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Starlike image of a class of analytic functions[1]

Szász Róbert

Abstract

It is proved that a subclass of the class of close-to-convex functions it is mapped by the Alexander Operator to the class of starlike functions.

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1 Introduction

We introduce the notation $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disc U with normalization of the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$.

The subclass of $\mathcal{A}$ consisting of functions, for which the domain $f(U)$ is starlike with respect to 0, is denoted by S^*. An analytic description of S^* is given by

$$S^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

The subset of $\mathcal{A}$ defined by

$$C = \left\{ f \in \mathcal{A} \mid \exists\, g \in S^* : \mathrm{Re}\,\frac{zf'(z)}{g(z)} > 0,\ z \in U \right\},$$

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is called the class of close – to – convex functions.

We mention that C and S^* contain univalent functions.

The Alexander integral operator is defined by the equality:

$$A(f)(z)=\int_0^z \frac{f(t)}{t}\,dt.$$

Recall that if f and g are analytic in U and g is univalent, then the function f is said to be subordinate to g, written $f \prec g$ if $f(0)=g(0)$ and $f(U)\subset g(U)$.

In [2] the authors proved the following result:

Theorem 1. *Let A be the operator of Alexander and let $g\in\mathcal{A}$ satisfy*

$$\text{(1)}\qquad \operatorname{Re}\frac{zg'(z)}{g(z)} \geq \left|\operatorname{Im}\frac{z(zg'(z))'}{g(z)}\right|,\ \ z\in U.$$

If $f\in\mathcal{A}$ satisfies

$$\operatorname{Re}\frac{zf'(z)}{g(z)}>0,\ \ z\in U$$

then $F=A(f)\in S^*$.

This Theorem says that a subclass of C is mapped by the Alexander Operator to S^*. This result naturally rises the question whether the Alexander Operator can map the whole class of close-to convex functions in S^*. In [4] the author proved that this did not happen. In the followings we are going to determine another subclass of C which is mapped by the Alexander Operator in S^*.

2 Preliminaries

We need the following definitions and lemmas in our study .

Definition 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ be two analytic functions in U.The convolution of the functions f and g is defined by the equality*

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n.$$

Definition 2. *Let A_0 be the the class of analytic functions in U which satisfy $f(0) = 1$. If $V \subset A_0$, then the dual of V denoted by V^d consists of functions g which satisfy $g \in A_0$ and $(f * g)(z) \neq 0$ for every $f \in V$ and every $z \in U$.*

Let h_T be the function defined by the equality

$$h_T(z) = \frac{1}{1+iT}\left[iT\frac{z}{1-z} + \frac{z}{(1-z)^2}\right], \quad T \in \mathbb{R}.$$

It is simple to observe that h_T is an element of class $\mathcal{A}$.
The class $\mathcal{P}$ is the subset of A_0 defined by

$$\mathcal{P} = \{f \in A_0 : Re(f(z)) > 0, z \in U\}.$$

Lemma 1. *([3],p.23) (duality theorem) The dual of $\mathcal{P}$ is*

$$\mathcal{P}^d = \{f \in A_0 | Re(f(z)) > \frac{1}{2}\ , z \in U\}.$$

Lemma 2. *([3],p.94) The function $f \in \mathcal{A}$ belongs to the class of the starlike functions (denoted by S^*) if and only if $\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0$ for all $T \in \mathbb{R}$ and for all $z \in U$.*

Lemma 3. *[1](The Herglotz formula) For all $f \in \mathcal{P}$ there exists a probability measure μ on the interval $[0, 2\pi]$ so that*

$$f(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t)$$

or in developed form

$$f(z) = 1 + 2\int_0^{2\pi} \left(\sum_{n=1}^{\infty} z^n e^{-int}\right) d\mu(t)$$

The converse of the theorem is also valid.

Lemma 4. *([1] p. 54) Let $\alpha, \beta \in (0, \infty)$ be arbitrary numbers. Let*

$$G_\alpha = \{f \in A_0 : f(z) = \int_0^{2\pi} \frac{1}{(1 - ze^{-it})^\alpha} d\mu(t), \mu \text{ is a probability measure.}\}.$$

If $f \in G_\alpha$ and $g \in G_\beta$, then $fg \in G_{\alpha\beta}$.

Lemma 5. *([1] p. 51) Let $\alpha \in (0, \infty)$, $c \in \mathbb{C}$, $|c| \leq 1$, $c \neq -1$, $F_\alpha = \left(\frac{1+cz}{1-z}\right)^\alpha$. If $f \prec F_\alpha$, then there is a probability measure μ, so that*

$$f(z) = \int_0^{2\pi} \left(\frac{1 + cze^{-it}}{1 - ze^{-it}}\right)^\alpha d\mu(t).$$

Lemma 6. *([2] p. 22) Let $p(z) = a + \sum\limits_{k=n}^{\infty} a_k z^k$ be analytic in U with $p(z) \not\equiv a$, $n \geq 1$ and let $q : U(0,1) \to \mathbb{C}$ be a univalent function with $q(0) = a$. If $p \not\prec q$ then there are two points $z_0 \in U(0,1)$, $\zeta_0 \in \partial U(0,1)$ and a real number $m \in [n, +\infty)$ so that q is defined in ζ_0, $p(z_0) = q(\zeta_0)$, $p(U(0, r_0)) \subset q(U)$, $r_0 = |z_0|$, and*

(i) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

(ii) $\mathrm{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \mathrm{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right).$

We mention that $z_0 p'(z_0)$ is the outward normal to the curve $p(\partial U(0, r_0))$ at the point $p(z_0)$, ($\partial U(0, r_0)$ denotes the border of the disc $U(0, r_0)$)

3 The Main Result

Theorem 2. *Let A be the operator of Alexander and let $g \in \mathcal{A}$ satisfy*

$$\frac{zg'(z)}{g(z)} \prec \frac{2 - z}{2(1 - z)}, \ z \in U. \tag{2}$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{g(z)} \prec \frac{1}{\sqrt{1 - z}}, \ z \in U, \tag{3}$$

then $F = A(f) \in S^*$.

Proof. The first step is to show that the condition (2) implies the subordination

$$\frac{g(z)}{z} \prec \frac{1}{\sqrt{1-z}}. \tag{4}$$

Using the notation $p(z) = \dfrac{g(z)}{z}$, condition (2) becomes

$$\frac{zp'(z)}{p(z)} \prec \frac{z}{2(1-z)} = h(z). \tag{5}$$

If the subordination $p(z) \prec \frac{1}{\sqrt{1-z}}$ does not hold true then according to Lemma 6 there are two points $z_0 \in U$ and $\zeta_0 \in \partial U$ and a real number $m \in [1, \infty)$, so that

$$p(z_0) = \frac{1}{\sqrt{1-\zeta_0}} \quad \text{and} \quad z_0 p'(z_0) = \frac{m}{2}\zeta_0(1-\zeta_0)^{-\frac{3}{2}}.$$

This implies that

$$\frac{z_0 p'(z_0)}{p(z_0)} = mh(\zeta_0).$$

Since h is a starlike function with respect to 0, $m \geq 1$ and $h(\zeta_0)$ is on the border of $h(U)$, we obtain that

$$mh(\zeta_0) \notin h(U) \quad \text{and} \quad \text{so} \quad \frac{z_0 p'(z_0)}{p(z_0)} \notin h(U).$$

This contradicts (5) and consequently (4) holds true.
Lemma 5 implies that there are two probability measures μ and ν so that

$$\frac{g(z)}{z} = \int_0^{2\pi} \frac{1}{\sqrt{1-ze^{-it}}} d\mu(t)$$

and

$$\frac{zf'(z)}{g(z)} = \int_0^{2\pi} \frac{1}{\sqrt{1-ze^{-is}}} d\nu(s).$$

A simple computation leads to

$$f'(z) = \int_0^{2\pi} \frac{1}{\sqrt{1-ze^{-it}}} d\mu(t) \int_0^{2\pi} \frac{1}{\sqrt{1-ze^{-is}}} d\nu(s) =$$
$$\int_0^{2\pi}\int_0^{2\pi} \frac{1}{\sqrt{1-ze^{-it}}} \frac{1}{\sqrt{1-ze^{-is}}} d\mu(t) d\nu(s).$$

According to Lemma 4 there is a probability measure λ so that

$$f'(z) = \int_0^{2\pi} \frac{1}{1-ze^{-it}} d\lambda(t). \tag{6}$$

We get after integrating the equality (6) that

$$f(z) = \int_0^{2\pi} e^{-it} \log\left(\frac{1}{1-ze^{-it}}\right) d\lambda(t) =$$
$$\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^{2\pi} e^{-it(n-1)} d\lambda(t)$$

and

$$F(z) = A(f)(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \int_0^{2\pi} e^{-it(n-1)} d\lambda(t).$$

We obtain after a simple calculation that

$$h_T(z) = z + \sum_{n=1}^{\infty} \frac{n+1+iT}{1+iT} z^{n+1}, \ z \in U.$$

Lemma 2 says that the function F is starlike if and only if

$$\frac{F(z)}{z} * \frac{h_T(z)}{z} \neq 0 \ \text{ for all } T \in \mathbb{R} \text{ and for all } z \in U. \tag{7}$$

We have:

$$\frac{F(z)}{z} * \frac{h_T(z)}{z} = 1 + \sum_{n=1}^{\infty} \frac{z^n(n+1+iT)}{(n+1)^2(1+iT)} \int_0^{2\pi} e^{-itn} d\lambda(t) =$$
$$\left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-itn} d\lambda(t)\right) * \left(1 + \sum_{n=1}^{\infty} \frac{z^n(n+1+iT)}{2(n+1)^2(1+iT)}\right).$$

According to the Lemma 1, to prove (4) we have to show that

$$\operatorname{Re}\left(1+\sum_{n=1}^{\infty}\frac{z^n(n+1+iT)}{2(n+1)^2(1+iT)}\right)>\frac{1}{2},\ z\in U,\ T\in\mathbb{R}$$

or equivalently

$$\operatorname{Re}\left(1+\sum_{n=1}^{\infty}\frac{z^n(n+1+iT)}{(n+1)^2(1+iT)}\right)>0,\ z\in U,\ T\in\mathbb{R}. \tag{8}$$

A simple calculation leads to

$$\operatorname{Re}\left(1+\sum_{n=1}^{\infty}\frac{z^n(n+1+iT)}{(n+1)^2(1+iT)}\right)$$

$$=\operatorname{Re}\left(1+\frac{1}{1+iT}\sum_{n=1}^{\infty}\frac{z^n}{1+n}+\frac{iT}{1+iT}\sum_{n=1}^{\infty}\frac{z^n}{(1+n)^2}\right).$$

Because of the minimum principle to prove (8) it is enough to show that

$$\operatorname{Re}\left(1+\frac{1}{1+iT}\sum_{n=1}^{\infty}\frac{e^{in\theta}}{1+n}+\frac{iT}{1+iT}\sum_{n=1}^{\infty}\frac{e^{in\theta}}{(1+n)^2}\right)>0, \tag{9}$$

$\theta\in(0,2\pi),\ T\in\mathbb{R}$.

We consider the function $f(z)=\dfrac{e^{i\theta z}}{(\beta+z)(e^{2\pi iz}-1)}, \beta>0$, where $\theta\in(0,2\pi)$ is a fixed number.

Let $\Gamma(r,n)$ be the contour constructed in the following way: $\Gamma(r,n)=\gamma_1\cup\gamma_3\cup\gamma_2\cup\gamma_4$, where $\gamma_1(t)=R_ne^{i(\pi t-\frac{\pi}{2})}, \gamma_2(t)=re^{i(-\pi t+\frac{\pi}{2})}, t\in[0,1]$, $\gamma_3(t)=iR_n+t(ir-iR_n), \gamma_4(t)=-ir+t(ir-iR_n), t\in[0,1], r\in(0,1)$ and $R_n=n+\frac{1}{2}$, where n belongs the set of natural numbers. We obtain from the residue theorem that

$$\int_{\Gamma(r,n)}f(z)dz=2\pi i\sum_{0<k<n+\frac{1}{2}}Res(f,k).$$

A straightforward computation yields

$$\lim_{r\to 0}\int_{\gamma_2} f(z)dz = -i\pi\cdot \mathrm{Res}(f,0)$$

$$\mathrm{Res}(f,z_k)=\mathrm{Res}(f,k)=\frac{e^{i\theta k}}{2\pi i(k+\beta)},\quad k\in\mathbb{N}.$$

We finally get that if $\theta\in(0,2\pi)$ and $\beta>0$, then the following identity holds true:

$$\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\beta^2+x^2)(e^{2\pi x}-1)}dx + i\beta\int_0^\infty \frac{e^{(2\pi-\theta)x}-e^{\theta x}}{(\beta^2+x^2)(e^{2\pi x}-1)}dx = \frac{1}{2\beta}+\sum_{k=1}^{\infty}\frac{e^{i\theta k}}{k+\beta}. \tag{11}$$

If we differentiate this equality with respect to β it results that

$$2\beta\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\beta^2+x^2)^2(e^{2\pi x}-1)}dx + i\int_0^\infty \frac{(\beta^2-x^2)(e^{(2\pi-\theta)x}-e^{\theta x})}{(\beta^2+x^2)^2(e^{2\pi x}-1)}dx = \frac{1}{2\beta^2}+\sum_{k=1}^{\infty}\frac{e^{i\theta k}}{(k+\beta)^2}. \tag{12}$$

Using (10) and (11) the expression from (9) becomes

$$\frac{1}{1+T^2}\Bigg(\frac{1}{2}+\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(1+x^2)(e^{2\pi x}-1)}dx+2T\int_0^\infty \frac{x^2(e^{(2\pi-\theta)x}-e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}dx$$
$$+T^2(\frac{1}{2}+2\int_0^\infty \frac{x(e^{(2\pi-\theta)x}+e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}dx)\Bigg)\geq 0, \tag{13}$$

for all $\theta\in(0,2\pi)$, $T\in\mathbb{R}$.

If we prove that

$$\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(1+x^2)(e^{2\pi x}-1)}dx + 2T\int_0^\infty \frac{x^2(e^{(2\pi-\theta)x}-e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}dx$$
$$+2T^2\int_0^\infty \frac{x(e^{(2\pi-\theta)x}+e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}dx\geq 0,\ \text{for all } \theta\in(0,2\pi),\ T\in\mathbb{R}. \tag{14}$$

then (12) results . The expression in (13) is a polynomial of degree two with respect to T. The discriminant of the polynomial is

$$\Delta_T = 4\Big(\int_0^\infty \frac{x^2(e^{(2\pi-\theta)x} - e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}dx\Big)^2 -$$
$$8\int_0^\infty \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(1+x^2)^2(e^{2\pi x}-1)}dx \int_0^\infty \frac{x(e^{\theta x} + e^{(2\pi-\theta)x})}{(1+x^2)(e^{2\pi x}-1)}dx.$$

The condition (14) holds true if $\Delta_T \leq 0,\ \theta \in (0, 2\pi),\ T \in \mathbb{R}$. This inequality is a simple consequence of the Cauchy-Schwarz inequality.

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Universitatea Sapientia
Facultatea de Ştiinţe Tehnice şi Umaniste
Şoseaua Sighişoarei, 1C
540485, Tg. Mureş, Romania
E-mail: szasz_robert2001@yahoo.com