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Weyl's type theorems for algebraically Class A operators[1]

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Abstract

Let T be a bounded linear operator acting on a Hilbert space $\mathcal{H}$. The semi-B-Fredholm spectrum is the set $\sigma_{SBF_+^-}(T)$ of all $\lambda \in \mathbb{C}$ such that $T-\lambda = T-\lambda I$ is not a semi-B-Fredholm. Let $E^a(T)$ be the set of all isolated eigenvalues in $\sigma_a(T)$. The aim of this paper is to show if T is algebraically class A, then T satisfies generalized a-Weyl's theorem $\sigma_{SBF_+^-}(T) = \sigma_a(T) - E^a(T)$, and the semi-Fredholm spectrum of T satisfies the spectral mapping theorem. We also consider commuting finite rank perturbations of operators satisfying generalized a-Weyl's theorem.

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1 Introduction

Throughout this note let $\mathbf{B}(\mathcal{H})$, $\mathbf{F}(\mathcal{H})$, $\mathbf{K}(\mathcal{H})$, denote, respectively, the algebra of bounded linear operators, the ideal of finite rank operators and

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the ideal of compact operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in \mathbf{B}(\mathcal{H})$ we shall write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null space and range of T, respectively. Also, let $\alpha(T) := \dim \mathcal{N}(T)$, $\beta(T) := \dim \mathcal{R}(T)$, and let $\sigma(T), \sigma_a(T), \sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in \mathbf{B}(\mathcal{H})$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

T is called Weyl if it is Fredholm of index 0, and Browder if it is Fredholm "of finite ascent and descent". The essential spectrum $\sigma_F(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T) = \{\lambda \in \mathbb{C} : T - \lambda \, \text{is not Fredholm}\}$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \, \text{is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \, \text{is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T) \subseteq \sigma_W(T) \subseteq \sigma_b(T) \subseteq \sigma_F(T) \cup acc\sigma(T)$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$. If we write $E(K) = K - accK$ then we let

$$E_0(T) := \{\lambda \in E(T) : 0 < \alpha(T - \lambda) < \infty\} \tag{1}$$

for the isolated eigenvalues of finite multiplicity and

$$\Pi_0(T) := \sigma(T) - \sigma_b(T) \tag{2}$$

for the set of poles of finite rank.

Following [3], We say that Weyl's theorem holds for T if

$$\sigma(T) - \sigma_W(T) = E_0(T),$$

and Browder's theorem holds for T if

$$\sigma(T) - \sigma_W(T) = \Pi_0(T).$$

We consider the sets

$$SF_+(\mathcal{H}) = \{T \in \mathbf{B}(\mathcal{H}) : \mathcal{R}(T) \text{is closed and} \quad \alpha(T) < \infty\},$$

$$SF_-(\mathcal{H}) = \{T \in \mathbf{B}(\mathcal{H}) : \mathcal{R}(T) \text{is closed and} \quad \beta(T) < \infty\}$$

and

$$SF_+^-(\mathcal{H}) = \{T \in \mathbf{B}(\mathcal{H}) : T \in SF_+(\mathcal{H}) \text{and} \quad i(T) \leq 0\},$$

For any $T \in \mathbf{B}(\mathcal{H})$ let

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(\mathcal{H})\}.$$

Let E_0^a be the set of all eigenvalues of T of finite multiplicity which are isolated in the approximate point spectrum. According to [17], we say that T satisfies a-Weyl's theorem if $\sigma_{SF_+^-}(T) = \sigma_a(T) - E_0^a(T)$. It follows from [24, corollary 2.5] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem.

In [9] Berkani define the class of B-Fredholm operators as follows. For each integer n, define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0 = T$). If for some n the range $\mathcal{R}(T^n)$ is closed and T_n is Fredholm (resp. Semi-B-Fredholm) operator, then T is called a B-Fredholm (resp. Semi-B-Fredholm) operator. In this case and from [8] T_m is a Fredholm operator and $i(T_m) = i(T_n)$ for each $m \geq n$.

According to Berkani [9] the index of a B-Fredholm operator T is defined as the index of the Fredholm operator T_n, where n is any integer such that the range $\mathcal{R}(T^n)$ is closed and T_n is Fredholm operator.

Let $BF(\mathcal{H})$ be the class of all B-Fredholm operators. In [8] Berkani has studied this class of operators and has proved that an operator $T \in \mathbf{B}(\mathcal{H})$ is a B-Fredholm if and only if $T = T_0 \oplus T_1$, where T_0 is a Fredholm and T_1

is a nilpotent operator.
Let $SBF_+(\mathcal{H})$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(\mathcal{H})$ the class of all $T \in SBF_+(\mathcal{H})$ such that $i(T) \leq 0$, and

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{H})\}$$

2 Preliminaries

Definition 2.1. *([9]) Let $T \in \mathbf{B}(\mathcal{H})$. Then T is called a B-Weyl's operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ is given by*

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \textit{is not B-Weyl}\}.$$

Berkani [9, Theorem 4.3] proved that if $T \in \mathbf{B}(\mathcal{H})$ such that T is a normal, then

$$\sigma_{BW}(T) = \sigma(T) - E(T),$$

where $E(T)$ is the set of isolated eigenvalues of T, which gives a generalization of a classical Weyl Theorem.

Definition 2.2. *([10])For any $T \in \mathbf{B}(\mathcal{H})$ we define the sequence $(c_n(T))$ and $(b_n(T))$ as follows:*

1. $c_n(T) = \dim(\mathcal{R}(T^n)/\mathcal{R}(T^{n+1}))$.
2. $b_n(T)) = \dim(\mathcal{N}(T^{n+1})/\mathcal{N}(T^{n+1}))$.

The descent $d(T)$ and ascent $a(T)$ are defined by

$$d(T) = inf\{n : c_n(T) = 0\} = inf\{n : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\},$$

$$a(T) = inf\{n : b_n(T) = 0\} = inf\{n : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}.$$

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [16] We say that $T \in \mathbf{B}(\mathcal{H})$ has the

single-valued extension property (SVEP) if for every open set $U \subseteq \mathbb{C}$ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation $(T - \lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} - \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [20, proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.
Recall that an operator $T \in \mathbf{B}(\mathcal{H})$ is Drazin invertible if it has a finite ascent and descent. The Drazin spectrum is given by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Drazin invertible}\}.$$

We observe that $\sigma_D(T) = \sigma(T) - \Pi(T)$, where $\Pi(T)$ is the set of all poles, while $\Pi_0(T)$ will denote the set of all poles of T of finite rank.

Definition 2.3. *([4, definition 2.4]) An operator $T \in \mathbf{B}(\mathcal{H})$is called left Drazin invertible if $a(T) < \infty$ and $\mathcal{R}(T^{a(T)+1})$ is closed. The left Drazin spectrum is given by*

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \textit{is not left Drazin invertible}\}.$$

Definition 2.4. *([4, definition 2.5]) We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I$ is left Drazin invertible and $\lambda \in \sigma_a(T)$ is a left pole of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$.*

We will denote $\Pi^a(T)$ the set of all left pole of T , and by $\Pi_0^a(T)$ the set of all left pole of T of finite rank. We have $\sigma_{LD}(T) = \sigma_a(T) - \Pi^a(T)$.
It is shown in [7] that Drazin invertibility is a good tool for the investigation of the class of B-Fredholm and of the induced B-Weyl spectrum.
Following [18] We say that $T \in \mathbf{B}(\mathcal{H})$ is Drazin invertible (with finite index) if there exists $B, U \in \mathbf{B}(\mathcal{H})$ such that U is nilpotent and

$$TB = BT, BTB = B, TBT = T + U.$$

It is well known that T is a Drazin invertible if and only if it has a finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is nilpotent and T_1 is invertible (see [18, Proposition A]).

Definition 2.5. *([10])Let $T \in \mathbf{B}(\mathcal{H})$ and let $s \in \mathbb{N}$. Then T has a uniform descent for $n \geq s$ if $\mathcal{R}(T) + \mathcal{N}(T^n) = \mathcal{R}(T) + \mathcal{N}(T^s)$ for all $n \geq s$. If in addition $\mathcal{R}(T) + \mathcal{N}(T^s)$ is closed, then T is said to have a topological uniform descent for $n \geq s$.*

Note that if $\lambda \in \Pi^a(T)$, then it is easily seen that $T - \lambda$ is an operator of topological uniform descent. Therefore it follows from ([10, Theorem 2.5]) that λ is isolated in $\sigma_a(T)$. Following [4] if $T \in \mathbf{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ is an isolated in $\sigma_a(T)$, then $\lambda \in \Pi^a(T)$ if and only if $\lambda \notin \sigma_{SBF_+^-}(T)$ and $\lambda \in \Pi_0^a(T)$ if and only if $\lambda \notin \sigma_{SF_+^-}(T)$.

Definition 2.6. *([10])Let $T \in \mathbf{B}(\mathcal{H})$. We will say that*

1. *T satisfies generalized Browder's theorem if $\sigma_W(T) = \sigma(T) - \Pi(T)$.*
2. *T satisfies a-Browder's theorem if $\sigma_{SF_+^-}(T) = \sigma_a(T) - \Pi_0^a(T)$.*
3. *T satisfies generalized a-Browder's theorem if $\sigma_{SBF_+^-}(T) = \sigma_a(T) - \Pi^a(T)$*
4. *T satisfies generalized a-Weyl's theorem if $\sigma_{SBF_+^-}(T) = \sigma_a(T) - E^a(T)$.*

Definition 2.7. *([4])An operator $T \in \mathbf{B}(\mathcal{H})$ is called polaroid (resp. a-polaroid)if all isolated points of the spectrum (resp. of the approximate point spectrum) of T are poles (resp. left poles) of the resolvent of T.*

Definition 2.8. *([15]) Let $T \in \mathbf{B}(\mathcal{H})$ and F be closed subset of $\mathbb{C}$.*
a)The glocal spectral is

$$\chi_T(F) := \{x \in \mathcal{H} : \exists \text{ analytic function} f : \mathbb{C} - F \longrightarrow \mathcal{H} \text{such that } (\lambda - T)f(\lambda) = x, \forall x \in \mathbb{C} - F\}.$$

b) The quasinilpotent part $H_0(T-\lambda)$ *is*

$$H_0(T-\lambda) := \{x \in \mathcal{H} : \lim_{n \longrightarrow \infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

c) The analytic core $K(T-\lambda)$ *of* $T-\lambda$ *are*

$$\begin{aligned} K(T-\lambda) = \{x \in \mathcal{H} : & \textit{ there exists a sequence } \{x_n\} \subset \mathcal{H} \textit{ and } \quad \delta > 0 \\ \textit{for which} \quad & x = x_0, (T-\lambda)x_{n+1} = x_n \textit{and} \quad \|x_n\| \leq \delta^n \|x\| \\ \textit{for all} \quad & n = 1, 2, \cdots\}. \end{aligned}$$

Note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are generally non-closed hyper-invariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-p}(0) \subseteq H_0(T-\lambda)$ for all $p = 0, 1, \cdots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$.
Recall that an operator T has a generalized Kato decomposition abbreviate GKD, if there exists a pair of T-invariant closed subspace (M, N) such that $\mathcal{H} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. Note that, an operator $T \in \mathbf{B}(\mathcal{H})$ has a GKD at every $\lambda \in E(T)$, namely $\mathcal{H} = H_0(T-\lambda) \oplus K(T-\lambda)$. We say that T is of Kato type at a point λ if $(T-\lambda)|_M$ is nilpotent in the GKD for $T-\lambda$.

Definition 2.9. *([11])*

1. *An operator* $X \in \mathbf{B}(\mathcal{H})$ *is said to be a quasiaffinity if it is an injective and has dense range.*

2. *An operator* $S \in \mathbf{B}(\mathcal{H})$ *is said to be quasiaffine transform of* T *(abbreviate* $S \prec T$ *) if there is a quasiaffinity* X *such that* $XS = TX$*.*

3. *Two operators* $T, S \in \mathbf{B}(\mathcal{H})$ *are said to be quasisimilar if there are a quasiaffinities* $X, Y \in \mathbf{B}(\mathcal{H})$ *such that* $XS = TX$ *and* $SY = YT$*.*

3 Properties of algebraically Class A

An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be class A if $|A|^2 \leq |A^2|$. We say that T is algebraically class A if there exists a non-constant complex polynomial $\mathcal{P}$ such that $\mathcal{P}(T)$ is class A.

In general,

hyponormal$\Rightarrow$ p-hyponormal $\Rightarrow$ ω-hyponormal $\Rightarrow$ class A $\Rightarrow$ algebraically class A.

Algebraically class A is preserved under translation by scalar and restriction to invariant subspaces. Moreover, if T is class A and invertible then T^{-1} is class A. Indeed,

$$T^*T = |T|^2 \leq |T^2| = (T^{*2}T^2)^{\frac{1}{2}} = T^{*2}(T^2T^{*2})^{\frac{-1}{2}}T^2$$

if and only if

$$T^{*-1}T^{-1} \leq (T^2T^{*2})^{\frac{-1}{2}} = (T^{-2*}T^{-2})^{\frac{1}{2}}$$

if and only if

$$|T^{-1}|^2 \leq |T^{-2}|.$$

We write $r(T)$ and $W(T)$ for the spectral radius and numerical range, respectively. It is well-known that $r(T) \leq \|T\|$ and that $W(T)$ is convex with convex hull $conv\sigma(T) \subseteq \overline{W(T)}$. T is called convexoid if $conv\sigma(T) = \overline{W(T)}$, and normaloid if $r(T) = \|T\|$.

Lemma 3.1. *([2]) If $T \in \mathbf{B}(\mathcal{H})$ is an algebraically class A, then T is polaroid (resp.a-polaroid).*

Definition 3.2. *([14]) An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be totally hereditarily normaloid,$T \in THN$ if every part of T (*i.e.*, its restriction to an invariant subspace), and T_p^{-1} for every invertible part T_p of T, is normaloid.*

Lemma 3.3. *Let $T \in THN$. Let $\lambda \in \mathbb{C}$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$*

Proof. We consider two cases:
case I. ($\lambda = 0$): Since T is normaloid. Therefore $T = 0$.
case II. ($\lambda \neq 0$): Here T is invertible, and since $T \in THN$, we see that T, T^{-1} are normaloid. On the other hand $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. This implies that $\frac{1}{\lambda}T$ is unitary with its spectrum $\sigma(\frac{1}{\lambda}T) = 1$. It follows that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda I$.

In [11], Curto and Han proved that quasinilpotent algebraically paranormal operators are nilpotent. We now establish a similar result for algebraically class A operators.

Lemma 3.4. *Let T be a quasinilpotent algebraically class A operator. Then T is nilpotent.*

Proof. Suppose $\mathcal{P}(T)$ is class A for some non-constant polynomial $\mathcal{P}$. Since $\sigma(\mathcal{P}(T)) = \mathcal{P}(\sigma(T))$, the operator $\mathcal{P}(T) - \mathcal{P}(0)$ is quasinilpotent. Since $\mathcal{P}(T) \in THN$, it follows from lemma 3.3 that $cT^m(T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n) \equiv \mathcal{P}(T) - \mathcal{P}(0)$, where $(m \geq 1)$. Since $T - \lambda_j$ is invertible for every $\lambda_j \neq 0, j = 1, \cdots n$, we must have $T^m = 0$.

It is well-known that every class A operator is isoloid (see [22]). We extend this result to algebraically class A operators.

Theorem 3.5. *Let $T \in \mathbf{B}(\mathcal{H})$ be algebraically class A operator. Then T is isoloid.*

Proof. Let $\lambda \in iso\sigma(T)$ and let $P := \frac{1}{2\pi i}\int_{\partial D}(\lambda - T)^{-1}\, d\lambda$ be the associated Riesz idempotent, where D is a closed disc centered at λ which contains no other points of $\sigma(T)$. We can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Since T is algebraically class A, $\mathcal{P}(T)$ is class A for some non-constant polynomial $\mathcal{P}$. Since $\sigma(T_1) = \{\lambda\}$, we must have $\sigma(\mathcal{P}(T_1)) = \mathcal{P}(\sigma(T_1)) = \mathcal{P}(\{\lambda\}) = \{\mathcal{P}(\lambda)\}$. Since $\mathcal{P}(T_1)$ is class A, it follows from lemma 3.4 that $\mathcal{P}(T_1) - \mathcal{P}(\lambda) = 0$. Put $Q(z) := \mathcal{P}(z) - \mathcal{P}(\lambda)$. Then $Q(T_1) = 0$, and hence T_1 is algebraically class A operator. Since $T_1 - \lambda$ is quasinilpotent and class A operator, it follows from lemma

3.4 that $T_1 - \lambda$ is nilpotent, therefore $\lambda \in \sigma_p(T_1)$, and hence $\lambda \in \sigma_p(T)$. This shows that T is an isoloid.

Lemma 3.6. *Let $T \in \mathbf{B}(\mathcal{H})$ be a class A operator, then T is of finite ascent.*

Proof. Let $x \in \mathcal{N}(T^2)$, then $\|Tx\|^2 \leq \|T^2x\| = 0$, and so $x \in \mathcal{N}(T)$. Since the non-zero eigenvalues of a a class A operators are normal eigenvalues of T, (see [23, lemma 8]), if $0 \neq \lambda \in \sigma_p(T)$ and $(T-\lambda)^2 = 0$, then $(T-\lambda)(T-\lambda)x = 0 = (T-\lambda)^*(T-\lambda)x$ and $\|(T-\lambda)x\| = \langle (T-\lambda)^*(T-\lambda)x, x\rangle = 0$. Hence, if T is class A, then $a(T-\lambda) = 1$.

Lemma 3.7. *Let $T \in \mathbf{B}(\mathcal{H})$ be algebraically class A operator. Let $\lambda \in \mathbb{C}$ be an isolated point in $\sigma(T)$, then λ is a simple pole of the resolvent $R_z(T) = (zI - T)^{-1}$.*

Proof. If $\lambda \in iso\sigma(T)$, then T has a direct sum decomposition $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Let $\mathcal{P}$ be a nonconstant polynomial such that $\mathcal{P}(T)$ is class A operator. Then $\mathcal{H}_1$ is a $\mathcal{P}(T)$-invariant subspace, and hence $\mathcal{P}(T_1$ is class A operator such that $\sigma(\mathcal{P}(T_1) = \mathcal{P}(\sigma(T_1) = \{\mathcal{P}(\lambda)\}$. But then $\mathcal{P}(\lambda) \in \Pi_0(T_1)$ and $\lambda \in \Pi_0(T_1)$. Hence, since $\lambda \notin \sigma(T_2)$, $\lambda \in \Pi_0(T)$.

The following result is a consequence of lemma3.7 and [12, theorem 1.52].

Corollary 3.8. *Let T be a an algebraically class A operator and $\lambda_0 \in iso\sigma(T)$. Let $\tau = \sigma(T) - \{\lambda_0\}$. Then λ_0 is an eigenvalue of T. The ascent and descent of $T - \lambda_0$ are both equal to 1. Also*

$$\mathcal{R}(P(\lambda_0)) = \mathcal{N}((T - \lambda_0)),$$

$$\mathcal{R}(P(\tau)) = \mathcal{R}((T - \lambda_0)).$$

Lemma 3.9. *Let $T \in \mathbf{B}(\mathcal{H})$ be an algebraically class A. Then $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T)$. Moreover T_1, the restriction of T to $\mathcal{R}(T)$ is one-one and onto.*

Proof. Suppose that $y \in \mathcal{R}(T) \cap \mathcal{N}(T)$ then $y = Tx$ for some $x \in \mathcal{H}$ and $Ty = 0$. It follows that $T^2x = 0$. However, $d(T) = 1$ and so $x \in \mathcal{N}(T^2) = \mathcal{N}(T)$. Hence $y = Tx = 0$ and so $\mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$. Also, $T\mathcal{R}(T) = \mathcal{R}(T)$.
If $x \in \mathcal{H}$ there is $u \in \mathcal{R}(T)$ such that $Tu = Tx$. Now if $z = x - u$ then $Tz = 0$. Hence

$$\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T).$$

Since $d(T) = 1$, T maps $\mathcal{R}(T)$ onto itself. If $y \in \mathcal{R}(T)$ and $Ty = 0$ then $y \in \mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$. Hence T_1 is one-one and onto.

Theorem 3.10. *Let $T \in \mathbf{B}(\mathcal{H})$ be algebraically class A operator. Then T is of Kato type at each $\lambda \in E(T)$.*

Proof. Let T be algebraically class A and $\lambda \in E(T)$. Then $\mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda)$, where $T|_{H_0(T-\lambda)} = T_1$ satisfies $\sigma(T_1) = \{\lambda\}$ and $T|_{K(T-\lambda)}$ is semi-regular. Since T is algebraically class A, then there exists a non-constant polynomial $\mathcal{P}$ such that $\mathcal{P}(T_1)$ is class A. Clearly, $\sigma(\mathcal{P}(T_1) = \mathcal{P}(\sigma(T_1)) = \{\mathcal{P}(\lambda)\}$. Applying lemma 3.3 it follows that $H_0(\mathcal{P}(T) - \mathcal{P}(\lambda)) = (\mathcal{P}(T_1) - \mathcal{P}(\lambda))^{-1}(0)$.

$$0 = \mathcal{P}(T_1) - \mathcal{P}(\lambda) = c(T_1 - \lambda)^m \prod_{j=1}^{n} (T_1 - \lambda_j),$$

for some complex numbers $c, \lambda_1, \cdots, \lambda_n$, then for each $j = 1, \cdots, n$, $T - \lambda_j$ is invertible, which implies $T_1 - \lambda$ is nilpotent. Hence $T - \lambda$ is of Kato type.

Lemma 3.11. *If T is class A operator and $S \prec T$. Then S has SVEP.*

Proof. Since T is class A operator, then it has a SVEP, then the result follows from [11, lemma 3.1].

4 Weyl's Type Theorems

Theorem 4.1. *If $T \in \mathbf{B}(\mathcal{H})$ is an algebraically class A operator. Then T and T^* satisfy Weyl's theorem.*

Proof. Since T is algebraically class A, then T has SVEP. Then T satisfies Browder's theorem if and only if T^* satisfies Browder's theorem if and only if $\Pi_0(T) = \sigma(T) - \sigma_W(T) \subseteq E_0(T)$ and $\Pi_0(T^*) = \sigma(T^*) - \sigma_W(T^*) \subseteq E_0(T^*)$. If $\lambda \in E_0(T^*)$, then both T and T^* has SVEP at λ and $0 < a((T-\lambda)^*) = b(T-\lambda) < \infty$. Thus the ascent and descent of $T-\lambda$ are finite and hence equal(see [12, prop.1.49]). Then $T-\lambda$ is a Fredholm of index zero and also $(T-\lambda)^*$ is a Fredholm of index zero, then $E_0(T) \subseteq \sigma(T) - \sigma_W(T)$ and $E_0(T^*) \subseteq \sigma(T^*) - \sigma_W(T^*)$. This implies that both T and T^* satisfy Weyl's theorem.

For $T \in \mathbf{B}(\mathcal{H})$, it is known that the inclusion $\sigma_{SF_+^-}(f(T)) \subseteq f(\sigma_{SF_+^-}(T))$ holds for every $f \in Hol(\sigma(T))$, with no restriction on T.
The next theorem shows that for algebraically class A operators the spectral mapping theorem holds for the semi-Fredholm spectrum.

Theorem 4.2. *If T or T^* is an algebraically class A operator. Then $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$ for all $f \in Hol(\sigma(T))$.*

Proof. Let $f \in Hol(\sigma(T))$. It suffices to show that $f(\sigma_{SF_+^-}(T)) \subseteq \sigma_{SF_+^-}(f(T))$. Suppose that $\lambda \notin \sigma_{SF_+^-}(f(T))$ then $f(T)-\lambda \in SF_+^-(\mathcal{H})$ and $i(f(T)-\lambda) \leq 0$ and

$$f(T) - \lambda = c(T-\alpha_1)\cdots(T-\alpha_n)g(T) \tag{3}$$

where $c, \alpha_1, \cdots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. If T is algebraically class A, then $0 \leq \sum_{j=1}^n i(T-\alpha_j) \leq 0$, then $i(T-\alpha_j) \leq 0$ for each $j = 1, 2, \cdots, n$, therefore $\lambda \notin f(\sigma_{SF_+^-}(T))$.
Suppose now that T^* is algebraically class A, then T^* has SVEP, and so $i(T-\alpha_j) \geq 0$ for each $j = 1, 2, \cdots, n$. since $0 \leq \sum_{j=1}^n i(T-\alpha_j) \leq 0$. Then $T-\alpha_j$ is Weyl for each $j = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_{SF_+^-}(T))$. This completes the proof.

as a consequence of [11, theorem 3.4] we have

Corollary 4.3. *Let $T \in \mathbf{B}(\mathcal{H})$ be a class A operator, then $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for each $f \in Hol(\sigma(T))$.*

Lemma 4.4. *If T or T^* is a class A operator. Then $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$ for all $f \in Hol(\sigma(T))$.*

Proof. This follows at once from [26, theorem 2.3].

Theorem 4.5. *If $T \in \mathbf{B}(\mathcal{H})$ is an algebraically class A operator. Then $\sigma(f(T)) - E(f(T)) = f(\sigma(T) - E(T))$ for every $f \in Hol(\sigma(T))$.*

Proof. It is suffices to show $f(\sigma(T) - E(T)) \subseteq \sigma(f(T)) - E(f(T))$, since the other inclusion holds with no restriction on T ([5, lemma 2.7]). If $\lambda \notin \sigma(f(T)) - E(f(T)$, then $f(T) - \lambda = \prod_{j=1}^{n} (T - \alpha_j)^{m_j}$, where $m_1, \cdots, m_n$ are integers and $\alpha_1, \cdots, \alpha_n$ are complex numbers, $g(T)$ is invertible operator, and $\alpha_i \neq \alpha_j$ when $i \neq j$. Since $f(T) - \lambda$ is not invertible, there exists $\alpha \in \{\alpha_1, \cdots, \alpha_n\}$ such that $\alpha \in \sigma(T)$. Since λ is isolated in $\sigma(f(T))$, α is isolated in $\sigma(T)$. Hence $\lambda = f(\alpha) \notin f(\sigma(T) - E(T))$. This completes the proof.

Lemma 4.6. *Let $T \in \mathbf{B}(\mathcal{H})$ be class A operator, then T satisfies the generalized Weyl's theorem.*

Proof. We shall show $\sigma(T) - \sigma_{BW}(T) = E(T)$. Let $\lambda \in \sigma(T) - \sigma_{BW}(T)$, then $T - \lambda$ is B-Weyl's. Then by [8, theoren 2.7] there exists two closed subspaces N and M of $\mathcal{H}$ such that $\mathcal{H} = M \oplus N$, $T_1 = (T - \lambda)|_M$ is Weyl's operator, $T_2 = (T - \lambda)|_N$ is nilpotent and $T - \lambda = T_1 \oplus T_2$.
we have two possibilities: either $\lambda \in \sigma(T|_M)$ or $\lambda \notin \sigma(T|_M)$.
case I: $\lambda \in \sigma(T|_M)$, since $T|_M$ is class A, then Weyl's theorem holds for $T|_M$, and so if $\lambda \in \sigma(T|_M)$, then $\lambda \in \Pi_0(T|_M) \subset iso\sigma(T|_M)$. Since $T - \lambda = (T|_M - \lambda I|_M) \oplus T_2$ and T_2 is nilpotent, $\sigma(T_1 - \lambda) - \{0\} = \sigma(T - \lambda) - \{0\}$ and $\lambda \in iso\sigma(T)$. this implies that $\lambda \in \Pi_0(T) \subset E(T)$.
case II: $\lambda \notin \sigma(T|_M)$, then λ is a pole of T which implies that $\lambda \in E(T)$.
Conversely, let $\lambda \in E(T)$. Let P be the spectral projection associated with λ, then $\mathcal{R}(P) = H_0(T - \lambda)$, $\mathcal{N}(P) = K(T - \lambda)$, $H_0(T - \lambda) \neq \{0\}$, $\mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda)$, $K(T - \lambda)$ is closed subspace(see [18, theorem 3], [21, lemma 1]). Since $0 \neq \mathcal{N}(T - \lambda) \subset H_0(T - \lambda)$, λ is a pole of the

resolvent $R_\lambda(T) = (T-\lambda)^{-1}$, then by [18, theorem 3.4] there is some $q > 0$ such that the space $(T-\lambda)^{-q}(0)$ is non-zero and complemented by a closed T-invariant subspace $\mathcal{R}((T-\lambda)^q) \subset \mathcal{R}(T-\lambda)$. Hence $T-\lambda$ is B-Weyl's, *i.e.*, $\lambda \notin \sigma_{BW}(T)$.

The following result is a consequence of theorem 4.5 and theorem 4.6.

Corollary 4.7. *Let $T \in \mathbf{B}(\mathcal{H})$ be class A operator. Then $f(T)$ satisfies generalized Weyl's theorem for every $f \in Hol(\sigma(T))$.*

Theorem 4.8. *Let $T \in \mathbf{B}(\mathcal{H})$ be class A operator, then generalized a-Weyl's theorem holds for T.*

Proof. We will show $\sigma_{SBF_+^-}(T) = \sigma_a(T) - E^a(T)$. In view of [10, theorem 3.1] it suffices to show $E^a(T) = \Pi^a(T)$ and $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$.
If $\lambda \in \sigma_a(T) - E^a(T)$, then λ is an isolated in $\sigma_a(T)$, then it follows from [10, lemma 2.12] that $\lambda \notin \sigma_{SBF_+^-}(T)$. Hence $T - \lambda \in SBF_+^-$, then by [10, theorem 2.8] λ is a left pole of T, and so $\lambda \in \Pi^a(T)$. As we have always true $\Pi^a(T) \subset E^a(T)$, then $E^a(T) = \Pi^a(T)$.
Now, assume $\lambda \notin \sigma_{SBF_+^-}(T)$. Then $T - \lambda \in SBF_+^-$. Hence $T - \lambda$ is a left Drazin invertible and $\sigma_{LD}(T) \subset \sigma_{SBF_+^-}(T)$. As it always true that $\sigma_{SBF_+^-}(T) \subset \sigma_{LD}(T)$, then $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$.

A bounded linear operator T is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Note that every a-isoloid operator is isoloid and the converse is not true in general(see [1]).

Theorem 4.9. *Let $T \in \mathbf{B}(\mathcal{H})$ be class A operator. Then $E(f(T)) = \Pi(f(T))$, for every $f \in Hol(\sigma(T))$*

Proof. Since T is isoloid operator, then from theorem 4.5, we have $\sigma(f(T)) - E(f(T)) = f(\sigma(T) - E(T))$. Since T satisfies generalized Weyl's theorem then $\sigma(T) = \Pi(T)$,so $\sigma(f(T)) - E(f(T)) = f(\sigma_D(T))$. From [7, corollary 2.4] we have $f(\sigma_D(T)) = \sigma_D(f(T))$. Hence $\sigma(f(T)) - E(f(T)) = \sigma_D(f(T))$.

Theorem 4.10. *Let $T \in \mathbf{B}(\mathcal{H})$ be class A operator, then $E^a(f(T)) = \Pi^a(f(T))$, for every $f \in Hol(\sigma(T))$.*

Proof. It is suffices to show $f(\sigma_a(T) - E^a(T)) \subset \sigma_a(f(T)) - E^a(f(T))$, since the other inclusion holds for T with no restriction on T (see [4, theorem 3.5]). If $\lambda \in f(\sigma_a(T) - E^a(T))$, then $\lambda \in \sigma_a(f(T)) = f(\sigma_a(T))$. Suppose $\lambda \in E^a(f(T))$, then λ is isolated in $\sigma_a(f(T))$.
Let $f(T) - \lambda = \prod_{j=1}^{n}(T - \mu_j)^{m_j} g(T)$, where $\mu_1, \cdots, \mu_n$ are complex numbers and $g(T)$ is invertible. If $\mu_j \in \sigma_a(T)$, then μ_j is an isolated in $\sigma_a(T)$. Since T is a-isoloid, μ_j is an eigenvalue of T. Therefore we have $\mu_j \in E^a(T)$. So $\lambda = f(\mu_j)$ and this contradicts to the fact that $\lambda \in f(\sigma_a(T) - E^a(T))$.

Theorem 4.11. *Let $T \in \mathbf{B}(\mathcal{H})$ be class A operator and $F \in \mathbf{F}(\mathcal{H})$ such that $FT = TF$, then $T + F$ satisfy generalized a-Weyl's theorem.*

Proof. Since T satisfies generalized a-Weyl's theorem, then $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$. Since F is a finite rank operator, then it follows from [10, theorem 4.1] that $\sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+F)$. Since $TF = FT$, then by [10, theorem 4.2] we have $\sigma_{LD}(T+F) = \sigma_{LD}(T)$. But $\Pi^a(T+F) = \Pi^a(T)$ (see [19]). Hence $\Pi^a(T) = E^a(T) = E^a(T+F)$. Then by [10, corollary 3.2] $T + F$ satisfies generalized a-Weyl's theorem.

As a consequence of theorem 4.11 and [4, theorem 3.8], we have

Corollary 4.12. *Let $T \in \mathbf{B}(\mathcal{H})$ be class A operator and $F \in \mathbf{F}(\mathcal{H})$ such that $FT = TF$, then $T + F$ is polaroid.*

Lemma 4.13. *Let $T \in \mathbf{B}(\mathcal{H})$ be algebraically class A operator and $S \prec T$. Then g-Browder's theorem holds for $f(S)$, for every $f \in Hol(\sigma(T))$.*

Proof. Since T is algebraically class A operator then T has SVEP, and so is S, consequently $f(S)$, because SVEP is stable under the functional calculus. (*i.e.*, if T has SVEP, then so does $f(T)$ for each $f \in Hol(\sigma(T))$). Observe that if $\lambda \in \Pi(T)$, then $T - \lambda$ is Drazin invertible and hence B-Weyl's. Thus $\Pi(T) \subseteq \sigma(T) - \sigma_{BW}(T)$.

Conversely, assume that $\lambda \in \sigma(T) - \sigma_{BW}(T)$. Then $T - \lambda$ is B-Fredholm, and hence of uniform topological descent (see [9]). We claim that $\lambda \in iso\sigma(T)$. If $\lambda \notin iso\sigma(T)$, there exists a sequence $\{\mu_n\} \subset \sigma(T)$ such that $\mu_n \longrightarrow \lambda$. But then $\dim(T - \mu_n)^{-1}(0) = \dim(T - \lambda)^{-1}(0) > 0$ and finite. So that $\lambda \in acc\sigma_p(T)$. Which is a contradiction to the fact that T has SVEP. Therefore $\lambda \in iso\sigma(T)$ which implies that λ is a pole of the resolvent of T. Thus $\lambda \in \Pi(T)$ and S satisfies g-Browder's theorem.

Theorem 2.4 of [26] affirms that if T^* or T has the SVEP and if T is a-isoloid and generalized a-Weyl's holds for T then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in Hol(\sigma(T))$. If T^* is algebraically class A, then we have:

Theorem 4.14. *Let T^* be an algebraically class A operator. Then generalized a-Weyl's holds for T.*

Proof. Since T^* has SVEP then $\sigma(T) = \sigma_a(T)$ and consequently $E(T) = E^a(T)$.

Let $\lambda \notin \sigma_{SBF_+^-}(T)$ be given, then $T - \lambda$ is semi-B-Fredholm and $i(T - \lambda) \leq 0$. Then [19, proposition 1.2] implies that $i(T - \lambda) = 0$ and consequently $T - \lambda$ is B-Weyl's. Hence $\lambda \notin \sigma_{BW}(T)$. Hence it follows from [26, theorem 3.1] that $\lambda \in E(T) = E^a(T)$.

For the converse, let $\lambda \in E^a(T)$. Then $\lambda \in iso\sigma_a(T)$. Since T^*, we have $\sigma(T) = \sigma_a(T)$. Hence $\overline{\lambda} \in \sigma(T^*)$. Now we represent T^* as the direct sum $T^* = T_1 \oplus T_2$, where $\sigma(T_1) = \{\overline{\lambda}\}$ and $\sigma(T_2) = \sigma(T) - \{\overline{\lambda}\}$. Since T is algebraically class A then so does T_1, and so we have two cases:

Case I:($\overline{\lambda} = 0$): then T_1 is quasinilpotent. Hence it follows from lemma 3.4 that T_1 is nilpotent. Since T_2 is invertible, Then T^* is a B-Weyl's.

Case II: ($\overline{\lambda} \neq 0$): Since $\sigma(T_1) = \{\overline{\lambda}\}$, then $T_1 - \overline{\lambda}$ is nilpotent and $T_2 - \overline{\lambda}$ is invertible, it follows from [26, theorem 3.1] that $T^* - \overline{\lambda}$ is B-Weyl's. Thus in any case $\lambda \in \sigma_a(T) - \sigma_{SBF_+^-}(T)$

Theorem 4.15. *Let $T \in \mathbf{B}(\mathcal{H})$. If T^* is a class A operator. Then generalized a-Browder's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. Let $\lambda \in \Pi^a(T)$ be given. then $\lambda \in iso\sigma_a(T)$ and it follows by [19, theorem 1.5] that $\lambda \notin \sigma_{SBF_+^-}(T)$ which shows that $\Pi^a(T) \subseteq \sigma_a(T) - \sigma_{SBF_+^-}(T)$.
Conversely if $\lambda \in \sigma_a(T) - \sigma_{SBF_+^-}(T)$, then $T - \lambda$ is semi-B-Fredholm and $i(T-\lambda) \leq 0$. Thus, since T^* has SVEP, then by [19, proposition 1.2] that $i(T-\lambda) = 0$. Therefore, $T-\lambda$ is Weyl's and $\lambda \notin \sigma_W(T) = \sigma_b(T)$ which shows that $\lambda \in \Pi(T)$. Consequently $\lambda \in iso\sigma_a(T)$ and hence $\lambda \in \Pi^a(T)$. Thus generalized a-Browder's theorem holds for T.

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