $L(z,t)/a_1(t)$ is the summation between z and an analytic function,we conclude that $\{L(z,t)/a_1(t)\}_{t\in I}$ is a normal family in U_{r_2}, $0<r_2<r_1$. By elementary computations, it can be shown that $\frac{\partial L(z,t)}{\partial t}$ can be expressed as the summation between $(2\beta-1)e^{(2\beta-1)t}z$ and an analytic function in U_r, $0<r<r_2$, and hence we obtain the absolute continuity requirement of Theorem 2.1. Let $p(z,t)$ be the analytic function defined in U_r by

$$p(z,t)=z\frac{\partial L(z,t)}{\partial z}\Bigg/\frac{\partial L(z,t)}{\partial t}$$

In order to prove that the function $p(z,t)$ has an analytic extension, with positive real part in U, for all $t\in I$, it is sufficient to show that the function $w(z,t)$ defined in U_r by

$$w(z,t)=\frac{p(z,t)-1}{p(z,t)+1}$$

can be continued analytically in U and that $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

By simple calculations, we obtain

$$w(z,t) = \frac{1}{\beta}\left(\frac{f'(e^{-t}z)}{g(e^{-t}z)} - \beta\right) e^{-2\beta t} + \tag{8}$$

$$\frac{1-e^{-2\beta t}}{\beta}\left(\frac{2e^{-t}zf'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} + \frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)} + 1 - \beta\right) +$$

$$\frac{(1-e^{-2\beta t})^2 e^{-2t}z^2}{\beta e^{-2\beta t}}\left(\frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} - h'(e^{-t}z)\right)$$

From (5) and (6) we deduce that the function $w(z,t)$ is analytic in the unit disk U. From (5) and since $\beta > 1/2$ we have

$$|w(z,0)| = \frac{1}{\beta}\left|\ \frac{f'(z)}{g(z)} - \beta\ \right| < 1 \tag{9}$$

$$|w(0,t)| = \left|\ \frac{1-\beta}{\beta}\ \right| < 1. \tag{10}$$

Let t be a fixed number, $t > 0$ and observing that $|e^{-t}z| \le e^{-t} < 1$ for all $z \in \overline{U} = \{z \in C : |z| \le 1\}$ we conclude that the function $w(z,t)$ is analytic in $\overline{U}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in R$ such that

$$|w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|, \tag{11}$$

We denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (8) we get

$$|w(e^{i\theta},t)| = \frac{1}{\beta}\left|\left(\frac{f'(u)}{g(u)} - \beta\right)|u|^{2\beta} + (1-|u|^{2\beta})\right.$$

$$\left(\frac{2uf'(u)h(u)}{g(u)} + \frac{ug'(u)}{g(u)} + 1 - \beta\right)$$

$$\left.+\frac{(1-|u|^{2\beta})^2 u^2}{|u|^{2\beta}}\left(\frac{f'(u)h^2(u)}{g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u)\right)\right|$$

The inequality (6) implies $|w(e^{i\theta},t)| \le 1$ and by using (9), (10) and (11) it follows that $|w(z,t)| < 1$ for all $z \in U$ and $t \ge 0$. From Theorem 2.1 we obtain that the function $L(z,t)$ has an analytic and univalent extension to the whole unit disk U, for all $t \ge 0$. For $t = 0$ we have $L(z,0) = f(z)$, $z \in U$ and therefore the function f is univalent in U.

Suitable choises of the functions g and h in Theorem 3.1 gives us various univalence criteria, between them being the very known Nehari's criterion, Becker's criterion and also Ozaki-Nunokawa's criterion.

Corollary 1. *Let β be a real number, $\beta > 1/2$ and $f \in A$. If for all $z \in U$*

$$\left| \frac{(1-|z|^{2\beta})^2}{|z|^{2\beta}} \cdot \frac{z^2\{f;z\}}{2} + 1 - \beta \right| \le \beta \tag{12}$$

where $\{f;z\}$ is defined by (2), then the function f is univalent in U.

Proof. It results from Theorem 3.1 with $g = f'$ and $h = \frac{-1}{2}\frac{f''}{f'}$.

Remark 1. *If we consider $\beta = 1$ in Corollary 1, the inequality (12) becomes (1) and then we obtain the univalence criterion due to Nehari [2].*

Corollary 2. *Let β be a real number, $\beta > 1/2$ and $f \in A$. If for all $z \in U$*

$$\left| (1-|z|^{2\beta}) \frac{zf''(z)}{f'(z)} + 1 - \beta \right| \le \beta \tag{13}$$

then the function f is univalent in U.

Proof. It results from Theorem 3.1 with $g = f'$ and $h = 0$.

Remark 2. *If we consider $\beta = 1$ in Corollary 2, the inequality (13) becomes (3) and then we obtain the univalence criterion due to Becker [1].*

Corollary 3. *Let β be a real number, $\beta > 1/2$ and $f \in A$. If for all $z \in U$*

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - (\beta - 1) \right| < \beta \tag{14}$$

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - (\beta - 1)|z|^{2\beta} \right| < \beta |z|^{2\beta} \tag{15}$$

then the function f is univalent in U.

Proof. It results from Theorem 3.1 with $g(z) = \left(\frac{f(z)}{z}\right)^2$ and $h(z) = \frac{1}{z} - \frac{f(z)}{z^2}$.

Remark 3. *If we consider $\beta = 1$ in Corollary 3, the inequalities (14) and (15) become (4) and then we obtain the univalence criterion due to Ozaki and Nunokawa [3].*

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