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Extremal problems with polynomials

Ana Maria Acu, Mugur Acu, Arif Rafiq

Abstract

Using quadrature formulae of the Gauss-Lobatto and Gauss-Radau type, we give some new results for extremal problems with polynomials. Let $\tilde{H}^{(\alpha,\beta)}$ be the class of real polynomials $p_{n-1} \in \prod_{n-1}$, such that

$$|p_{n-1}(x_i)| \leq \left|\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}(x_i)\right|, \ i = \overline{1,n},$$

where by $\tilde{\mathcal{P}}_n^{(\alpha,\beta)}$ we denote the nth Jacobi polynomial and the x_i are the zeroes of $\tilde{\mathcal{P}}_n^{(\alpha,\beta)}$. We give exact estimation of certain weighted L^2-norms of the kth derivative of polynomials with there are in the class $\tilde{H}^{(\alpha,\beta)}$.

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1 Introduction

Let $\prod_n$ be the space of polynomials of degree not greater than n. By $P_n^{(\alpha,\beta)}(x)$, where n is a non-negative whole number and $\alpha, \beta > -1$, we denote the n-th Jacobi polynomial. It is know that Jacobi polynomials with the same parameters α and β are orthogonal on $[-1,1]$ with respect to the weight function $\rho(x)=(1-x)^{\alpha}(1+x)^{\beta}$. We shall need the following properties of Jacobi polynomials ([9]):

$$P_n^{(\alpha,\beta)}(1)=\begin{pmatrix} n+\alpha \\ n \end{pmatrix}, \tag{1}$$

$$P_n^{(\alpha,\beta)}(-1)=(-1)^n\begin{pmatrix} n+\beta \\ n \end{pmatrix}, \tag{2}$$

$$\frac{d}{dx}\left\{P_n^{(\alpha,\beta)}\right\}(x)=\frac{1}{2}\cdot(n+\alpha+\beta+1)\cdot P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{3}$$

Let $\tilde{P}_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, normalized to have the leading coefficient equal 1. Then

$$\tilde{P}_n^{(\alpha,\beta)}(x)=2^n n!\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\cdot P_n^{(\alpha,\beta)}(x). \tag{4}$$

From the relations (3) and (4) we obtain

$$\frac{d}{dx}\left\{\tilde{P}_n^{(\alpha,\beta)}\right\}(x)=n\cdot\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{5}$$

Lemma 1. *The following formulae hold:*

$$\left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)=\frac{2^{n-k-2}(n-1)!\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\alpha+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(n-k-1)\Gamma(k+\alpha+3)}$$

and

$$\left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(-1)=(-1)^{n-k-2}\frac{2^{n-k-2}(n-1)!\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(n-k-1)\Gamma(k+\beta+3)}.$$

The following problem was raised by Turán:

Problem. *Let $\varphi(x) \geq 0$ for $-1 \leq x \leq 1$ and consider the class $\prod_{n,\varphi}$ of all polynomials of degree n such that $|p_n(x)| \leq \varphi(x)$ for $-1 \leq x \leq 1$. How large can $\max\limits_{x\in[-1,1]} \left|p_n^{(k)}(x)\right|$ be if p_n is an arbitrary polynomial in $\prod_{n,\varphi}$?*

He pointed out two cases: $\varphi(x) = \sqrt{1-x^2}$ and $\varphi(x) = 1-x^2$.

Let as denote by

$$G_{n-1}(x) = \frac{\sqrt{2}}{2n(2n+1)} \left[(2n\eta+\nu)W_n'(x) - (2n+1)\nu W_{n-1}'(x)\right],$$

where

$$W_n(x) = \frac{(2n)!!}{(2n-1)!!}\mathcal{P}_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x).$$

Let $Z_{\eta,\nu}^{W,\varphi}$ be the class of polynomials $p_{n-1} \in \prod_{n-1}$ such that

$$|p_{n-1}(x_i)| \leq |G_{n-1}(x_i)|,$$

where the x_i are the zeroes of $W_n(x)$.

In paper [5], the author obtains the following results:

Theorem 1.[5] *If $p_{n-1} \in Z_{\eta,\nu}^{W,\varphi}$, then we have*

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[p_{n-1}(x)\right]^2 dx \leq \frac{2\pi n\left[2(\eta^2+\nu^2+\eta\nu)(n+1)-3\nu^2\right]}{3(2n+1)},$$

$$\int_{-1}^{1} (1-x)\sqrt{1-x^2}\left[p_{n-1}'(x)\right]^2 dx \leq \frac{2\pi(n^3-n)}{15(2n+1)}\left[2(2\eta^2+\eta\nu+2\nu^2)n+8\eta^2+4\eta\nu-7\nu^2\right],$$

with equality for $p_{n-1} = G_{n-1}$.

Theorem 2.*[5] If $p_{n-1} \in Z_{1,0}^{W,\varphi}$ and $0 \le |b| \le a$, then we have*

$$\int_{-1}^{1} (a+bx)(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}} \left[p_{n-1}^{(k)}(x)\right]^2 dx \\ \le \frac{2\pi(n+k+1)!(2ak+2a-b)}{(n-k-1)!(2n+1)(2k+1)(2k+3)},$$

$k = 1, \ldots, n-2$ with equality for $p_{n-1} = \dfrac{\sqrt{2}}{2n+1} W_n'(x)$.

Let $\tilde{H}^{(\alpha,\beta)}$ be the class of real polynomials $p_{n-1} \in \prod_{n-1}$, such that

$$|p_{n-1}(x_i)| \le \left|\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x_i)\right|, \; i = \overline{1,n}\,,$$

where the x_i are the zeros of $\tilde{P}_n^{(\alpha,\beta)}$.

In this paper we want to give exact estimation of certain weighted L^2-norms of the kth derivative of polynomials with there are in the class $\tilde{H}^{(\alpha,\beta)}$.

To obtain our results, we need the following result of Duffin and Schaeffer [1].

Lemma 2.*(Duffin and Schaeffer) If $q(x) = c \cdot \prod_{i=1}^{n}(x - x_i)$ is a polynomial of degree n with n distinct real zeroes and if $p \in \prod_n$ is such that*

$$|p'(x_i)| \le |q'(x_i)|\,, \; i = \overline{1,n},$$

then for $k = \overline{1, n-1}$,

$$\left|p^{(k+1)}(x)\right| \le \left|q^{(k+1)}(x)\right|$$

whenever $q^{(k)}(x) = 0$.

Using the result of Duffin and Schaeffer can be obtained the following result:

Lemma 3. *If $p_{n-1} \in \tilde{H}^{(\alpha,\beta)}$, then for $k = \overline{0, n-1}$ we have*

$$\left| p_{n-1}^{(k+1)}(y_j^{(k)}) \right| \leq \left| \left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)} \right)^{(k+1)} (y_j^{(k)}) \right|, \tag{6}$$

whenever

$$\left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)} \right)^{(k)} (y_j^{(k)}) = 0 \text{ for } j = \overline{1, n-k-1} \text{ and}$$

$$\left| p_{n-1}^{(k+1)}(1) \right| \leq \left| \left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)} \right)^{(k+1)} (1) \right|, \tag{7}$$

$$\left| p_{n-1}^{(k+1)}(-1) \right| \leq \left| \left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)} \right)^{(k+1)} (-1) \right|. \tag{8}$$

2 Main results

Let

$$\int_a^b \rho(x) f(x) dx = \sum_{i=1}^{n} A_i f(a_i) + C f(c) + \mathcal{R}[f] \tag{9}$$

be a quadrature formula of the Gauss-Radau type, where ρ is a nonnegative weight function and $c = a$ or $c = b$. The nodes $a_i \in (a, b),\ i = \overline{1, n}$, will be determined from the condition that the quadrature formula (9) has maximal degree of exactness.

The maximal degree of exactness, $r = 2n$, of quadrature formula (9) is obtained if and only if the nodes a_i, $i = \overline{1, n}$ are the zeroes of an orthogonal polynomial of degree n with respect to the weight function $w(x) = \rho(x)\,|x - c|$, $x \in (a, b)$.

Let

$$\int_a^b \rho(x) f(x) dx = \sum_{i=1}^{n} \tilde{A}_i f(a_i) + \tilde{C} f(p_1) + \tilde{D} f'(p_1) + \tilde{E} f(p_2) + \tilde{\mathcal{R}}[f] \tag{10}$$

be a quadrature formula of the Gauss-Lobatto type, where ρ is a nonnegative weight function and $p_1 = a$, $p_2 = b$ or $p_1 = b$, $p_2 = a$.

The maximal degree of exactness, $r = 2n + 2$, of quadrature formula (10) is obtained if and only if the nodes a_i, $i = \overline{1, n}$, are the zeroes of an orthogonal polynomial of degree n with respect to the weight function $w(x) = \rho(x)(x - p_1)^2\,|x - p_2|$, $x \in (a, b)$.

Lemma 4. *For any given n and k, $0 \leq k \leq n-1$, let $y_i^{(k)}$, $i = \overline{1, n-k-1}$ be the zeroes of $\mathcal{P}_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$. Then the quadrature formulae*

$$\text{(11)}\quad \int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1} f(x)dx = C_1 f(1) + \sum_{i=1}^{n-k-1} A_{1,i} f\left(y_i^{(k)}\right) + \mathcal{R}[f],$$

where

$$\begin{aligned} A_{1,i} &> 0, \\ \text{(12)}\quad C_1 &= 2^{2k+\alpha+\beta+2} \frac{\Gamma(k+\alpha+1)\Gamma(n+\beta+1)\Gamma(n-k)\Gamma(\alpha+k+2)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+2)}; \end{aligned}$$

$$\begin{aligned} \text{(13)}\int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1} f(x)dx &= \tilde{C}_1 f(1) + \tilde{D}_1 f'(1) + \tilde{E}_1 f(-1) \\ &+ \sum_{i=1}^{n-k-2} \tilde{A}_{1,i} f\left(y_i^{(k+1)}\right) + \tilde{\mathcal{R}}[f], \end{aligned}$$

where

$$
\begin{aligned}
\tilde{A}_{1,i} &> 0, \\
(14)\ \tilde{C}_1 &= 2^{2k+\alpha+\beta+2}\cdot\frac{\Gamma(k+\alpha+1)\Gamma(n+\beta+1)\Gamma(n-k-1)\Gamma(\alpha+k+3)}{(\alpha+k+3)\Gamma(n+\beta+k+\alpha+3)\Gamma(n+\alpha+1)} \\
&\cdot\ \{(n-k-1)(n+k+\alpha+\beta+2)(\alpha+k+3)+(k+\alpha+1)(\alpha+k+3) \\
&+\ (k+\alpha+1)(n+\alpha+\beta+k+3)(n-k-2)\}\,, \\
(15)\ \tilde{D}_1 &= -2^{2k+\alpha+\beta+3}\frac{\Gamma(n-k-1)\Gamma(k+\alpha+3)\Gamma(k+\alpha+2)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+k+\alpha+3)}, \\
(16)\ \tilde{E}_1 &= 2^{2k+\alpha+\beta+2}\frac{\Gamma(n-k-1)\Gamma(\beta+k+3)\Gamma(k+\beta+2)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+k+3)},
\end{aligned}
$$

have degree of exactness $2n-2k-2$.

Proof. If in the quadrature formula of the Gauss-Radau type (9) we consider $a=-1$, $b=1$, $\rho(x)=(1-x)^{k+\alpha}(1+x)^{k+\beta+1}$, $n\to n-k-1$, $c=1$, then the quadrature formula (11) has the maximal degree of exactness $r=2n-2k-2$.

In order to compute the coefficient C_1, we need the following formulae:

$$
(17)\qquad \begin{aligned}
&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\lambda}P_m^{(\alpha,\beta)}(x)dx= \\
&\frac{(-1)^m 2^{\alpha+\lambda+1}\Gamma(\lambda+1)\Gamma(m+\alpha+1)\Gamma(\beta-\lambda+m)}{\Gamma(m+1)\Gamma(\beta-\lambda)\Gamma(m+\alpha+\lambda+2)},\ \lambda<\beta,
\end{aligned}
$$

and

$$
(18)\qquad \begin{aligned}
&\int_{-1}^{1}(1-x)^{\lambda}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)dx= \\
&\frac{2^{\beta+\lambda+1}\Gamma(\lambda+1)\Gamma(m+\beta+1)\Gamma(\alpha-\lambda+m)}{\Gamma(m+1)\Gamma(\alpha-\lambda)\Gamma(m+\beta+\lambda+2)},\ \lambda<\alpha.
\end{aligned}
$$

If in the quadrature formula (11) we consider $f(x)=\mathcal{P}_{n-k-1}^{(\alpha+k+1,\beta+k+1)}(x)$, then by using the relation (18) we obtain (12).

If in the quadrature formula of Gauss-Lobatto type (10), we consider $a = -1$, $b = 1$, $\rho(x) = (1-x)^{k+\alpha}(1+x)^{k+\beta+1}$, $n \to n-k-2$, $p_1 = 1$, $p_2 = -1$, then the quadrature formula (11) has the maximal degree of exactness $r = 2n-2k-2$.

If in the quadrature formula (13) we consider $f(x) = (1-x)^2 \mathcal{P}_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, $f(x) = (1-x)(1+x)\mathcal{P}_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, respective $f(x) = (1+x)\mathcal{P}_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, then by using the formulae (17) and (18) we obtain the coefficients (16), (15), respective (14).

Theorem 3. *If* $p_{n-1} \in \tilde{H}^{(\alpha,\beta)}$, *then*

$$\int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx \le 2^{2n+\alpha+\beta-2}(n-1)!^2 \cdot \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+3)}{\Gamma^2(2n+\alpha+\beta+1)\Gamma(n-k-1)} \cdot \left\{\frac{1}{k+\alpha+2} + \frac{1}{k+\beta+2} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{(k+\alpha+2)(k+\alpha+1)} - \frac{(n+\alpha+\beta+k+3)(n-k-2)}{(k+\alpha+3)(k+\alpha+2)}\right\} \tag{19}$$

holds for all $k = \overline{0, n-2}$, *with equality for* $p_{n-1} = \tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}$.

Proof. According to Lemma 3 and positivity of the coefficients in the quadrature formulae of Gauss type, we have

$$\begin{aligned} &\int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx \\ &= C_1\left[p_{n-1}^{(k+1)}(1)\right]^2 + \sum_{i=1}^{n-k-1} A_{1,i}\left[p_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right]^2 \\ &\le C_1\left[\left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\right]^2 + \sum_{i=1}^{n-k-1} A_{1,i}\left[\left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_i^{(k)}\right)\right]^2 \end{aligned}$$

$$= \int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1}\left[\left(\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x)\right)^{(k+1)}\right]^2 dx$$
$$= \tilde{C}_1\left[\left(\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\right]^2 + 2\tilde{D}_1\left(\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\cdot\left(\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+2)}(1)$$
$$+\tilde{E}_1\left[\left(\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(-1)\right]^2 + \sum_{i=1}^{n-k-2}\tilde{A}_{i,1}\left[\left(\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_i^{(k+1)}\right)\right]^2.$$

Since $\left(\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_i^{(k+1)}\right) = 0$, $i = \overline{1, n-k-2}$ and by using Lemma 1 we obtain the inequality (19).

If in Theorem 3 we choose $\alpha = \beta = -\dfrac{1}{2}$ and $k = 0$, we obtain the following result:

Corollary 1. *If* $|p_{n-1}(x_i)| \le \dfrac{1}{2^{n-1}\sqrt{1-x_i^2}}$, $i = 1, \ldots, n$, *where* x_i *are the zeroes of* $\mathcal{P}_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$, *then*

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\left[p'_{n-1}(x)\right]^2 \le \frac{n^5-n}{15\cdot 2^{2n-3}}\cdot\pi.$$

Lemma 5. *For any given* n *and* k, $0 \le k \le n-1$, *let* $y_i^{(k)}$, $i = \overline{1, n-k-1}$ *be the zeroes of* $\mathcal{P}_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$. *Then the quadrature formulae*

$$(20)\quad \int_{-1}^{1}(1-x)^{k+\alpha+1}(1+x)^{k+\beta}f(x)dx = C_2 f(-1) + \sum_{i=1}^{n-k-1} A_{2,i} f\left(y_i^{(k)}\right) + \mathcal{R}[f],$$

where

$$A_{2,i} > 0,$$
$$(21)\quad C_2 = 2^{2k+\alpha+\beta+2}\frac{\Gamma(k+\beta+1)\Gamma(n+\alpha+1)\Gamma(n-k)\Gamma(\beta+k+2)}{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+k+2)};$$

$$(22)\quad \int_{-1}^{1}(1-x)^{k+\alpha+1}(1+x)^{k+\beta}f(x)dx = \tilde{C}_2 f(-1) + \tilde{D}_2 f'(-1) + \tilde{E}_2 f(1) + \sum_{i=1}^{n-k-2}\tilde{A}_{2,i} f\left(y_i^{(k+1)}\right) + \tilde{\mathcal{R}}[f],$$

where

$$
\begin{aligned}
\tilde{A}_{2,i} &> 0,\\
(23)\ \tilde{C}_2 &= 2^{2k+\alpha+\beta+2}\cdot\frac{\Gamma(k+\beta+1)\Gamma(n+\alpha+1)\Gamma(n-k-1)\Gamma(\beta+k+3)}{(\beta+k+3)\Gamma(n+\alpha+k+\beta+3)\Gamma(n+\beta+1)}\\
&\cdot\ \{(n-k-1)(n+k+\beta+\alpha+2)(\beta+k+3)+(k+\beta+1)(\beta+k+3)\\
&+\ (k+\beta+1)(n+\alpha+\beta+k+3)(n-k-2)\},\\
(24)\ \tilde{D}_2 &= 2^{2k+\alpha+\beta+3}\frac{\Gamma(n-k-1)\Gamma(k+\beta+3)\Gamma(k+\beta+2)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+\alpha+k+\beta+3)},\\
(25)\ \tilde{E}_2 &= 2^{2k+\alpha+\beta+2}\frac{\Gamma(n-k-1)\Gamma(\alpha+k+3)\Gamma(k+\alpha+2)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+3)},
\end{aligned}
$$

have degree of exactness $2n-2k-2$.

Proof. If in the quadrature formula of the Gauss-Radau type (9) we consider $a=-1$, $b=1$, $\rho(x)=(1-x)^{k+\alpha+1}(1+x)^{k+\beta}$, $n\to n-k-1$, $c=-1$, then the quadrature formula (20) has the maximal degree of exactness $r=2n-2k-2$.

If in the quadrature formula (20) we consider $f(x)=\mathcal{P}_{n-k-1}^{(\alpha+k+1,\beta+k+1)}(x)$, then by using the relation (17) we obtain (21).

If in the quadrature formula of Gauss-Lobatto type (10), we consider $a=-1$, $b=1$, $\rho(x)=(1-x)^{k+\alpha+1}(1+x)^{k+\beta}$, $n\to n-k-2$, $p_1=-1$, $p_2=1$, then the quadrature formula (22) has the maximal degree of exactness $r=2n-2k-2$.

If in the quadrature formula (22) we consider $f(x)=(1+x)^2\mathcal{P}_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, $f(x)=(1-x)(1+x)\mathcal{P}_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, respective $f(x)=(1-x)\mathcal{P}_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, then by using the formulae (17) and (18) we obtain the coefficients (25), (24), respective (23).

Theorem 4. *If* $p_{n-1} \in \tilde{H}^{(\alpha,\beta)}$*, then*

$$(26)\quad \int_{-1}^{1}(1-x)^{k+\alpha+1}(1+x)^{k+\beta}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx \le 2^{2n+\alpha+\beta-2}(n-1)!^2 \cdot\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+k+3)}{\Gamma^2(2n+\alpha+\beta+1)\Gamma(n-k-1)}\cdot\left\{\frac{1}{k+\beta+2}+\frac{1}{k+\alpha+2} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{(k+\beta+2)(k+\beta+1)}-\frac{(n+\alpha+\beta+k+3)(n-k-2)}{(k+\beta+3)(k+\beta+2)}\right\}$$

holds for all $k=\overline{0,n-2}$*, with equality for* $p_{n-1}=\tilde{\mathcal{P}}_{n-1}^{(\alpha+1,\beta+1)}$.

Proof. The proof follows in a similar way with the proof of Theorem 3.

If in Theorem 4 we choose $\alpha=\beta=-\dfrac{1}{2}$ and $k=0$, we obtain the following result:

Corollary 2. *If* $|p_{n-1}(x_i)| \le \dfrac{1}{2^{n-1}\sqrt{1-x_i^2}}$*,* $i=1,\dots,n$*, where* x_i *are the zeroes of* $\mathcal{P}_n^{(-\frac{1}{2},-\frac{1}{2})}$*, then*

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\left[p_{n-1}'(x)\right]^2 \le \frac{n^5-n}{15\cdot 2^{2n-3}}\cdot\pi.$$

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Ana Maria Acu, Mugur Acu
"Lucian Blaga" University
Department of Mathematics
Sibiu, Romania
e-mail: acuana77@yahoo.com

Arif Rafiq
COMSATS Institute of Information Technology
Department of Mathematics
Islamabad, Pakistan
e-mail: arafiq@comsats.edu.pk