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A criteria of ϕ-like functions[1]

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Abstract

In this paper, we obtain some sufficient conditions for a normalized analytic function to be ϕ-like and starlike of order α.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the unit disc $E = \{z : |z| < 1\}$ and are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Denote by $S^*(\alpha)$ and $K(\alpha)$, the classes of starlike functions of order α and convex functions of order α respectively, which are analytically defined as follows

$$S^*(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in E \right\}$$

and

$$K(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in E \right\}$$

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where α is a real number such that $0 \leq \alpha < 1$. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.
Let f and g be analytic in E. We say that f is subordinate to g in E, written as $f(z) \prec g(z)$ in E, if g is univalent in E, $f(0) = g(0)$ and $f(E) \subset g(E)$. Denote by $S^*[A, B]$, $-1 \leq B < A \leq 1$, the class of functions $f \in \mathcal{A}$ which satisfy

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz},\ z \in E.$$

Note that $S^*[1 - 2\alpha, -1] = S^*(\alpha),\ 0 \leq \alpha < 1$ and $S^*[1, -1] = S^*$.
A function $f, f'(0) \neq 0$, is said to be close-to-convex in E, if and only if, there is a starlike function h (not necessarily normalized) such that

$$\Re\, \frac{zf'(z)}{h(z)} > 0,\ z \in E.$$

Let ϕ be analytic in a domain containing $f(E)$, $\phi(0) = 0$ and $\Re\, \phi'(0) > 0$, then, the function $f \in \mathcal{A}$ is said to be ϕ-like in E if

$$\Re\, \frac{zf'(z)}{\phi(f(z))} > 0,\ z \in E.$$

This concept was introduced by L. Brickman [1]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is ϕ-like for some ϕ. Later, Ruscheweyh [8] investigated the following general class of ϕ-like functions: Let ϕ be analytic in a domain containing $f(E)$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(E) - \{0\}$, then the function $f \in \mathcal{A}$ is called ϕ-like with respect to a univalent function $q, q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in E.$$

In the present note, we obtain some sufficient conditions for a normalized analytic function to be ϕ-like. In [9], Silverman defined the class G_b as

$$G_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b,\ z \in E \right\}$$

and proved that the functions in the class G_b are starlike in E. Later on, this class was studied extensively by Tuneski [4,11,12,13,14,15]. As particular cases, we obtain many interesting results for the class G_b. Most of the results proved by Tuneski follow as corollaries to our theorem.

2 Preliminaries

We shall need following definition and lemmas to prove our results.

Definition 2.1. *A function $L(z,t), z \in E$ and $t \geq 0$ is said to be a subordination chain if $L(.,t)$ is analytic and univalent in E for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in E$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.*

Lemma 2.1 [5, page 159]. *The function $L(z,t) : E \times [0,\infty) \to \mathbb{C}$, ($\mathbb{C}$ is the set of complex numbers), of the form $L(z,t) = a_1(t)z + \ldots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t\to\infty} |a_1(t)| = \infty$, is said to be a subordination chain if and only if $\operatorname{Re}\left[\frac{z\partial L/\partial z}{\partial L/\partial t}\right] > 0$ for all $z \in E$ and $t \geq 0$.*

Lemma 2.2 [3]. *Let F be analytic in E and let G be analytic and univalent in $\overline{E}$ except for points ζ_0 such that $\lim_{z\to\zeta_0} F(z) = \infty$, with $F(0) = G(0)$. If $F \not\prec G$ in E, then there is a point $z_0 \in E$ and $\zeta_0 \in \partial E$ (boundary of E) such that $F(|z| < |z_0|) \subset G(E)$, $F(z_0) = G(\zeta_0)$ and $z_0F'(z_0) = m\zeta_0G'(\zeta_0)$ for some $m \geq 1$.*

3 Main Result

Lemma 3.1. *Let $\gamma, \Re\ \gamma \geq 0$, be a complex number. Let q be univalent function such that either $\frac{zq'(z)}{q^2(z)}$ is starlike in E or $\frac{1}{q(z)}$ is convex in E. If an analytic function p, satisfies the differential subordination*

$$(3.1) \qquad 1 - \frac{\gamma}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec 1 - \frac{\gamma}{q(z)} + \frac{zq'(z)}{q^2(z)},\ p(0) = q(0) = 1,\ z \in E,$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Let us define a function

$$h(z) = 1 - \frac{\gamma}{q(z)} + \frac{zq'(z)}{q^2(z)}, \; z \in E. \tag{3.2}$$

Firstly, we will prove that $h(z)$ is univalent in E so that the subordination (3.1) is well-defined in E. Differentiating (3.2) and simplifying a little, we get

$$\frac{zh'(z)}{Q(z)} = \gamma + \frac{zQ'(z)}{Q(z)}, \; z \in E,$$

where $Q(z) = \frac{zq'(z)}{q^2(z)}$. In view of the given conditions, we obtain

$$\Re \frac{zh'(z)}{Q(z)} > 0, \; z \in E.$$

Thus, $h(z)$ is close-to-convex and hence univalent in E. We need to show that that $p \prec q$. Suppose to the contrary that $p \not\prec q$ in E. Then by Lemma 2.2, there exist points $z_0 \in E$ and $\zeta_0 \in \partial E$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta q'(\zeta_0), \; m \geq 1$. Then

$$1 - \frac{\gamma}{p(z_0)} + \frac{z_0 p'(z_0)}{p^2(z_0)} = 1 - \frac{\gamma}{q(\zeta_0)} + \frac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}, \; z \in E. \tag{3.3}$$

Consider a function

$$L(z,t) = 1 - \frac{\gamma}{q(z)} + (1+t)\frac{zq'(z)}{q^2(z)}, \; z \in E. \tag{3.4}$$

The function $L(z,t)$ is analytic in E for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in E$. Now,

$$a_1(t) = \left(\frac{\partial L(z,t)}{\partial z}\right)_{(0,t)} = q'(0)(\gamma + 1 + t).$$

In view of the condition that $\Re\,\gamma \geq 0$, we get $|\arg(\gamma + 1 + t)| \leq \pi/2$. Also, as q is univalent in E, so, $q'(0) \neq 0$. Therefore, it follows that $a_1(t) \neq 0$ and

$\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$z\frac{\partial L/\partial z}{\partial L/\partial t} = \gamma + (1+t)\frac{zQ'(z)}{Q(z)},\ z \in E.$$

Clearly

$$\Re\ z\frac{\partial L/\partial z}{\partial L/\partial t} > 0,\ z \in E,$$

in view of given conditions. Hence, $L(z,t)$ is a subordination chain. Therefore, $L(z,t_1) \prec L(z,t_2)$ for $0 \le t_1 \le t_2$. From (3.4), we have $L(z,0) = h(z)$, thus we deduce that $L(\zeta_0, t) \notin h(E)$ for $|\zeta_0| = 1$ and $t \ge 0$. In view of (3.3) and (3.4), we can write

$$1 - \frac{\gamma}{p(z_0)} + \frac{z_0 p'(z_0)}{p^2(z_0)} = L(\zeta_0, m-1) \notin h(E),$$

where $z_0 \in E$, $|\zeta_0| = 1$ and $m \ge 1$ which is a contradiction to (3.1). Hence, $p \prec q$. This completes the proof of the Lemma.

Theorem 3.1. *Let $\gamma, \Re\ \gamma \ge 0$, be a complex number. Let $q, q(0) = 1$, be a univalent function such that $\frac{zq'(z)}{q^2(z)}$ is starlike in E or, equivalently, $\frac{1}{q(z)}$ is convex in E. If an analytic function $f \in \mathcal{A}$ satisfies the differential subordination*

$$1 + \frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/\phi(f(z))} - \frac{(\phi(f(z)))'}{f'(z)} \prec 1 - \frac{\gamma}{q(z)} + \frac{zq'(z)}{q^2(z)},\ z \in E,$$

for some function ϕ, analytic in a domain containing $f(E)$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \ne 0$ for $w \in f(E) - \{0\}$, then $\frac{zf'(z)}{\phi(f(z))} \prec q(z)$ and $q(z)$ is the best dominant.

Proof. The proof of the theorem follows by writing $p(z) = \frac{zf'(z)}{\phi(f(z))}$ in Lemma 3.1.

In particular, for $\phi(w) = w$ and $q(z) = \frac{zg'(z)}{g(z)}$ in Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let $\gamma, \Re\ \gamma \ge 0$, be a complex number. Let $g \in \mathcal{A}$ be such that $\frac{zg'(z)}{g(z)} = q(z)$ is univalent in E. Assume that either $\frac{zq'(z)}{q^2(z)}$ is starlike* *in E or $\frac{1}{q(z)}$ is convex in E. If an analytic function $f \in \mathcal{A}$ satisfies the differential subordination*

$$\frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \frac{1-\gamma+zg''(z)/g'(z)}{zg'(z)/g(z)},\ z \in E,$$

then $\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$.

4 Applications to univalent functions

In this section, we obtain a criterion for a normalized analytic function to be ϕ-like. As an application of Theorems 3.1 and 3.2, we obtain some new conditions and also few existing conditions for a function to be in the class S^* and $S^*(\alpha)$.
When the dominant is $q(z) = \frac{1+Az}{1+Bz}$. We observe that q is univalent in E and $\frac{1}{q(z)}$ is convex in E where $-1 \leq B < A \leq 1$. From Theorem 3.1, we deduce the following result.

Theorem 4.1. *Let $\gamma, \Re\, \gamma \geq 0$, be a complex number and A and B be real numbers $-1 \leq B < A \leq 1$. Let $f \in \mathcal{A}$ satisfy the differential subordination*

$$1+\frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/\phi(f(z))} - \frac{(\phi(f(z)))'}{f'(z)} \prec 1-\gamma\frac{1+Bz}{1+Az}+\frac{(A-B)z}{(1+Az)^2},\ z \in E,$$

for some function ϕ, analytic in a domain containing $f(E)$, $\phi(0)=0, \phi'(0)=1$ and $\phi(w) \neq 0$ for $w \in f(E)-\{0\}$, then $\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+Az}{1+Bz},\ z \in E$.

As an example, if we take $\gamma = i, A = 0, B = -1$ in Theorem 4.1, we obtain the following result.
Example 4.1. Let $f \in \mathcal{A}$ satisfy

$$\left|\frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/\phi(f(z))} - \frac{(\phi(f(z)))'}{f'(z)} + i\right| < \sqrt{2},\ z \in E,$$

then $\frac{zf'(z)}{\phi(f(z))} \prec \frac{1}{1-z},\ z \in E.$

In particular, for $\gamma = 0$ and $A = 1, B = -1$, Theorem 4.1, reduces to the following result.

Corollary 4.1. *Let $f \in \mathcal{A}$ satisfy the differential subordination*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/\phi(f(z))} - \frac{(\phi(f(z)))'}{f'(z)} \prec \frac{2z}{(1+z)^2}, \; z \in E,$$

for some function ϕ, analytic in a domain containing $f(E)$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(E) - \{0\}$, then $\mathrm{Re}\frac{zf'(z)}{\phi(f(z))} > 0, \; z \in E$.

Note that several such results are available for different substitutions of constants A, B.

For the dominant $q(z) = \frac{1+Az}{1+Bz}$, Theorem 3.2 gives us the following result.

Theorem 4.2. Let $\gamma, \Re\, \gamma \geq 0$, *be a complex number and A and B be real numbers $-1 \leq B < A \leq 1$. Let $f \in \mathcal{A}$ satisfy the differential subordination*

$$\frac{1 - \gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 - \gamma\frac{1 + Bz}{1 + Az} + \frac{(A - B)z}{(1 + Az)^2}, \; z \in E,$$

then $f \in S^*[A, B]$.

Writing $\gamma = 1$ in Theorem 4.2, we obtain the following result.

Corollary 4.2. *If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\frac{f''(z)f(z)}{f'^2(z)} \prec 1 - \frac{1 + Bz}{1 + Az} + \frac{(A - B)z}{(1 + Az)^2}, \; z \in E, \; -1 \leq B < A \leq 1$$

then $f \in S^*[A, B]$.

Writing $A = 0$ in Theorem 4.2, we obtain the following result.

Corollary 4.3. *Let $f \in \mathcal{A}$ satisfy*

$$\left|\frac{1 - \gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} - (1 - \gamma)\right| < (1 + \gamma)B, \; z \in E, \gamma \geq 0, \; 0 < B \leq 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1 + Bz}, \; z \in E.$$

In particular, for $\gamma = 1$, in Corollary 4.3, we obtain the following result.

Corollary 4.4. *Let* $f \in \mathcal{A}$ *satisfy*

$$\left|\frac{f(z)f''(z)}{f'^2(z)}\right| < 2B,\ z \in E,\ 0 < B \leq 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1+Bz},\ z \in E.$$

The selection of $B = 0$ in Theorem 4.2 gives us the following result.

Corollary 4.5. *Let* $f \in \mathcal{A}$ *satisfy*

$$\frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1-\frac{\gamma}{1+Az}+\frac{Az}{(1+Az)^2},\ z \in E,\ \gamma \geq 0,\ 0 < A \leq 1,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < A,\ z \in E.$$

In particular, for $\gamma = 0$ in Corollary 4.5, we obtain the following result.

Corollary 4.6. *Let* $f \in \mathcal{A}$ *satisfy*

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1+\frac{Az}{(1+Az)^2},\ z \in E,\ 0 < A \leq 1,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < A,\ z \in E.$$

Taking $\gamma = 1$ in corollary 4.5, we obtain the following result.

Corollary 4.7. *If*

$$\frac{f(z)f''(z)}{f'^2(z)} \prec 1-\frac{1}{(1+Az)^2},\ z \in E,\ 0 < A \leq 1,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < A,\ z \in E.$$

Remark 4.1. (i) *Writing* $\gamma = 0$ *in Theorem* 4.2, *we obtain the Theorem 2.3 in* [14].

(ii) *Writing* $A = -1, B = 1$ *in Theorem* 4.2, *we obtain Theorem 1 of* [15].
(iii) *Taking* $A = 1, B = -1, \gamma = 0$ *in Theorem* 4.2, *we obtain Theorem* 3 *in* [4].
(iv) *Taking* $A = -1, B = 1, \gamma = 1$ *in Theorem* 4.2, *we get Theorem* 1 *in* [12].
(v) *Taking* $A = 0, \gamma = 0$ *in Theorem* 4.2, *we obtain Theorem* 1 *in* [4].
(vi) *Writing* $A = 0, B = -1, \gamma = 1$ *in Theorem* 4.2, *we obtain the following result*:
If $f \in \mathcal{A}$ *satisfies*, $\frac{f''(z)f(z)}{f'^2(z)} \prec 2z,\ z \in E$, *then* $f \in S^*(1/2)$.
This is an improvement of Corollary 2 proved in [12].
(vii) *Taking* $A = -(1 - 2\alpha), B = 1, 0 \leq \alpha < 1$ *in Theorem* 4.2, *we get the Theorem* 3 *in* [15].
(viii) *Writing* $A = -(1 - 2\alpha), B = 1, 0 \leq \alpha < 1$ *and* $\gamma = 0$ *in Theorem* 4.2, *we obtain Corollary* 4(i) *in* [15].
(ix) *Writing* $A = -(1 - 2\alpha), B = 1, 0 \leq \alpha < 1$ *and for* $\gamma = 1$ *in Theorem 4.2, Corollary* 4(ii) *in* [15] *follows.*
(x) *For* $B = \frac{1-\beta}{\beta}, 1/2 \leq \beta < 1$ *in Corollary* 4.4, *we obtain the result of Robertson* [7].
(xi) *Taking* $q(z) = \frac{2\alpha}{1+z}$ *in Theorem* 3.2, *we obtain Theorem* 2 *in* [15].

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