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About Fejér′s sum

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Abstract

In this paper we will show an improvement of Fejér inequality,

$$\sum_{k=1}^{n} \frac{\sin k\phi}{k} \geq \frac{16\pi(1 - P_n(\cos\phi))}{(n+1)\sin\phi} \quad or$$

$$\sum_{k=1}^{n} \frac{\sin k\phi}{k} \geq \frac{\pi(1 - \cos n\phi)}{2^{2n-5}(n+1)\sin\phi}, \ where$$

$P_k(x) = \frac{1}{2^k k!}\left(\frac{d}{dx}\right)^n (x^2 - 1)^n$ is Legendre polynomial

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1 Main results

Using the quadrature formulas of Bouzitat, we present an improvement of Fejér inequality:

$$\sum_{k=1}^{n} \frac{\sin k\phi}{k} > 0, \ \forall\phi \in (0, \pi), \ n \in \mathbb{N}^*.$$

For $(\alpha,\beta)\in(-1,+\infty)\times(-1,+\infty)$ we define the functional $I^{(\alpha,\beta)}:\Pi\to\mathbb{R}$. It is known that

$$I^{(\alpha,\beta)}(f)=\int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta}dx,\ \ f\in\Pi_n$$

where Π_n is the set of all polynomials of degree less or equal to n. The following result is well-known.

Theorem 1. *Let $P\in\Pi_n$ with degree $[P]=n$ and $P\ge 0$ on $[-1,1]$. If $m=\left[\frac{n}{2}\right]$, $d=\left[\frac{n+1}{2}\right]$ then*

$$I^{(\alpha,\beta)}(P)\ge 2^{\alpha+\beta+1}B(\alpha+1,\beta+1)\frac{\Gamma(m+1)(\beta+1)_d}{(\alpha+2)_m(\alpha+\beta+2)_d}$$

We note with P_k the Legendre polynomial, and with U_k the Cebyshev second species polynomial. We have

$$P_k(x)=\frac{1}{2^k k!}\left(\frac{d}{dx}\right)^n(x^2-1)^n,$$

$$U_k(x)=\frac{\sin((k+1)\arccos x)}{(k+1)\sqrt{1-x^2}}.$$

It is well-known (see [9]) that $|P_n(x)|\le 1$, $x\in(-1,1)$ and $|P_n(\pm)|=1$.

Theorem 2. *For all $x\in(-1,1)$ and $n\in\mathbb{N}$ the inequality*

$$\sum_{k=0}^{n} U_k(x)\ge\frac{16\pi}{n+2}\cdot\frac{1-P_{n+1}(x)}{1-x^2}\quad \textit{holds.}$$

Proof. For $x\in(-1,1)$. A. Lupaş [6] established the identity

$$\sum_{k=1}^{n}\frac{\sin(k\arccos x)}{k}=\frac{\sqrt{1-x}}{2}\int_{-1}^{1}\frac{1-P_n(y)}{1-y}\cdot\frac{dy}{\sqrt{x-y}}.$$

For $y(t,x) := \frac{x+1}{2}t + \frac{x-1}{2}$, we have

$$\sum_{k=0}^{n} U_k(x) = \frac{\sqrt{2}}{2(1+x)} \int_{-1}^{1} H_n(t,x) \frac{dt}{\sqrt{1-t}},\ H_n(t,x) := \frac{1 - P_{n+1}(y(t,x))}{1 - y(t,x)}.$$

We observe that the $H_n(\cdot,x)$ is a polynomial of effective degree n, and, additionally $H_n(\cdot,x) \geq 0$ on $[-1,1]$. But $H_n(1,x) = \frac{1-P_{n+1}(x)}{1-x}$. We have

$$\sum_{k=0}^{n} U_k(x) \geq \mu_n \cdot \frac{1 - P_{n+1}(x)}{1 - x^2}$$

where denoting $m = [n/2]$, $d = [(n+1)/2] = n - m$ we have

$$\mu_n := \frac{2^{2n+3} m!(m+1)!(d+1)!d!}{(2m+2)!(2d+2)!} = \frac{8\pi\Gamma(m+1)\Gamma(d+1)}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(d+\frac{3}{2}\right)}.$$

From the convexity of $\log\Gamma : (0,\infty) \to (0,\infty)$ we have the next evaluations

$$\frac{1}{\sqrt{x+\frac{1}{2}}} \leq \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} \leq \frac{1}{\sqrt{x}}.$$

which implies:

$$\frac{8\pi}{\sqrt{(m+1)(d+1)}} \leq \mu_n \leq \frac{8\pi}{\sqrt{\left(m+\frac{1}{2}\right)\left(d+\frac{1}{2}\right)}},\ \mu_n \geq \frac{16\pi}{n+2},$$

So $\sum_{k=0}^{n} U_k(x) \geq \frac{16\pi}{n+2} \cdot \frac{1-P_{n+1}(x)}{1-x^2}$ and thus, the assertion is proved.

Corollary 1. *For all $\phi \in [0,\pi]$ the following inequality is true:*

$$\sum_{k=1}^{n} \frac{\sin k\phi}{k} \geq \frac{16\pi(1 - P_n(\cos\phi))}{(n+1)\sin\phi}. \tag{1}$$

For proving our main result we will need the following lemmas:

Lemma 1. (T. Koorwinder [1]). *Let* $g_{k,n}\binom{\alpha,\beta}{a,b}$ *be the coefficients of the following development:*

$$R_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} g_{k,n}\binom{\alpha,\beta}{a,b} R_k^{(a,b)}(x).$$

If $a \le b, \alpha+\beta \ge a+b$ *'si* $\beta-\alpha \le b-a$, *then* $g_{n,k}\binom{\alpha,\beta}{a,b} \ge 0,\ 0 \le k \le n.$

Lemma 2. *If* $\alpha \ge a \ge -\frac{1}{2}$, *and*

$$T_n^{(\alpha)}(x) := \frac{(\alpha+1)_n}{(n+2\alpha+1)_n}(1 - R_n^{(\alpha,\alpha)}(x)),$$

then $T_n^{(\alpha)}(x) \ge T_n^{(a)}(x)$ *for all* $x \in [-1,1]$.

Proof. Taking into account the hypergeometric form of the Jacobi polynomials, through identification of x^n coefficients we deduce:

$$g_{n,n}\binom{\alpha,\beta}{a,b} = \frac{(a+1)_n(n+\alpha+\beta+1)_n}{(\alpha+1)_n(n+a+b+1)_n}. \tag{2}$$

On the other hand, for $x=1$ and taking into account that $R_k^{(a,b)}(1)=1$ we deduce that $\sum\limits_{k=0}^{n} g_{k,n}\binom{\alpha,\beta}{a,b} = 1$. We consider $a=b \ge -\frac{1}{2}$ and $\alpha=\beta$. For $\alpha \ge a \ge -\frac{1}{2}$ and $x \in [-1,1]$ we have

$$\begin{aligned} 1 - R_n^{(\alpha,\alpha)}(x) &= \sum_{k=0}^{n} g_{k,n}\binom{\alpha,\alpha}{a,a}\left(1 - R_k^{(a,a)}(x)\right) \\ &\ge g_{n,n}\binom{\alpha,\alpha}{a,a}\left(1 - R_n^{(a,a)}(x)\right) \end{aligned}$$

and $T_n^{(\alpha)}(x) \ge T_n^{(a)}(x)$, $\forall x \in [-1,1]$. So, if we choose $\alpha=0$ and $a \in [-1/2, 0]$ for $x \in [-1,1]$ we have

$$1 - P_n(x) \ge \frac{(a+1)_n(n+1)_n}{(n+2a+1)_n n!}(1 - R_n^{(a,a)}(x))$$

and

$$1 - P_n(x) \geq \frac{1}{2^{2n-1}}(1 - T_n(x)).$$

Corollary 2. *For all $\phi \in [0, \pi]$ the next inequality is true*

$$\sum_{k=1}^{n} \frac{\sin k\phi}{k} \geq \frac{\pi(1 - \cos n\phi)}{2^{2n-5}(n+1)\sin\phi}$$

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