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Cauchy problem for a class of elliptic systems of third order in the plane with Fuchsian differential operator[1]

S.K.Burgumbayeva, A.B.Tungatarov

Abstract

The solutions of a class of complex partial differential equations of third order in the plane with Fuchs type differential operator are constructed in explicit form and the Cauchy problem with prescribed growth at infinity is solved in unbounded angular domains within specified function classes.

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1 Introduction

Let $0 < \varphi \leq 2\pi$, $G = \{z = re^{i\varphi} : 0 \leq r < \infty,\ \ 0 \leq \varphi \leq \varphi_1\}$. Consider in G the equation

$$(1)\qquad 8f_1(\varphi)\bar{z}^3\frac{\partial^3 V}{\partial\bar{z}^3} + 4f_2(\varphi)\bar{z}^2\frac{\partial^2 V}{\partial\bar{z}^2} + 2f_3(\varphi)\bar{z}\frac{\partial V}{\partial\bar{z}} + f_4(\varphi)\overline{V} = f_5(\varphi)r^{\nu},$$

$\nu > 0$ is a real parameter, $f_l(\varphi) \in C[0,\, \varphi_1]$, $(l = \overline{1,5})$, $f_1(\varphi) \neq 0$;

$$\frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),\ \frac{\partial^k V}{\partial\bar{z}^k} = \frac{\partial}{\partial\bar{z}}\left(\frac{\partial^{k-1} V}{\partial\bar{z}^{k-1}}\right),\ (k = \overline{2,3}).$$

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Equation (1) is investigation for $f_1(\varphi) \equiv 0,\ f_2(\varphi) \equiv const \neq 0, f_3(\varphi) \equiv const \neq 0$ in [1, 2] and for $f_2(\varphi) \equiv f_2(\varphi) \equiv 0,\ f_3 \equiv const \neq 0$ in [3].

2 Solutions of the equation

Using these operators in polar coordinates

$$\frac{\partial}{\partial \bar{z}} = \frac{e^{i\varphi}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right),$$

$$\frac{\partial^2}{\partial \bar{z}^2} = \frac{e^{2i\varphi}}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{2i}{r}\frac{\partial^2}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{2i}{r^2}\frac{\partial}{\partial \varphi}\right),$$

$$\frac{\partial^3}{\partial \bar{z}^3} = \frac{e^{3i\varphi}}{8}\left(\frac{\partial^3}{\partial r^3} + \frac{3i}{r}\frac{\partial^3}{\partial r^2 \partial \varphi} - \frac{3}{r^2}\frac{\partial^3}{\partial \varphi^2 \partial r} - \frac{i}{r^3}\frac{\partial^3}{\partial \varphi^3} - \frac{3}{r}\frac{\partial^2}{\partial r^2} - \frac{9i}{r^2}\frac{\partial^2}{\partial r \partial \varphi} + \right.$$

$$\left. + \frac{6}{r^3}\frac{\partial^2}{\partial \varphi^2} + \frac{3}{r^2}\frac{\partial}{\partial r} + \frac{8i}{r^3}\frac{\partial}{\partial \varphi}\right)$$

equation (1) is written in the form

$$f_1(\varphi) r^3 \frac{\partial^3 V}{\partial r^3} + 3i f_1(\varphi) r^2 \frac{\partial^3 V}{\partial r^2 \partial \varphi} - 3 f_1(\varphi) r \frac{\partial^3 V}{\partial r \partial \varphi^2} -$$

$$-i f_1(\varphi) \frac{\partial^3 V}{\partial \varphi^3} + (f_2(\varphi) - 3 f_1(\varphi)) r^2 \frac{\partial^2 V}{\partial r^2} +$$

(2)

$$(2f_2(\varphi) - 9f_1(\varphi)) i r \frac{\partial^2 V}{\partial r \partial \varphi} + (6 f_1(\varphi) - f_2(\varphi)) \frac{\partial^2 V}{\partial \varphi^2} + (3 f_1(\varphi) - f_2(\varphi) + f_3(\varphi)) r \frac{\partial V}{\partial r} +$$

$$+(8 f_1(\varphi) - 2 f_2(\varphi) + f_3(\varphi)) i \frac{\partial V}{\partial \varphi} + f_4(\varphi) \overline{V} = f_5(\varphi) r^{\nu}$$

The solutions of equation (2) are searched for in the Sobolev class [4]

(3) $$W_p^3(G),$$

where $1 < p < \frac{2}{3-\nu}$, if $\nu < 3$ and $p > 1$, if $\nu \geq 3$.

One can see easily, that the function

$$V(r,\ \varphi) = r^{\nu}\psi(\varphi) \tag{4}$$

represents a solution of equation (2) from class (3), if $\psi(\varphi) \in C^3[0,\ \varphi_1]$ is satisfying the equation

$$\psi''' + a_1(\varphi)\psi'' + a_2(\varphi)\psi' + a_3(\varphi)\psi = f_5(\varphi) - f_4(\varphi)\overline{\psi}, \tag{5}$$

where

$$a_1(\varphi) = \frac{6f_1(\varphi) - f_2(\varphi) - 3\nu f_1(\varphi)}{f_1(\varphi)} \cdot i,$$

$$a_2(\varphi) = \frac{\nu(9f_1(\varphi) - 2f_2(\varphi)) - 3\nu(\nu-1)f_1(\varphi) - 8f_1(\varphi) + 2f_2(\varphi) - f_3(\varphi)}{f_1(\varphi)},$$

$$a_3(\varphi) = \frac{\nu(\nu-1)(\nu-2)f_1(\varphi) + \nu(\nu-1)(f_2(\varphi) - 3f_1(\varphi)) + 3f_1(\varphi) - f_2(\varphi) + f_3(\varphi)}{f_1(\varphi)} \cdot i$$

Let $\theta(\varphi) = \{\psi_1(\varphi),\ \psi_2(\varphi),\ \psi_3(\varphi)\}$ be a fundamental system of solutions of homogeneous equation

$$\psi''' + a_1(\varphi)\psi'' + a_2(\varphi)\psi' + a_3(\varphi)\psi = 0 \tag{6}$$

Using the general solution of this equation

$$\psi(\varphi) = c_1\psi_1(\varphi) + c_2\psi_2(\varphi) + c_3\psi_3(\varphi),$$

where c_l, $(l = \overline{1,3})$ are arbitrary constants, in that case by applying the method of variation of constant equation (5) becomes the integral equation

$$\psi(\varphi) = (B\psi)(\varphi) + cJ_0(\varphi) + G_0(\varphi), \tag{7}$$

where

$$J_k(\varphi) = \{J_{1,k}(\varphi),\ J_{2,k}(\varphi),\ J_{3,k}(\varphi)\},\ J_{1,0}(\varphi) = \psi_1(\varphi)$$

$$J_{2,0}(\varphi) = \psi_2(\varphi),\ J_{3,0}(\varphi) = \psi_3(\varphi),\ c = \{c_1,\ c_2,\ c_3\},$$

$$(B\psi)(\varphi) = \int_0^{\varphi} b(\varphi,\tau)\overline{\psi(\tau)}d\tau, \quad G_0(\varphi) = \int_0^{\varphi} g(\varphi,\tau)d\tau,$$

$$b(\varphi,\tau)=f_4(\tau)\cdot\gamma(\varphi,\tau),\quad g(\varphi,\tau)=f_5(\tau)\cdot\gamma(\varphi,\tau),$$

$$\gamma(\varphi,\tau)=\frac{1}{|\triangle(\varphi)|}((\psi_2(\tau)\psi_3'(\tau)-\psi_3(\tau)\psi_2'(\tau))J_{1,0}(\varphi)-(\psi_1(\tau)\psi_3'(\tau)-$$

$$-\psi_3(\tau)\psi_1'(\tau))J_{2,0}(\varphi)+(\psi_1(\tau)\psi_2'(\tau)-\psi_2(\tau)\psi_1'(\tau))J_{3,0}(\varphi)),$$

$\triangle(\varphi)=\begin{pmatrix}\psi_1 & \psi_2 & \psi_3\\ \psi_1' & \psi_2' & \psi_3'\\ \psi_1'' & \psi_2'' & \psi_3''\end{pmatrix}$, $|\triangle(\varphi)|$ is the determinant of the matrix $\triangle(\varphi)$.

For solving equation (7) the functions and operators

$$J_{k,j}(\varphi)=\int\limits_0^{\varphi}b(\varphi,\tau)\overline{J_{k,j-1}(\tau)}d\tau,\quad G_k(\varphi)=\int\limits_0^{\varphi}b(\varphi,\tau)\overline{G_{k-1}(\tau)}d\tau,\quad (j=\overline{1,\infty}),$$

$$(B(B^{k-1}f)(\varphi))(\varphi)=(B^kf)(\varphi),\quad (k=\overline{1,\infty}),\quad (B^0f)(\varphi)=(Bf)(\varphi)$$

are used.

Applying the operator B to both sides of equation (7) gives an expression for the function $(Bf)(\varphi)$. Inserting it again into (7), we have

$$(8)\quad \psi(\varphi)=(B_3\psi)(\varphi)+c(J_0(\varphi)+J_2(\varphi))+\overline{c}J_1(\varphi)+G_0(\varphi)+G_1(\varphi)+G_2(\varphi),$$

Continuing this process $2k+1$ times, we get

$$(9)\qquad \psi(\varphi)=(B^{2k+1}\psi)(\varphi)+c\sum_{n=0}^{k}J_{2n}(\varphi)+\overline{c}\sum_{n=1}^{k}J_{2n-1}(\varphi)+\sum_{n=0}^{2k}G_n(\varphi).$$

As a consequence it is easy to check the inequalities

$$(10)\qquad |(B^k\psi)(\varphi)|\le|b(\varphi,\,\tau)|_1^k\cdot\frac{\varphi^k}{k!},\quad |J_{k,\,j}(\varphi)|\le|b(\varphi,\,\tau)|_1^j\cdot\frac{\varphi^j}{j!},$$

where

$$|b(\varphi,\,\tau)|_1=\max_{0\le\varphi,\,\tau\le\varphi_1}|b(\varphi,\,\tau)|.$$

If pass to the limit as $k\longrightarrow\infty$ in the representations (9), by virtue (10) we receive

$$(11)\qquad \psi(\varphi)=cQ(\varphi)+\overline{c}P(\varphi)+G(\varphi),$$

where

$$Q(\varphi) = (Q_1(\varphi),\, Q_2(\varphi),\, Q_3(\varphi)), \quad P(\varphi) = (P_1(\varphi),\, P_2(\varphi),\, P_3(\varphi)),$$

$$Q_j(\varphi) = \sum_{n=o}^{\infty} J_{j,\,2n}(\varphi), \quad P_j(\varphi) = \sum_{n=1}^{\infty} J_{j,\,2n-1}(\varphi), \quad G(\varphi) = \sum_{n=o}^{\infty} G_n(\varphi), \quad (j = 1,\, 3).$$

For these functions $Q_j(\varphi)$, $P_j(\varphi)$, $(j = \overline{1,3})$ and $G(\varphi)$ it is easy to check the relations

$$Q_j(\varphi) = J_{j,\,0}(\varphi) + \int\limits_0^{\varphi} b(\varphi,\tau)\overline{P_j(\tau)}d\tau, \quad P_j(\varphi) = \int\limits_0^{\varphi} b(\varphi,\tau)\overline{Q_j(\tau)}d\tau,$$

$$G(\varphi) = G_0(\varphi) + \int\limits_0^{\varphi} b(\varphi,\tau)\overline{G(\tau)}d\tau,$$

$$Q_j^{(k)}(\varphi) = \psi_j^{(k)}(\varphi) + \int\limits_0^{\varphi} b_{\varphi^k}^{(k)}(\varphi,\,\tau)\overline{P_j(\tau)}d\tau, \quad P_j^{(k)}(\varphi) = \int\limits_0^{\varphi} b_{\varphi^k}^{(k)}(\varphi,\,\tau)\overline{Q_j(\tau)}d\tau,$$

$$G_j^{(k)}(\varphi) = \int\limits_0^{\varphi} g_{\varphi^k}^{(k)}(\varphi,\,\tau)d\tau + \int\limits_0^{\varphi} b_{\varphi^k}^{(k)}(\varphi,\,\tau)\overline{G(\tau)}d\tau, \quad (k = 1, 2), \tag{12}$$

$$Q_j'''(\varphi) = \psi_j'''(\varphi) - f_4(\varphi)\overline{P_j(\varphi)} + \int\limits_0^{\varphi} b_{\varphi^3}'''(\varphi,\,\tau)\overline{P_j(\tau)}d\tau,$$

$$P_j'''(\varphi) = -f_4(\varphi)\overline{Q_j(\varphi)} + \int\limits_0^{\varphi} b_{\varphi^3}'''(\varphi,\,\tau)\overline{Q_j(\tau)}d\tau, \quad (j = \overline{1,\,3}),$$

$$G_{\varphi^3}'''(\varphi) = f_5(\varphi) + \int\limits_0^{\varphi} g_{\varphi^3}'''(\varphi,\,\tau)d\tau - f_4(\varphi)\overline{G(\varphi)} + \int\limits_0^{\varphi} b_{\varphi^3}'''(\varphi,\,\tau)\overline{G(\tau)}d\tau.$$

It is also easy to check the equalities

$$b(\varphi,\varphi)=0,\quad b'_{\varphi}(\varphi,\varphi)=0,\quad b''_{\varphi^2}(\varphi,\varphi)=-b(\varphi),$$

(13)

$$g(\varphi,\varphi)=0,\quad g'_{\varphi}(\varphi,\varphi)=0,\quad g''_{\varphi}(\varphi,\varphi)=f_5(\varphi)$$

By using formula (12) and (13) we receive the equalities

$$P_j'''(\varphi)+a_1P_j''(\varphi)+a_2P_j'(\varphi)+a_3P_j(\varphi)=-f_4(\varphi)\overline{Q_j(\varphi)},$$

$$Q_j'''(\varphi)+a_1Q_j''(\varphi)+a_2Q_j'(\varphi)+a_3Q_j(\varphi)=-f_4(\varphi)\overline{P_j(\varphi)},$$

$$G'''(\varphi)+a_1G''(\varphi)+a_2G'(\varphi)+a_3G(\varphi)=f_5(\varphi)-f_4(\varphi)\overline{G(\varphi)}.$$

Hence, we receive

$$\psi'(\varphi)=c\theta'_{\varphi}+c\int\limits_0^{\varphi}b'_{\varphi}(\varphi,\,\tau)\overline{P_j(\tau)}d\tau+\bar{c}\int\limits_0^{\varphi}b'_{\varphi}(\varphi,\,\tau)\overline{Q_j(\tau)}d\tau+$$

$$+\int\limits_0^{\varphi}g'_{\varphi}(\varphi,\,\tau)d\tau+\int\limits_0^{\varphi}b'_{\varphi}(\varphi,\,\tau)\overline{G(\tau)}d\tau,$$

$$\psi''(\varphi)=c\theta''_{\varphi^2}+c\int\limits_0^{\varphi}b''_{\varphi^2}(\varphi,\,\tau)\overline{P_j(\tau)}d\tau+\bar{c}\int\limits_0^{\varphi}b''_{\varphi^2}(\varphi,\,\tau)\overline{Q_j(\tau)}d\tau+$$

$$+\int\limits_0^{\varphi}g''_{\varphi^2}(\varphi,\,\tau)d\tau+\int\limits_0^{\varphi}b''_{\varphi^2}(\varphi,\,\tau)\overline{G(\tau)}d\tau,$$

(14) $$\psi'''(\varphi)=c\theta'''_{\varphi^3}-cf_4(\varphi)\overline{P_j(\varphi)}+c\int\limits_0^{\varphi}b'''_{\varphi^3}(\varphi,\,\tau)\overline{P_j(\tau)}d\tau-\bar{c}f_4(\varphi)\overline{Q_j(\varphi)}+$$

$$+\bar{c}\int\limits_0^{\varphi}b'''_{\varphi^3}(\varphi,\,\tau)\overline{Q_j(\tau)}d\tau+f_5(\varphi)+\int\limits_0^{\varphi}g'''_{\varphi^3}(\varphi,\,\tau)d\tau-f_4(\varphi)\overline{G(\varphi)}+\int\limits_0^{\varphi}b'''_{\varphi^3}(\varphi,\,\tau)\overline{G(\tau)}d\tau.$$

Since $J_{j,0}(\varphi)$, $b(\varphi,\tau)$, $g(\varphi,\,\tau)$ represent a solution of equation (6), then by virtue of formula (14) we receive, that the function $\psi(\varphi)$, given by formula (11) is satistying equation (5).

Using inequality (10), it is easy to receive the estimates

$$|Q_j(\varphi)| \leq |\psi_j(\varphi_1)|ch(|b(\varphi,\,\tau)|_1), \quad |P_j(\varphi)| \leq |\psi_j(\varphi_1)|sh(|b(\varphi,\,\tau)|_1),$$

(15)

$$|G(\varphi)| \leq \varphi_1 |g(\varphi,\,\tau)|_1 \exp(|b(\varphi,\,\tau)|_1), \quad (j = \overline{1,\,3}).$$

By estimate (15) we can assure that the function $V(r,\varphi)$, given by formulas (4), (11), is solving equation (1) in the class (3).

Thus, the following result holds:

Theorem 1. *Equation* (1) *has a solution in the class* (3)*, which is given by formula* (4)*,* (11)*.*

3 Cauchy problem

Consider the Cauchy problem with prescribed growth at infinity for system (1).

Problem C. *Find a solution of equation* (1) *from the class* (3)*, satisfying the conditions*

$$\alpha_{k1}V(r,\,0) + \alpha_{k2}\frac{\partial V}{\partial\varphi}(r,\,0) + \alpha_{k3}\frac{\partial^2 V}{\partial\varphi^2}(r,\,0) = \beta_k r^{\nu}, \quad (k = \overline{1,\,3}) \tag{16}$$

$$|V(r,\,\varphi)| = O(r^{\nu}), \quad r \longrightarrow \infty, \tag{17}$$

where a_{kj}, $(k = (\overline{1,3}),\ j = (\overline{1,3}))$ *are given real numbers.*

For solving problem **C** formulas (4), (11) are used. In that case (17) holds automatically. The constants c_1, c_2, c_3 in formula (11) are determined, in order that the solution of equation (1), represented in the form (4) and

(11), satisfies condition (16). For that, insert function $V(r,\varphi)$ according to formulas (4), (11) into (16). Thus we receive a system of linear algebraic equations in c_1, c_2, c_3:

$$(\alpha\triangle(0))c^T = \beta, \tag{18}$$

where

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \quad c^T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Solving system (18) under $|\alpha\triangle(0)| \neq 0$, we receive

$$c^T = (\alpha\triangle(0))^{-1}\beta \tag{19}$$

Thus, the following result holds:

Theorem 2. *Let the roots the characteristic equation* (6) *be different and different from zero. Under* $|\alpha\triangle(0)| \neq 0$ *the Cauchy problem has a unique solution, which is given by formulas* (4), (11) *and* (19).

If $|\alpha\triangle(0)| = 0$ for the solvability of the algebraic systems (18) the conditions:

$$|A_1| = 0, \quad |A_2| = 0, \quad |A_3| = 0 \tag{20}$$

are necessary and sufficient. Here A_j ia a matrix, which is received by replacing the i matrix column of the matrix $\alpha\triangle(0)$ by the column β.

Thus, the following result holds:

Theorem 3. *Let* $|\alpha\triangle(0)| = 0$*, then for the solvability of the Cauchy problem the condition* (20) *is necessary and sufficiently. In that case the Cauchy problem has an infinity number of solutions. They are given by formulas* (4), (11)*, where c is determined from equation* (18) *under condition* (20).

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Eurasian National University
Munaitpasov st.,5 010008 Astana
KAZAKHSTAN
E-mail: tun-mat@list.ru
E-mail: saulenai@yandex.ru