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A Note on Salagean-type Harmonic Univalent Functions[1]

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Abstract

A class of Salagean-type harmonic univalent functions is defined and investigated. Coefficient conditions, extreme points, distortion bounds, convex combination and radii of convexity for this class, are obtained.

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1 Introduction

Let S_H denote the family of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$(1) \qquad h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1.$$

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The class S_H was introduced by Clunie and Sheil-Small [1].

Subclasses of harmonic univalent functions have been studied by many authors and in particular Salagean-type harmonic functions have been investigated in [2, 4].

For $f = h + \bar{g}$ given by (1), Jahangiri et al. [2] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \tag{2}$$

where $D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k$ and $D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k$.
In this paper we introduce a new class $G_H(m, n, \gamma)$ of harmonic functions that includes the class in [3] for specific values of m and n. For $0 \leq \gamma < 1$, α real, $m \in N$, $n \in N_0$, $m > n$ and $z \in U$, $G_H(m, n, \gamma)$ is the family of harmonic functions f of the form (1) such that

$$Re\left\{(1 + e^{i\alpha})\frac{D^m f(z)}{D^n f(z)} - e^{i\alpha}\right\} \geq \gamma, \tag{3}$$

where $D^m f$ is defined by (2).

Let $G_{\bar{H}}(m, n, \gamma)$ denote the subclass of $G_H(m, n, \gamma)$ consisting of harmonic functions $f_m = h + \overline{g_m}$ such that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k; \quad a_k, b_k \geq 0. \tag{4}$$

We note that when $m = 1$ and $n = 0$, $G_{\bar{H}}(m, n, \gamma)$ reduces to the class $G_{\bar{H}}(\gamma)$ [3].

Here we obtain coefficient condition, extreme points, distortion bounds, convolution conditions and convex combinations for $G_{\bar{H}}(m, n, \gamma)$ following the techniques in [4]. We also note that, α being real, when $\alpha = 0$, the class $G_{\bar{H}}(m, n, \gamma)$ coincides with the class $\bar{S}_H(m, n; \beta)$studied in [4] with $\beta = \frac{1+\gamma}{2}$.

2 Main Results

We begin with a sufficient coefficient condition for functions in $G_H(m, n, \gamma)$.

Theorem 1. *Let $f = h+\bar{g}$ be so that h and g are given by (1). Furthermore,*

$$\sum_{k=1}^{\infty}\left[\frac{2k^m-k^n(1+\gamma)}{1-\gamma}|a_k|+\frac{2k^m-(-1)^{m-n}k^n(1+\gamma)}{1-\gamma}|b_k|\right]\leq 2 \tag{5}$$

where $a_1 = 1$, $m \in N$, $n \in N_0$, $m > n$ and $0 \leq \gamma < 1$. Then f is sense preserving, harmonic univalent in U and $f \in G_H(m,n,\gamma)$.

Proof. If $z_1 \neq z_2$, then on using (5), we have

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq$$

$$\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)-\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right|$$

$$\geq 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \geq 1-\frac{\sum_{k=1}^{\infty}\frac{2k^m-(-1)^{m-n}k^n(1+\gamma)}{1-\gamma}|b_k|}{1-\sum_{k=2}^{\infty}\frac{2k^m-k^n(1+\gamma)}{1-\gamma}|a_k|} \geq 0$$

which proves univalence. Also f is sense preserving in U since

$$\begin{aligned}|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty}\frac{2k^m-k^n(1+\gamma)}{1-\gamma}|a_k| \\ &\geq \sum_{k=1}^{\infty}\frac{2k^m-(-1)^{m-n}k^n(1+\gamma)}{1-\gamma}|b_k| \geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.\end{aligned}$$

According to the condition (3) we only need to show that if (5) holds, then

$$Re\left\{\frac{(1+e^{i\alpha})D^m f(z)-e^{i\alpha}D^n f(z)}{D^n f(z)}\right\} = Re\frac{A(z)}{B(z)} \geq \gamma$$

where $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $0 \leq r < 1$ and $0 \leq \gamma < 1$.

Note that $A(z) = (1+e^{i\alpha})D^m f(z) - e^{i\alpha}D^n f(z)$ and $B(z) = D^n f(z)$. Using the fact that $Re\ w \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$ it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \geq 0. \tag{6}$$

Substituting for $A(z)$ and $B(z)$ in the left side of (6) we obtain

$$\begin{aligned}
&|(1+e^{i\alpha})D^m f(z) - e^{i\alpha}D^n f(z) + (1-\gamma)D^n f(z)| \\
&\quad -|(1+\gamma)D^n f(z) - ((1+e^{i\alpha})D^m f(z) - e^{i\alpha}D^n f(z))| \\
= &\left|(2-\gamma)z + \sum_{k=2}^{\infty}\left[(k^m+(1-\gamma)k^n) + e^{i\alpha}(k^m-k^n)\right] a_k z^k\right. \\
&\left. -(-1)^n \sum_{k=1}^{\infty}\left[(k^n(\gamma-1)-(-1)^{m-n}k^m) + e^{i\alpha}(k^n-(-1)^{m-n}k^m)\right]\overline{b_k z^k}\right| \\
&-\left|\gamma z - \sum_{k=2}^{\infty}\left[(k^m-(1+\gamma)k^n) + e^{i\alpha}(k^m-k^n)\right] a_k z^k\right. \\
&\left. +(-1)^n \sum_{k=1}^{\infty}\left[(1+\gamma)k^n-(-1)^{m-n}k^m) + e^{i\alpha}(k^n-(-1)^{m-n}k^m)\right]\overline{b_k z^k}\right| \\
\geq &(2-\gamma)|z| - \sum_{k=2}^{\infty}[2k^m-\gamma k^n]|a_k||z|^k - \sum_{k=1}^{\infty}|\gamma k^n-(-1)^{m-n}2k^m||b_k||z|^k \\
&-\gamma|z| - \sum_{k=2}^{\infty}[2k^m-(2+\gamma)k^n)]|a_k||z|^k - \sum_{k=1}^{\infty}|k^n(2+\gamma)-(-1)^{m-n}2k^m||b_k||z|^k
\end{aligned}$$

$$= \begin{cases} 2(1-\gamma)|z| - 2\sum_{k=2}^{\infty}[2k^m-\gamma k^n-k^n]|a_k||z|^k \\ \qquad -2\sum_{k=1}^{\infty}[2k^m+k^n(1+\gamma)]|b_k||z|^k\ , & \text{if } m-n \text{ is odd} \\ 2(1-\gamma)|z| - 2\sum_{k=2}^{\infty}[2k^m-\gamma k^n-k^n]|a_k||z|^k \\ \qquad -2\sum_{k=1}^{\infty}[2k^m-k^n(1+\gamma)]|b_k||z|^k\ , & \text{if } m-n \text{ is even} \end{cases}$$

$$> 2\left\{1-\gamma-\left[\sum_{k=2}^{\infty}2k^m-k^n(1+\gamma)|a_k|+\sum_{k=1}^{\infty}2k^m-(-1)^{m-n}k^n(1+\gamma)|b_k|\right]\right\}$$

≥ 0, on using equation (5).

The harmonic univalent functions

(7)

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\gamma}{2k^m-(1+\gamma)k^n}x_kz^k+\sum_{k=1}^{\infty}\frac{1-\gamma}{2k^m-(-1)^{m-n}k^n(1+\gamma)}\overline{y_kz^k}$$

where $m \in N$, $n \in N_0$, $m > n$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (5) is sharp.

We now show that the condition (5) is also necessary for functions $f_m = h+\overline{g_m}$ where h and g_m are given by (4).

Theorem 2. *Let $f_m = h+\overline{g_m}$ be given by (4). Then $f_m \in G_{\bar{H}}(m,n,\gamma)$ if and only if the coefficient condition (5) holds.*

Proof. Since $G_{\bar{H}}(m,n,\gamma) \subset G_H(m,n,\gamma)$, we only need to prove the "only if" part of the theorem. For functions f_m of the form (4), the inequality (3) with $f = f_m$ is equivalent to

$$Re\left\{\frac{\begin{array}{l}(1-\gamma)z-\sum\limits_{k=2}^{\infty}[(k^m-\gamma k^n)+e^{i\alpha}(k^m-k^n)a_kz^k\\ \qquad +(-1)^{2m-1}\sum\limits_{k=1}^{\infty}[k^m-\gamma(-1)^{m-n}k^n+e^{i\alpha}(k^m-(-1)^{m-n}k^n)]b_k\bar{z}^k\end{array}}{z-\sum\limits_{k=2}^{\infty}k_na_kz^k+(-1)^{m+n-1}\sum\limits_{k=1}^{\infty}k_nb_k\bar{z}^k}\right\}$$

(8) $$\geq 0$$

which must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$ we must have

$$\frac{1-\gamma-\sum_{k=2}^{\infty}[2k^m-(1+\gamma)k^n]a_kr^{k-1}-\sum_{k=1}^{\infty}[2k^m-(-1)^{m-n}k^n(1+\gamma)]b_kr^{k-1}}{1-\sum_{k=2}^{\infty}k_na_kr^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}k_nb_kr^{k-1}}$$

$$\ge 0 \tag{9}$$

If the condition (5) does not hold, then the numerator in (9) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in (9) is negative. This contradicts the required condition for $f_m \in G_{\bar{H}}(m,n,\gamma)$ and the proof is complete.

Theorem 3. *Let f_m be given by (4). Then $f_m \in G_{\bar{H}}(m,n,\gamma)$ if and only if $f_m(z) = \sum_{k=1}^{\infty}(x_kh_k(z)+y_kg_{m_k}(z))$ where $h_1(z) = z$, $h_k(z) = z-\frac{1-\alpha}{2k^m-k^n(1+\gamma)}z^k$, $(k = 2,3,\dots)$ and $g_{m_k}(z) = z+(-1)^{m-1}\frac{1-\gamma}{2k^m-(-1)^{m-n}k^n(1+\gamma)}\bar{z}^k$, $(k = 1,2,3,\dots)$ $x_k \ge 0$, $y_k \ge 0$, $x_1 = 1-\sum_{k=2}^{\infty}(x_k+y_k) \ge 0$. In particular the extreme points of $G_{\bar{H}}(m,n,\gamma)$ are $\{h_k\}$ and $\{g_{m_k}\}$.*

Proof. Suppose

$$f_m(z) = \sum_{k=1}^{\infty}(x_kh_k(z)+y_kg_{m_k}(z)) = \sum_{k=1}^{\infty}(x_k+y_k)z-\sum_{k=2}^{\infty}\frac{1-\gamma}{2k^m-k^n(1+\gamma)}x_kz^k$$
$$+(-1)^{m-1}\sum_{k=1}^{\infty}\frac{1-\gamma}{2k^m-(-1)^{m-n}k^n(1+\gamma)}y_k\bar{z}^k.$$

Then

$$\sum_{k=2}^{\infty}\frac{2k^m-k^n(1+\gamma)}{1-\gamma}\left(\frac{1-\gamma}{2k^m-k^n(1+\gamma)}x_k\right)$$
$$+\sum_{k=1}^{\infty}\frac{2k^m-(-1)^{m-n}k^n(1+\gamma)}{1-\gamma}\left(\frac{1-\gamma}{2k^m-(-1)^{m-n}k^n(1+\gamma)}y_k\right)$$
$$=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k=1-x_1\leq 1$$

and so $f_m \in G_{\bar{H}}(m,n,\gamma)$.

Conversely if $f_m \in clcoG_{\bar{H}}(m,n,\gamma)$ then

$$a_k \leq \frac{1-\gamma}{2k^m-k^n(1+\gamma)} \quad \text{and} \quad b_k \leq \frac{1-\gamma}{2k^m-(-1)^{m-n}k^n(1+\gamma)}$$

Set $x_k = \frac{2k^m-k^n(1+\gamma)}{1-\gamma}a_k, (k=2,3,\dots)$ and $y_k = \frac{2k^m-(-1)^{m-n}k^n(1+\gamma)}{1-\gamma}b_k$, $(k=1,2,\dots)$.
Then by Theorem 2, $0 \leq x_k \leq 1, (k = 2,3,\dots)$ and $0 \leq y_k \leq 1, (k = 1,2,\dots)$. We define $x_1 = 1-\sum_{k=2}^{\infty}x_k-\sum_{k=1}^{\infty}y_k$ and again by Theorem 2, $x_1 \geq 0$. Consequently we obtain $f_m(z)$
as required.

We now obtain the distortion bounds for functions in $G_{\bar{H}}(m,n,\gamma)$.

Theorem 4. *Let f_m be given by (4) and $f_m \in G_{\bar{H}}(m,n,\gamma)$. Then for $|z|=r<1$ we have*

$$|f_m(z)| \leq (1+b_1)r+\frac{1}{2^n}\left(\frac{1-\gamma}{2^{m-n+1}-(1+\gamma)}-\frac{2-(-1)^{m-n}(1+\gamma)}{2^{m-n+1}-(1+\gamma)}b_1\right)r^2,$$

and

$$|f_m(z)| \geq (1-b_1)r-\frac{1}{2^n}\left(\frac{1-\gamma}{2^{m-n+1}-(1+\gamma)}-\frac{2-(-1)^{m-n}(1+\gamma)}{2^{m-n+1}-(1+\gamma)}b_1\right)r^2.$$

Proof. We prove the first inequality. The proof of the second is similar. Let $f_m \in G_{\bar{H}}(m,n,\gamma)$. We have

$$|f_m(z)| \leq$$

$$\begin{aligned}
&\leq (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^k \leq (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^2 \\
&= (1+b_1)r + \frac{1-\gamma}{2^n(2^{m-n+1}-(1+\gamma))}\sum_{k=2}^{\infty}\frac{2^n(2^{m-n+1}-(1+\gamma))}{1-\gamma}(a_k+b_k)r^2 \\
&\leq (1+b_1)r + \frac{(1-\gamma)r^2}{2^n(2^{m-n+1}-(1+\gamma))} \\
&\quad \sum_{k=2}^{\infty}\left[\frac{2k^m-k^n(1+\gamma)}{1-\gamma}a_k + \frac{2k^m-(-1)^{m-n}k^n(1+\gamma)}{1-\gamma}b_k\right] \\
&\leq (1+b_1)r + \frac{1}{2^n}\left[\frac{1-\gamma}{(2^{m-n+1}-(1+\gamma))} - \frac{2-(-1)^{m-n}(1+\gamma)}{(2^{m-n+1}-(1+\gamma))}b_1\right]r^2.
\end{aligned}$$

The bounds given in Theorem 4 for the functions $f = h + \bar{g}$ of the form (4) also hold for functions of the form (1) if the coefficient condition (5) is satisfied. The upper bound given for $f \in G_{\bar{H}}(m,n,\gamma)$ is sharp and the equality occurs for the function

$$f(z) = z + |b_1|\bar{z} + \frac{1}{2^n}\left[\frac{1-\gamma}{(2^{m-n+1}-(1+\gamma))} - \frac{2-(-1)^{m-n}(1+\gamma)}{(2^{m-n+1}-(1+\gamma))}|b_1|\right]\bar{z}^2,$$

$$|b_1| \leq \frac{1-\gamma}{2-(-1)^{m-n}(1+\gamma)}.$$

The following covering result is a consequence of the second inequality in Theorem 4.

Theorem 5. *Let f_m of the form (4) be so that $f_m \in G_{\bar{H}}(m,n,\gamma)$. Then*

$$\left\{w : |w| < \frac{2^{m+1}-2^n-1-(2^n-1)\gamma}{2^{m+1}-(1+\gamma)2^n} - \right.$$

$$\left. -\frac{2^{m+1}-2^n-2+(-1)^{m-n}-\gamma(2^n-(-1)^{m-n})}{2^{m+1}-(1+\gamma)2^n}b_1\right\} \subset f_m(U)$$

We now consider the convolution of two harmonic functions

$$f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k \bar{z}^k \quad \text{and}$$

$$F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} B_k \bar{z}^k$$

as

$$\begin{aligned} (f_m * F_m)(z) &= f_m(z) * F_m(z) \\ &= z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k B_k \bar{z}^k. \end{aligned} \tag{10}$$

With this definition, we show that the class $G_{\bar{H}}(m,n,\gamma)$ is closed under convolution.

Theorem 6. *For $0 \le \beta \le \gamma < 1$ let $f_m \in G_{\bar{H}}(m,n,\gamma)$ and $F_m \in G_{\bar{H}}(m,n,\beta)$. Then $f_m * F_m \in G_{\bar{H}}(m,n,\gamma) \subset G_{\bar{H}}(m,n,\beta)$.*

Let $f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k \bar{z}^k$ be in $G_{\bar{H}}(m,n,\gamma)$ and $F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} B_k \bar{z}^k$ be in $G_{\bar{H}}(m,n,\beta)$. Then the convolution $f_m * F_m$ is given by (10). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2. For $F_m \in G_{\bar{H}}(m,n,\beta)$ we note that $A_k < 1$ and $B_k < 1$. Now for the convolution function $f_m * F_m$ we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{2k^m - k^n(1+\beta)}{1-\beta} a_k A_k + \sum_{k=1}^{\infty} \frac{2k^m - (-1)^{m-n} k^n (1+\beta)}{1-\beta} b_k B_k \\ &\le \sum_{k=2}^{\infty} \frac{2k^m - k^n(1+\beta)}{1-\beta} a_k + \sum_{k=1}^{\infty} \frac{2k^m - (-1)^{m-n} k^n (1+\beta)}{1-\beta} b_k \\ &\le \sum_{k=2}^{\infty} \frac{2k^m - (1+\gamma)k^n}{1-\gamma} a_k + \sum_{k=1}^{\infty} \frac{2k^m - (-1)^{m-n} k^n (1+\gamma)}{1-\gamma} b_k \le 1 \end{aligned}$$

since $0 \le \beta \le \gamma < 1$ and $f_m \in G_{\bar{H}}(m,n,\gamma)$. Therefore $f_m * F_m \in G_{\bar{H}}(m,n,\gamma) \subset G_{\bar{H}}(m,n,\beta)$.

Now we show that $G_{\bar{H}}(m,n,\gamma)$ is closed under convex combination of its members.

Theorem 7. *The class $G_{\bar{H}}(m,n,\gamma)$ is closed under convex combination.*

Proof. For $i=1,2,3,\ldots$. Let $f_{m_i} \in G_{\bar{H}}(m,n,\gamma)$, where f_{m_i} is given by

$$f_{m_i} = z - \sum_{k=2}^{\infty} a_{k,i} z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k,i} \bar{z}^k.$$

Then by (5)

$$\sum_{k=1}^{\infty} \left[\frac{2k^m - (1+\gamma)k^n}{1-\gamma} a_{k,i} + \frac{2k^m - (-1)^{m-n} k^n (1+\gamma)}{1-\gamma} b_{k,i} \right] \le 2 \tag{11}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$ the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k,i} \right) z^k + (-1)^{m-1} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k,i} \right) \bar{z}^k$$

Then by (11)

$$\begin{aligned} &\sum_{k=1}^{\infty} \left[\frac{2k^m - k^n(1+\gamma)}{1-\gamma} \sum_{i=1}^{\infty} t_i a_{k,i} + \frac{2k^m - (-1)^{m-n} k^n (1+\gamma)}{1-\gamma} \sum_{i=1}^{\infty} t_i b_{k.i} \right] \\ = &\sum_{i=1}^{\infty} t_i \left\{ \sum_{k=1}^{\infty} \left[\frac{2k^m - k^n(1+\gamma)}{1-\gamma} a_{k,i} + \frac{2k^m - (-1)^{m-n} k^n (1+\gamma)}{1-\gamma} b_{k,i} \right] \right\} \\ \le &\ 2 \sum_{i=1}^{\infty} t_i = 2, \end{aligned}$$

which is the required coefficient condition.

Theorem 8. *If $f_m \in G_{\bar{H}}(m,n,\gamma)$ then f_m is convex in the disc*

$$|z| \le \min_k \left\{ \frac{(1-\gamma)(1-b_1)}{k[1-\gamma-(2-(-1)^{m-n}(1+\gamma))b_1]} \right\}^{\frac{1}{k-1}}, \quad k = 2,3,\ldots$$

Proof. Let $f_m \in G_{\bar{H}}(m, n, \gamma)$ and let r, $0 < r < 1$, be fixed. Then $r^{-1} f_m(rz) \in G_{\bar{H}}(m, n, \gamma)$ and we have

$$\sum_{k=2}^{\infty} k^2(a_k + b_k) \leq$$

$$\leq \sum_{k=2}^{\infty} \left(\frac{2k^m - k^n(1+\gamma)}{1-\gamma} a_k + \frac{2k^m - (-1)^{m-n} k^n (1+\gamma)}{1-\gamma} b_k \right) kr^{k-1} \leq 1 - b_1$$

if $kr^{k-1} \leq \frac{1-b_1}{1-\frac{2-(-1)^{m-n}(1+\gamma)}{1-\gamma} b_1}$ or $r \leq \min_k \left\{ \frac{(1-\gamma)(1-b_1)}{k[1-\gamma-(2-(-1)^{m-n}(1+\gamma))b_1]} \right\}^{\frac{1}{k-1}}$, $k = 2, 3, \ldots$

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