General Mathematics Vol. 14, No. 4 (2006), 47–50

The properties of Laguerre polynomials [1]

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Abstract

In this paper we prove a property of the Laguerre polynomials L_n^α using the interpolation polynomial of Hermite.

2000 Mathematics Subject Classification: 33C45

Let $\alpha > -1, x \geq 0$ and $L_n^\alpha(x) = \dfrac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} e^x x^{-\alpha}(e^{-x}x^{n+\alpha})^{(n)}$ be the polynomials of degree n normalized by $L_n^{(\alpha)}(0) = 1$.
In the following we shall use the notation $L_n(x) = L_n^{(\alpha)}(x)$. The following formulas are known:

$$(1) \qquad xy''(x) + (1+\alpha-x)y'(x) + ny(x) = 0, \quad y(x) = L_n(x)\ ,$$

$$(2) \qquad (n+\alpha+1)L_{n+1}(x) + (x-\alpha-2n-1)L_n(x) + nL_{n-1}(x) = 0\ ,$$

$$(3) \qquad x\frac{d}{dx}L_n(x) = nL_n(x) - nL_{n-1}(x), \quad (\forall)\ \alpha > -1, (\forall)\ x \geq 0\ .$$

Theorem 1. *Let $\omega(x) = \lambda(x-x_1)(x-x_2)\dots(x-x_n)$ by $\omega(0) = 1$ and $x_i \neq x_j$ for $i \neq j$.*

If

$$2x \sum_{1\leq i<j\leq n} \frac{1}{(x-x_i)(x-x_j)} + (\alpha+1-x)\sum_{k=1}^{n}\frac{1}{x-x_k} + n = 0 \quad , \quad \alpha > -1$$

is verified, then

$$\omega(x) = L_n(x)\ .$$

[1] *Received ...., 2006*
Accepted for publication (in revised form) ....., 2006

Proof. We consider $\Delta_{2n}(x) = L_n^2(x) - L_{n+1}(x)L_{n-1}(x)$ and observe that $\Delta_{2n}(0) = 0$. According to Hermite interpolation formula

$$\begin{aligned}\Delta_{2n}(x) &= H_{2n}(x_1x_1, x_2x_2, \ldots, x_nx_n, c; \Delta_{2n}(x)) = \\ &= \left[\frac{L_n(x)}{L_n(c)}\right]^2 \Delta_{2n}(c) + (x-c)\sum_{k=1}^{n} \frac{\varphi_k(x)}{x_k - c} B_k(\Delta_{2n}; x)\end{aligned}$$

where $x_1, x_2, \ldots, x_n$ are the roots of $L_n(x)$ and

$$\varphi_k(x) = \left[\frac{L_n(x)}{(x-x_k)L_n'(x_k)}\right]^2 ,$$

$$B_k(\Delta_{2n}; x) = \Delta_{2n}(x_k) + (x-x_k)\left[\Delta_{2n}'(x_k) - \frac{L_n''(x_k)}{L_n'(x_k)}\Delta_{2n}(x_k) - \frac{1}{x_k - c}\Delta_{2n}(x_k)\right] .$$

For $c = 0$, we obtain

$$\Delta_{2n}(0) = 0 ,$$

$$\Delta_{2n}(x) = L_n^2(x) - L_{n+1}(x)L_{n-1}(x) = x\sum_{k=1}^{n}\left[\frac{L_n(x)}{(x-x_k)L_n'(x_k)}\right]^2 \cdot \frac{1}{x_k} \cdot B_k(\Delta_{2n}; x) -$$

$$-1 - \frac{L_{n+1}(x)}{L_n(x)} \cdot \frac{L_{n-1}(x)}{L_n(x)} = x\sum_{k=1}^{n}\left[\frac{l}{(x-x_k)L_n'(x_k)}\right]^2 \cdot \frac{B_k(\Delta_{2n}; x)}{x_k} .$$

Further, we investigate $B_k(\Delta_{2n}; x)$. Observe that

$$\Delta_{2n}(x_k) = -L_{n+1}(x_k)L_{n-1}(x_k) .$$

From (2) we have

$$L_{n+1}(x_k) = -\frac{n}{n+\alpha+1} L_{n-1}(x_k) \tag{4}$$

$$\Delta_{2n}(x_k) = -\frac{n}{n+\alpha+1} L_{n-1}^2(x_k) \tag{5}$$

$$\Delta_{2n}'(x_k) = -L_{n+1}'(x_k)L_{n-1}(x_k) - L_{n+1}(x_k)L_{n-1}'(x_k) .$$

Using (2) and (3) one finds

$$L_{n+1}'(x_k) = \frac{n+1}{x_k} L_{n+1}(x_k) \quad , \quad L_{n-1}'(x_k) = \frac{x_k - \alpha - n}{x_k} L_{n-1}(x_k) .$$

Therefore

(6) $$\Delta'_{2n}(x_k) = \frac{n}{n+\alpha+1} \cdot \frac{x_k - \alpha + 1}{x_k} L^2_{n-1}(x_k)\ .$$

From (1), (5) and (6) we obtain

(7) $$\frac{\Delta'_{2n}(x_k)}{\Delta_{2n} x_k} = \frac{x_k - \alpha + 1}{x_k}$$

and

(8) $$\frac{L''_n(x_k)}{L'_n(x_k)} = -\frac{1 + \alpha - x_k}{x_k}\ .$$

By means (7), (8) we have

$$B_k(\Delta_{2n}; x) = \Delta_{2n}(x_k) \left\{ 1 + (x - x_k) \left[\frac{\Delta'_{2n}(x_k)}{\Delta_{2n}(x_k)} - \frac{L''_n(x_k)}{L'_n(x_k)} - \frac{1}{x_k} \right] \right\}\ ,$$

(9) $$B_k(\Delta_{2n}; x) = \frac{x}{x_k} \Delta_{2n}(x_k)\ .$$

Therefore

$$\Delta_{2n}(x) = L^2_n(x) - L_{n+1}(x) L_{n-1}(x) = x \sum_{k=1}^{n} \frac{L^2_n(x)}{[(x - x_k) L'_n(x_k)]^2} \cdot \frac{x}{x_k^2} \Delta_{2n}(x_k)\ ,$$

$$1 - \frac{L_{n+1}(x)}{L_n(x)} \cdot \frac{L_{n-1}(x)}{L_n(x)} = x \sum_{k=1}^{n} \frac{1}{[(x - x_k) L'_n(x_k)]^2} \cdot \frac{x}{x_k^2} \Delta_{2n}(x_k)$$

From (3), we have

(10) $$\frac{L_{n-1}(x)}{L_n(x)} = 1 - \frac{x}{n} \cdot \frac{L'_n(x)}{L_n(x)}$$

(11) $$\frac{L_{n+1}(x)}{L_n(x)} = 1 + \frac{x}{n+\alpha+1} \left[\frac{L'_n(x)}{L_n(x)} - 1 \right]\ .$$

(12) $$\frac{L_{n-1}(x_k)}{L'_n(x_k)} = -\frac{x_k}{n}$$

$$1 - \left\{ 1 + \frac{x}{n+\alpha+1} \left[\frac{L'_n(x)}{L_n(x)} - 1 \right] \right\} \cdot \left\{ 1 - \frac{x}{n} \cdot \frac{L'_n(x)}{L_n(x)} \right\} =$$

$$= \frac{n}{n\alpha+1} x^2 \sum_{k=1}^{n} \frac{1}{(x - x_k)^2} \cdot \left[\frac{L_{n-1}(x_k)}{L'_n(x_k)} \right]^2\ ,$$

$$\frac{\alpha+1-x}{n}\cdot\frac{L_n'(x)}{L(x)}+\frac{x}{n}\left[\frac{L_n'(x)}{L_n(x)}\right]^2+1=\frac{x}{n}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}\ ,$$

$$\frac{x}{n}\left(\sum_{k=1}^{n}\frac{1}{x-x_k}\right)^2+\frac{\alpha+1-x}{n}\sum_{k=1}^{n}\frac{1}{x-x_k}+1=\frac{x}{n}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}\ ,$$

$$2x\sum_{1\le i<j\le n}\frac{1}{(x-x_i)(x-x_j)}+(\alpha+1-x)\sum_{k=1}^{n}\frac{1}{x-x_k}+n=0\ .$$

In conclusion $\omega(x)=L_n(x)$.

References

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