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On New Classes of Sălăgean-type p-valent Functions with Negative and Missing Coefficients [1]

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Abstract

We define and investigate new classes of Sălăgean-type p-valent functions with negative and missing coefficients. We obtain coefficient estimates, distortion bounds, integral operators of functions belonging to these classes, extreme points, convex combinations and radius of convexity for these classes of p-valent functions. Furthermore, we give modified Hadamard product of several functions and some distortion theorems for fractional calculus of p-valent functions with negative and missing coefficients belonging to these generalized classes.

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1 Introduction

Let A be class of functions $f(z)$of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

For $f(z)$ belong to A, Sălăgean [9] has introduced the following operator called the Sălăgean operator:

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z),$$

$$D^n f(z) = D(D^{n-1} f(z)),\ \ n \in N = \{1, 2, \cdots\}.$$

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \ \ ; \ \ n \in N_0 = \{0\} \cup N.$$

Let $S_p(p \geq 1)$denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

that are holomorphic and p-valent in the unit disc U.

Also, let T_p denote the subclass of $S_p(p \geq 1)$ consisting of functions that can be expressed in the form,

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n} \ ; \ a_{p+n} \geq 0 \ , \ k \geq 2.$$

We can write the following equalities for the functions $f(z)$ belonging to the class T_p:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = \frac{z}{p} f'(z) = z^p - \sum_{n=k}^{\infty} \frac{(p+n)}{p} a_{p+n} z^{p+n},$$

$$\vdots$$

$$D^\lambda f(z) = D(D^{\lambda-1} f(z)) = z^p - \sum_{n=k}^{\infty} \frac{(p+n)^\lambda}{p^\lambda} a_{p+n} z^{p+n} \quad ; \ \lambda \in N_0 = \{0\} \cup N.$$

A function $f(z) \in T_p$is in $T_p^*(\alpha, A, B, k, \beta, \lambda)$ if and only if

$$\left| \frac{z(D^\lambda f(z))' - pD^\lambda f(z)}{[(A-B)(p-\alpha) + pB]D^\lambda f(z) - Bz(D^\lambda f(z))'} \right| < \beta,$$

for

$$\lambda \in N_0, 0 \leq \alpha < p, 0 < \beta \leq 1, -1 \leq B < A \leq 1 \ , -1 \leq B < 0, n \geq k \geq 2$$

and $z \in U$.

Further f said to belong to the class $C_p(\alpha, A, B, k, \beta, \lambda)$ if and only if $\frac{zf'}{p} \in T_p^*(\alpha, A, B, k, \beta, \lambda)$.

We note that by specializing the parameters $\alpha, A, B, k, \beta \ and \, \lambda$, we obtain the following interesting subclasses including those that were studied by various earlier authors.

(i) In [1], Ahuja and Jain defined $T_1^*(\alpha, -1, 1, q, \beta, 0),\ q \geq 1$, the subclass of starlike functions of order α and type-β and also defined $C_1(\alpha, -1, 1, q, \beta, 0)\ , q \geq 1$, the class of convex functions of order α and type-β.

(ii) The subclass $T_1^*(\alpha, A, B, q, \beta, 0),\ q \geq 1$ has been studied by Aouf [6].

(iii) The subclasses $T_1^*(\alpha,-1,1,1,\beta,0)$ and $C_1(\alpha,-1,1,1,\beta,0)$, especially the class $T_1^*(\alpha,(2\alpha-1)\beta,\beta,1,1,0)$, have been studied by Gupta and Jain [2].

(iv) The classes $T^*(\alpha) = T_1^*(\alpha,-1,1,1,1,0)$ and $C(\alpha) = = C_1(\alpha,-1,1,1,1,0)$ which are subclasses starlike of order α and convex of of order α, respectively, have been studied by Silverman [3]. Evidently $T^*(0) = T_1^*(0,-1,1,1,1,0)$.

(v) The subclass $T_p^*(0,A,B,1,1,0) = T_p^*(A,B)$ was defined by Goel and Sohi [7].

(vi) The subclass $T_p^*(0,A,B,k,1,0) = P_k(p,A,B,0)$ has been investigated by Sarangi and Patel [5].

(vii) The subclass $T_p^*(\alpha,A,B,k,1,0) = P_k(p,A,B,\alpha)$ was defined by Aouf and Darwish [4].

Finally, we generalized the results of Aouf and Darwish [4] and investigated the class $C_p(\alpha,A,B,k,\beta,\lambda)$ which is generalization of the results of Ahuja and Jain [1] ,Gupta and Jain [2] and Silverman [3]. Furthermore, we give modified Hadamard product of several functions and some distortion theorems for fractional calculus of analytic functions with negative and missing coefficients belonging to a certain generalized classes $T_p^*(\alpha,A,B,k,\beta,\lambda)$ and $C_p(\alpha,A,B,k,\beta,\lambda)$.

2 Coefficients Estimates

Theorem 2.1. *A function*

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n} \ ; \ \ a_{p+n} \geq 0,$$

belongs to $T_p^*(\alpha, A, B, k, \beta, \lambda)$ *if and only if*

$$(1)\qquad \sum_{n=k}^{\infty}[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}a_{p+n} \le (A-B)p^{\lambda}\beta(p-\alpha)$$

The result is sharp.

Proof. Assume that the inequality (1) holds true and let $|z| = 1$. Then we obtain

$$\left|z(D^{\lambda}f(z))' - pD^{\lambda}f(z)\right| - \beta\left|[(A-B)(p-\alpha)+pB]D^{\lambda}f(z) - Bz(D^{\lambda}f(z))'\right| =$$

$$\left|-\sum_{n=k}^{\infty}[(p+n)^{\lambda+1} - p(p+n)^{\lambda}]a_{p+n}z^{p+n}\right| -$$

$$-\beta\left|(A-B)(p-\alpha)p^{\lambda}z^{p} - \sum_{n=k}^{\infty}[(A-B)(p-\alpha)(p+n)^{\lambda}+\right.$$

$$\left.+pB(p+n)^{\lambda} - B(p+n)^{\lambda+1}]a_{p+n}z^{p+n}\right| \le$$

$$\le \sum_{n=k}^{\infty}[(p+n)^{\lambda+1} - p(p+n)^{\lambda} + (A-B)\beta(p-\alpha)(p+n)^{\lambda} - B\beta n(p+n)^{\lambda}]a_{p+n} -$$

$$-(A-B)\beta(p-\alpha)p^{\lambda} =$$

$$= \sum_{n=k}^{\infty}[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}a_{p+n} - (A-B)p^{\lambda}\beta(p-\alpha) \le 0$$

by hypothesis. Hence, by the maximum modulus theorem, we have

$$f \in T_p^*(\alpha, A, B, k, \beta, \lambda).$$

Conversely, assume that

$$\left|\frac{z(D^{\lambda}f(z))' - pD^{\lambda}f(z)}{[(A-B)(p-\alpha)+pB]D^{\lambda}f(z) - Bz(D^{\lambda}f(z))'}\right|$$

$$= \left| \frac{-\sum_{n=k}^{\infty}[\frac{(p+n)^{\lambda+1}}{p^{\lambda}} - \frac{(p+n)^{\lambda}}{p^{\lambda-1}}]a_{p+n}z^{p+n}}{(A-B)(p-\alpha)z^p - \sum_{n=k}^{\infty}[(A-B)(p-\alpha)-Bn]\frac{(p+n)^{\lambda}}{p^{\lambda}}a_{p+n}z^{p+n}} \right| < \beta.$$

Since $|Re(z)| \leq | z |$ for all z, we have

$$Re\left\{ \frac{\sum_{n=k}^{\infty}[\frac{(p+n)^{\lambda+1}}{p^{\lambda}} - \frac{(p+n)^{\lambda}}{p^{\lambda-1}}]a_{p+n}z^{p+n}}{(A-B)(p-\alpha)z^p - \sum_{n=k}^{\infty}[(A-B)(p-\alpha)-Bn]\frac{(p+n)^{\lambda}}{p^{\lambda}}a_{p+n}z^{p+n}} \right\} < \beta. \tag{2}$$

Choose values of z on the real axis and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=k}^{\infty}[(p+n)^{\lambda+1} - p(p+n)^{\lambda}]a_{p+n} \leq (A-B)\beta(p-\alpha)p^{\lambda} -$$

$$-\sum_{n=k}^{\infty}[(A-B)\beta(p-\alpha) - B\beta n](p+n)^{\lambda}a_{p+n}$$

which obviously is required assertion (1).

Finally, the function

$$f(z) = z^p - \sum_{n=k}^{\infty}\frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}}z^{p+n}. \tag{3}$$

is an extremal function.

Corollary 2.1. *If $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$, then*

$$a_{p+n} \leq \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}}$$

with equality only for functions of the form

$$f(z) = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}}z^{p+n}.$$

Theorem 2.2. *A function*

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{n+p} z^{p+n}$$

belongs to $C_p(\alpha, A, B, k, \beta, \lambda)$ *if and only if*

$$(4) \qquad \sum_{n=k}^{\infty} [(1 - B\beta)n + (A - B)\beta(p - \alpha)](p + n)^{\lambda+1} a_{p+n} \leq$$

$$\leq (A - B)p^{\lambda+1}\beta(p - \alpha).$$

The result is sharp.

Proof. $f \in C_p(\alpha, A, B, k, \beta, \lambda)$ is equivalent $\frac{zf'}{p} \in T_p^*(\alpha, A, B, k, \beta, \lambda)$. Since

$$\frac{zf'(z)}{p} = z^p - \sum_{n=k}^{\infty} \left(\frac{p+n}{p}\right) a_{n+p} z^{p+n},$$

we may replace a_{p+n} by $\frac{p+n}{p} a_{p+n}$ in Theorem 2.1.

3 Distortion Properties

Theorem 3.1. *If* $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$, *then for* $|z| = r < 1$

$$(5) \qquad r^p - \frac{(A - B)\beta(p - \alpha)p^{\lambda}}{[(1 - B\beta)k + (A - B)\beta(p - \alpha)](p + k)^{\lambda}} r^{p+k} \leq |f(z)| \leq$$

$$\leq r^p + \frac{(A - B)\beta(p - \alpha)p^{\lambda}}{[(1 - B\beta)k + (A - B)\beta(p - \alpha)](p + k)^{\lambda}} r^{p+k}$$

and

$$(6) \; pr^{p-1} - \frac{(A - B)\beta(p - \alpha)p^{\lambda}}{[(1 - B\beta)k + (A - B)\beta(p - \alpha)](p + k)^{\lambda-1}} r^{p+k-1} \leq |f'(z)| \leq$$

$$\leq pr^{p-1} + \frac{(A - B)\beta(p - \alpha)p^{\lambda}}{[(1 - B\beta)k + (A - B)\beta(p - \alpha)](p + k)^{\lambda-1}} r^{p+k-1}.$$

All the inequalities are sharp.

Proof. Let $f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}$, $a_{p+n} \geq 0$.

From Theorem 2.1, we have

$$[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda} \sum_{n=k}^{\infty} a_{p+n} \leq$$

$$\leq \sum_{n=k}^{\infty} [(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda} a_{p+n} \leq (A-B)\beta(p-\alpha)p^{\lambda}$$

which implies that

$$\sum_{n=k}^{\infty} a_{p+n} \leq \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda}} \tag{7}$$

and

$$\sum_{n=k}^{\infty} (p+n) a_{p+n} \leq \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda-1}}. \tag{8}$$

Consequently, for $\mid z \mid = r < 1$, we obtain

$$\mid f(z) \mid \leq \mid z \mid^p + \sum_{n=k}^{\infty} \mid a_{p+n} \mid \mid z \mid^{p+n} \leq r^p + r^{p+k} \sum_{n=k}^{\infty} a_{p+n} \leq$$

$$\leq \ r^p + \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda}} r^{p+k}$$

and

$$\mid f(z) \mid \geq \mid z \mid^p - \sum_{n=k}^{\infty} \mid a_{p+n} \mid \mid z \mid^{p+n} \geq$$

$$\geq r^p - r^{p+1} \sum_{n=k}^{\infty} a_{p+n} \geq$$

$$\geq r^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda}} r^{p+k}$$

which prove that the assertion (5) of Theorem 3.1.

Furthermore, for $\mid z \mid= r < 1$ and (8) , we have

$$\mid f'(z) \mid\leq p \mid z \mid^{p-1} + \sum_{n=k}^{\infty}(p+n) \mid a_{p+n} \mid\mid z \mid^{p+n-1}\leq$$

$$\leq pr^{p-1} + r^{p+k-1}\sum_{n=k}^{\infty}(p+n)a_{p+n} \leq$$

$$\leq pr^{p-1} + \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda-1}}r^{p+k-1}$$

and

$$\mid f'(z) \mid\geq p \mid z \mid^{p-1} - \sum_{n=k}^{\infty}(p+n) \mid a_{p+n} \mid\mid z \mid^{p+n-1}\geq$$

$$\geq pr^{p-1} - r^{p}\sum_{n=k}^{\infty}(p+n)a_{p+n} \geq$$

$$\geq pr^{p-1} - \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda-1}}r^{p+k-1}$$

which prove that the assertion (6) of Theorem 3.1.

The bounds in (5) and (6) are attained for the function $f$given by

$$f(z) = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda}}z^{p+k}. \tag{9}$$

Letting $r \to 1^-$ in the left hand side of (5), we have the following:

Corollary 3.1. *Let $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$. Then the unit disk U is mapped by f onto a domain that contains the disk*

$$|w| < \frac{(p+k)^{\lambda}(1-B\beta)k + (A-B)\beta(p-\alpha)[(p+k)^{\lambda} - p^{\lambda}]}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda}}.$$

The result is sharp with the extremal function f being given by (6).

Theorem 3.2. *Let $f \in C_p(\alpha, A, B, k, \beta, \lambda)$, then $\mid z \mid= r < 1$*

$$r^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda+1}} r^{p+k} \le |f(z)| \le \tag{10}$$

$$\le r^p + \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda+1}} r^{p+k}$$

and

$$pr^{p-1} - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda}} r^{p+k-1} \le |f'(z)| \le \tag{11}$$

$$\le pr^{p-1} + \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda}} r^{p+k-1}$$

All the inequalities are sharp with the extremal function

$$f(z) = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda+1}} z^{p+k}. \tag{12}$$

Proof. Using the arguments as in the Theorem 3.1, the required results for $C_p(\alpha, A, B, k, \beta, \lambda)$ is established.

Letting $r \to 1^-$in the left hand side of (10), we have:

Corollary 3.2. *Let $f \in C_p(\alpha, A, B, k, \beta, \lambda)$. Then the unit disk U is mapped by $f$onto a domain that contains the disk*

$$|w| < \frac{(p+k)^{\lambda+1}(1-B\beta)k + (A-B)\beta(p-\alpha)[(p+k)^{\lambda+1} - p^{\lambda+1}]}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda+1}}.$$

The result is sharp with the extremal function $f$being given by (12).

4 Integral Operators

Theorem 4.1. *Let c be a real number such that $c > -p$.*

If, $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$, *then the function* F *defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \tag{13}$$

also belongs to $T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Proof. Let $f(z) = z^p - \sum\limits_{n=k}^{\infty} a_{n+p} z^{p+n}$; $a_{p+n} \geq 0$. Then from representation of F, it follows that

$$F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n} \quad ; b_{p+n} \geq 0$$

where

$$b_{p+n} = \left(\frac{c+p}{c+p+n} \right) a_{p+n}.$$

Therefore using Theorem 2.1 for the coefficients of F, we have

$$\sum_{n=k}^{\infty} [(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda} b_{p+n} =$$

$$= \sum_{n=k}^{\infty} [(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda} \left(\frac{c+p}{c+p+n} \right) a_{p+n} \leq$$

$$\leq \sum_{n=k}^{\infty} [(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda} a_{p+n} \leq (A-B)p^{\lambda}\beta(p-\alpha)$$

since $\frac{c+p}{c+p+n} < 1$ and $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Hence $F \in T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Theorem 4.2. *Let* c *be a real number such that* $c > -p$.

If $F \in T_p^*(\alpha, A, B, k, \beta, \lambda)$, *then the function* $f(z) = z^p - \sum\limits_{n=k}^{\infty} a_{p+n} z^{p+n}$, $a_{p+n} \geq 0$ *is p-valent in* $|z| < R^*$, *where*

$$R^* = \inf_{n \geq k \geq 2} \left\{ \left[\left(\frac{c+p}{c+p+n} \right) a_{p+n} \leq \right. \right. \tag{14}$$

$$\leq \frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda-1}}{(A-B)p^{\lambda-1}\beta(p-\alpha)}\Bigg]^{\frac{1}{n}}\Bigg\}.$$

The result is sharp.

Proof. Let $F(z) = z^p - \sum_{n=k}^{\infty} a_{p+n}z^{p+n}; a_{p+n} \geq 0$. It follows then from (13) that

$$f(z) = \frac{z^{1-c}}{c+p}\frac{d}{dz}\left[z^c F(z)\right] = z^p - \sum_{n=k}^{\infty}\left(\frac{c+p+n}{c+p}\right)a_{p+n}z^{p+n}.$$

In order to obtain the required result it sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p$$

for $\mid z \mid < R^*$ where R^* is defined by (14).

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=k}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right)a_{p+n}z^n\right| \leq$$

$$\leq \sum_{n=k}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right)\left|a_{p+n}\right|\left|z\right|^n.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p \ \text{ if } \ \sum_{n=k}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right)a_{p+n}\left|z\right|^n < p. \tag{15}$$

But Theorem 2.1 confirms that

$$\sum_{n=k}^{\infty}\frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda-1}\beta(p-\alpha)}a_{p+n} \leq p.$$

Hence (15) will be satisfied if

$$(p+n)\left(\frac{c+p+n}{c+p}\right)a_{p+n}\left|z\right|^n \leq$$

$$\leq \frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda-1}\beta(p-\alpha)}a_{p+n};$$

$n \geq k \geq 2$ or if

$$\mid z \mid \leq \left\{ \left(\frac{c+p}{c+p+n} \right) \frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda-1}}{(A-B)p^{\lambda-1}\beta(p-\alpha)} \right\}^{\frac{1}{n}};$$

$n \geq k \geq 2$.

Therefore f is p-valent in $\mid z \mid < R^*$.

Sharpness follows if we take

$$F(z) = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}z^{p+n};$$

$n \geq k \geq 2$.

5 Closure Properties

In this section we show that the classes $T_p^*(\alpha, A, B, k, \beta, \lambda)$ and $C_p(\alpha, A, B, k, \beta, \lambda)$ are closed under "arithmetic mean" and "convex linear combinations".

Theorem 5.1. *The class $T_p^*(\alpha, A, B, k, \beta, \lambda)$ is closed under convex linear combinations.*

Proof. Suppose that

$$f^{(i)}(z) = z^p - \sum_{n=k}^{\infty} a_{p+n}^{(i)} z^{p+n} \ ; \ i = 1, 2 \ ; \ a_{p+n}^{(i)} \geq 0$$

are in the class $T_p^*(\alpha, A, B, k, \beta, \lambda)$. Let $f(z) = (1-\varsigma)f^{(1)}(z) + \varsigma f^{(2)}(z)$ with $0 \leq \varsigma \leq 1$. It is easy to satisfy, by Theorem 2.1, that $f(z)$ is in $T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Theorem 5.2. *The class $C_p(\alpha, A, B, k, \beta, \lambda)$ is closed under convex linear combinations.*

6 Radius of Convexity

Theorem 6.1. *If $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$, then f is p-valent convex function $\mid z \mid < R_1$, where*

$$(16)\quad R_1 = \inf_{n\geq k\geq 2}\left\{\left[\frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda-2}}{(A-B)p^{\lambda-2}\beta(p-\alpha)}\right]^{\frac{1}{n}}\right\}.$$

The result is sharp with the extremal function f given by (3).

Proof. It is sufficient to show that

$$\left|[1+\frac{zf''(z)}{f'(z)}]-p\right| \leq p \text{ for } \mid z \mid < R_1.$$

We have

$$\left|[1+\frac{zf''(z)}{f'(z)}]-p\right| = \left|\frac{f'(z)+zf''(z)-pf'(z)}{f'(z)}\right| =$$

$$= \left|\frac{-\sum\limits_{n=k}^{\infty} n(p+n)a_{p+n}z^n}{p-\sum\limits_{n=k}^{\infty}(p+n)a_{p+n}z^n}\right| \leq \frac{\sum\limits_{n=k}^{\infty} n(p+n)a_{p+n}\mid z\mid^n}{p-\sum\limits_{n=k}^{\infty}(p+n)a_{p+n}\mid z\mid^n}.$$

Therefore

$$\left|[1+\frac{zf''(z)}{f'(z)}]-p\right| \leq p$$

if

$$\sum_{n=k}^{\infty}(p+n)^2 a_{p+n}\left|z\right|^n \leq p^2$$

or

$$\sum_{n=k}^{\infty}(\frac{p+n}{p})^2 a_{p+n}\left|z\right|^n \leq 1.$$

By Theorem 2.1, we have

$$\sum_{n=k}^{\infty}\frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\alpha)}a_{p+n} \leq 1.$$

Hence f is p-valently convex if

$$\left(\frac{p+n}{p}\right)^2 |z|^n \leq \frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^\lambda}{(A-B)p^\lambda\beta(p-\alpha)}$$

or if

$$|z| \leq \left[\frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda-2}}{(A-B)p^{\lambda-2}\beta(p-\alpha)}\right]^{\frac{1}{n}} \;;\; n \geq k \geq 2.$$

7 The extreme points of $T_p^*(\alpha, A, B, k, \beta, \lambda)$ and $C_p(\alpha, A, B, k, \beta, \lambda)$

Theorem 7.1. *Let* $f_p(z) = z^p$ *and*

$$f_{p+n} = z^p - \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^\lambda} z^{p+n} \,;\, n \geq k \geq 2.$$

Then $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$ *if and only if it can be expressed in the form* $f(z) = \xi_p f_p(z) + \sum\limits_{n=k}^{\infty} \xi_n f_{p+n}(z)$, $z \in U$, *where* $\xi_n \geq 0$ *and* $\xi_p = 1 - \sum\limits_{n=k}^{\infty} \xi_n$.

Proof. Let us assume that

$$f(z) = \xi_p f_p(z) + \sum_{n=k}^{\infty} \xi_n f_{p+n}(z) =$$

$$= [1 - \sum_{n=k}^{\infty} \xi_n] z^p + \sum_{n=k}^{\infty} \xi_n \left\{ z^p - \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^\lambda} z^{p+n} \right\} =$$

$$= z^p - \sum_{n=k}^{\infty} \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^\lambda} \xi_n z^{p+n}.$$

Then from Theorem 2.1, we have

$$\sum_{n=k}^{\infty} [(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^\lambda \cdot$$

$$\cdot\frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}\xi_n \leq (A-B)p^{\lambda}\beta(p-\alpha)$$

Hence $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Conversely, let $f \in T_p^*(\alpha, A, B, k, \beta, \lambda)$. Using Corollary 2.1 , setting

$$\xi_n = \frac{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\alpha)}a_{p+n} \ ; \ n=k,k+1,\cdots;k\geq 2,$$

and letting $\xi_p = 1-\sum\limits_{n=k}^{\infty}\xi_n$, we have

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n}z^{p+n} = z^p - \sum_{n=k}^{\infty}\xi_n z^p + \sum_{n=k}^{\infty}\xi_n z^p -$$

$$-\sum_{n=k}^{\infty}\xi_n\frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}z^{p+n} =$$

$$= [1-\sum_{n=k}^{\infty}\xi_n]z^p + \sum_{n=k}^{\infty}\xi_n[z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}z^{p+n}] =$$

$$= \xi_p f_p(z) + \sum_{n=k}^{\infty}\xi_n f_{p+n}(z).$$

This completes the proof of Theorem 7.1.

Corollary 7.1. *The extreme points of* $T_p^*(\alpha, A, B, k, \beta, \lambda)$ *are the functions* $f_p(z) = z^p$ *and*

$$f_{p+n} = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}}z^{p+n}; n\geq k\geq 2.$$

Theorem 7.2. *Let* $f_p(z) = z^p$ *and*

$$f_{p+n} = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda+1}}z^{p+n} \ ; \ n\geq k\geq 2.$$

Then $f \in C_p(\alpha, A, B, k, \beta, \lambda)$ *if and only if it can be expressed in the form* $f(z) = \xi_p f_p(z) + \sum\limits_{n=k}^{\infty}\xi_n f_{p+n}(z)$, $z \in U$, *where* $\xi_n \geq 0$ *and* $\xi_p = 1-\sum\limits_{n=k}^{\infty}\xi_n$.

Corollary 7.2. *The extreme points of* $C_p(\alpha, A, B, k, \beta, \lambda)$ *are the functions* $f_p(z) = z^p$ *and*

$$f_{p+n} = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}}{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda+1}} z^{p+n}\,;\; n \geq k \geq 2.$$

8 Modified Hadamard Products

Let the functions $f_j(z)(j = 1, 2)$ be defined by

(17) $$f_j(z) = z^p - \sum_{n=k}^{\infty} a_{p+n,j} z^{p+n} \qquad (a_{p+n,j} \geq 0).$$

The Modified Hadamard product $f_1 * f_2$ of f_1 and f_2 is denoted by

(18) $$(f_1 * f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n}$$

Theorem 8.1. *Let the functions* $f_j(z)(j = 1, 2)$ *defined by be in the class* $T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Then $(f_1 * f_2)(z)$ *belongs to the class* $T_p^*(\gamma(p, \alpha, A, B, k, \beta, \lambda), A, B, k, \beta, \lambda)$ *where*

(19) $$\gamma = \gamma(p, \alpha, A, B, k, \beta, \lambda) =$$

$$= p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}(p-\alpha)^2(1-B\beta)k}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda} - (A-B)^2\beta^2 p^{\lambda}(p-\alpha)^2}.$$

The result is sharp for the functions $f_j(z)$*given by*

$$f_j(z) = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}(p-\alpha)}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda}} z^{p+k}; \qquad k \geq 2.$$

Proof. Employing the tecnique used earlier by Schild and Silverman [14], we need to find the largest $\gamma = \gamma(p, \alpha, A, B, k, \beta, \lambda)$ such that

$$(20) \qquad \sum_{n=k}^{\infty} \frac{[(1-B\beta)n + (A-B)\beta(p-\gamma)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\gamma)} a_{p+n,1}a_{p+n,2} \leq 1.$$

Since

$$(21) \qquad \sum_{n=k}^{\infty} \frac{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\alpha)} a_{p+n,1} \leq 1.$$

and

$$(22) \qquad \sum_{n=k}^{\infty} \frac{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\alpha)} a_{p+n,2} \leq 1.$$

by means of Cauchy-Schwarz inequality, we have

$$(23) \qquad \sum_{n=k}^{\infty} \frac{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\alpha)} \sqrt{a_{p+n,1}a_{p+n,2}} \leq 1.$$

Therefore, it is sufficient to show that

$$\frac{[(1-B\beta)n + (A-B)\beta(p-\gamma)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\gamma)} a_{p+n,1}a_{p+n,2} \leq$$

$$\leq \frac{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}}{(A-B)p^{\lambda}\beta(p-\alpha)} \sqrt{a_{p+n,1}a_{p+n,2}}$$

that is, that

$$(24) \quad \sqrt{a_{p+n,1}a_{p+n,2}} \leq \frac{(p-\gamma)[(1-B\beta)n + (A-B)\beta(p-\alpha)]}{(p-\alpha)[(1-B\beta)n + (A-B)\beta(p-\gamma)]} \ ; \ n \geq k \geq 2.$$

Note that

$$(25) \qquad \sqrt{a_{p+n,1}a_{p+n,2}} \leq \frac{(A-B)p^{\lambda}\beta(p-\alpha)}{[(1-B\beta)n + (A-B)\beta(p-\alpha)](p+n)^{\lambda}} \quad ;$$

$n \geq k \geq 2$.

Consequently, we need only to prove that

$$\frac{(A-B)p^{\lambda}\beta(p-\alpha)}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}} \leq$$

$$\leq \frac{(p-\gamma)[(1-B\beta)n+(A-B)\beta(p-\alpha)]}{(p-\alpha)[(1-B\beta)n+(A-B)\beta(p-\gamma)]} \quad ;\ n \geq k \geq 2.$$

or , equivalently, if

(26)
$$\gamma = \gamma(p,\alpha,A,B,k,\beta,\lambda) =$$
$$= p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}(p-\alpha)^2(1-B\beta)n}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}-(A-B)^2\beta^2p^{\lambda}(p-\alpha)^2}.$$

Since $\Phi(n)$by

(27)
$$\Phi(n) =$$
$$= p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}(p-\alpha)^2(1-B\beta)n}{[(1-B\beta)n+(A-B)\beta(p-\alpha)](p+n)^{\lambda}-(A-B)^2\beta^2p^{\lambda}(p-\alpha)^2},$$

is an increasing function of n, $n \geq k \geq 2$, letting $n = k$ in (8.11) we obtain, therefore,

(28)
$$\gamma \leq \Phi(k) =$$
$$= p - \frac{(A-B)\beta(p-\alpha)p^{\lambda}(p-\alpha)^2(1-B\beta)k}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda}-(A-B)^2\beta^2p^{\lambda}(p-\alpha)^2}.$$

which completes the assertion of theorem.

Corollary 8.1. For $f_1(z)$and $f_2(z)$ as in Theorem 8.1, the function

$$h(z) = z^p - \sum_{n=k}^{\infty} \sqrt{a_{p+n,1}a_{p+n,2}}\, z^{p+n}$$

belongs to the class $T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Proof. It follows from the Cauchy-Shwarz inequality (8.7). It is sharp for the same functions as in Theorem 8.1.

Theorem 8.2. *Let the functions $f_j(z)(j = 1, 2)$ defined by be in the class $C_p(\alpha, A, B, k, \beta, \lambda)$.*

Then $(f_1 * f_2)(z)$ belongs to the class $C_p(\psi(p, \alpha, A, B, k, \beta, \lambda), A, B, k, \beta, \lambda)$

$$\psi = \psi(p, \alpha, A, B, k, \beta, \lambda) = \tag{29}$$

$$= p - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}(p-\alpha)^2(1-B\beta)k}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda+1} - (A-B)^2\beta^2 p^{\lambda+1}(p-\alpha)^2}.$$

The result is sharp for the functions $f_j(z)$given by

$$f_j(z) = z^p - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}(p-\alpha)}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda+1}} z^{p+k}; \qquad k \geq 2.$$

9 Definitions and Applications of The Fractional Calculus

In this section, we shall prove several distortion theorems for functions to general classes $T_p^*(\alpha, A, B, k, \beta, \lambda)$ and $C_p(\alpha, A, B, k, \beta, \lambda)$.

Each of these theorems would involve certain operators of fractional calculus we find it to be convenient to recall here the following definition which were used recently by Owa [10] (and more recently, by Owa and Srivastava [11], and Srivastava and Owa [12], ; see also Srivastava et all. [13])

Definition 9.1. *The fractional integral of order μ is defined, for a function f, by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta, (\mu > 0) \tag{30}$$

where f *is an analytic function in a simply – connected region of the* z*-plane containing the origin, and the multiplicity of* $(z-\zeta)^{\mu-1}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $z-\zeta>0$.

Definition 9.2. *The fractional derivative of order* μ *is defined, for a function* f, *by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta, \ (0 \leq \mu < 1) \tag{31}$$

where f *is constrained, and the multiplicity of* $(z-\zeta)^{-\mu}$ *is removed, as in Definition 9.1.*

We remark that in Definition 9.1 and 9.2, Γ *denotes the Gamma function.*

Definition 9.3. *Under the hypotheses of Definition 9.2, the fractional derivative of order* $(n+\mu)$ *is defined by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \tag{32}$$

where $0 \leq \mu < 1$ *and* $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

From Definition 9.2, we have

$$D_z^0 f(z) = f(z) \tag{33}$$

which, in view of Definition 9.3 yields,

$$D_z^{n+0} f(z) = \frac{d^n}{dz^n} D_z^0 f(z) = f^n(z). \tag{34}$$

Thus, it follows from (33) and (34) that $\lim\limits_{\mu\to 0} D_z^{-\mu} f(z) = f(z)$ *and* $\lim\limits_{\mu\to 0} D_z^{1-\mu} f(z) = f'(z)$.

Theorem 9.1. *Let the function f defined by*

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n} \qquad ; a_{p+n} \geq 0$$

be in the class $T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Then

$$\left|D_z^{-\mu}(D^i f(z))\right| \geq |z|^{p+\mu} \left\{ \frac{\Gamma(p+1)}{\Gamma(\mu+p+1)} - \right. \tag{35}$$

$$\left. - \frac{(A-B)\beta(p-\alpha)p^\lambda \Gamma(p+k+1)}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda-i}\Gamma(\mu+p+k+1)} |z|^k \right\}$$

and

$$\left|D_z^{-\mu}(D^i f(z))\right| \leq |z|^{p+\mu} \left\{ \frac{\Gamma(p+1)}{\Gamma(\mu+p+1)} + \right. \tag{36}$$

$$\left. + \frac{(A-B)\beta(p-\alpha)p^\lambda \Gamma(p+k+1)}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda-i}\Gamma(\mu+p+k+1)} |z|^k \right\}$$

for $\mu > 0$, $0 \leq i \leq \lambda$ and $z \in U$. The equalities in (35) and (36) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)k + (A-B)\beta(p-\alpha)]}(p+k)^{-\lambda} z^{p+k}. \tag{37}$$

Proof. We note that

$$\frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu}(D^i f(z)) =$$

$$= z^p - \sum_{n=k}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+n+\mu+1)} (p+n)^i a_{p+n} z^{p+n}.$$

Defining the function $\varphi(n)$

$$\varphi(n) = \frac{\Gamma(p+n+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+n+\mu+1)},$$

$\mu > 0$; $n \geq k \geq 2$,

We can see that $\varphi(n)$ is decreasing in n , that is , that

$$0 < \varphi(n) \leq \varphi(k) = \frac{\Gamma(p+k+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+k+\mu+1)}.$$

On the other hand, from [8],

$$\sum_{n=k}^{\infty}(p+n)^i a_{p+n} \leq \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)k+(A-B)\beta(p-\alpha)]}(p+k)^{-(\lambda-i)} \;\; ; \;\; 0 \leq i \leq \lambda.$$

Therefore,

$$\left|\frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu}(D^i f(z))\right| \geq |z|^p - \varphi(k)\,|z|^{p+k} \sum_{n=k}^{\infty}(p+n)^i a_{p+n} \geq$$

$$\geq |z|^p - \frac{\Gamma(p+k+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+k+\mu+1)} \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)k+(A-B)\beta(p-\alpha)]} \cdot$$

$$\cdot (p+k)^{-(\lambda-i)}\,|z|^{p+k}$$

and

$$\left|\frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu}(D^i f(z))\right| \leq |z|^p + \varphi(k)\,|z|^{p+k} \sum_{n=k}^{\infty}(p+n)^i a_{p+n} \leq$$

$$\leq |z|^p + \frac{\Gamma(p+k+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+k+\mu+1)} \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)k+(A-B)\beta(p-\alpha)]} \cdot$$

$$\cdot (p+k)^{-(\lambda-i)}\,|z|^{p+k}$$

which completes the proof of theorem.

Remark. *By letting $\mu \to 0$, taking $i = 0$ and $\lambda = 0$ in Theorem 9.1 , we have the former results by Aouf and Darwish [4].*

Next, we prove,

Theorem 9.2. *Let the function f defined by*

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n} \qquad ; a_{p+n} \geq 0$$

be in the class $T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Then,

$$\left|D_z^{\mu}(D^i f(z))\right| \geq |z|^{p-\mu} \left\{ \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \right. \tag{38}$$

$$\left. - \frac{(A-B)\beta(p-\alpha)p^{\lambda}\Gamma(p+k+1)}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda-i-1}\Gamma(p+k-\mu+1)} |z|^k \right\}$$

$$\left|D_z^{\mu}(D^i f(z))\right| \leq |z|^{p-\mu} \left\{ \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \right. \tag{39}$$

$$\left. + \frac{(A-B)\beta(p-\alpha)p^{\lambda}\Gamma(p+k+1)}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda-i-1}\Gamma(p+k-\mu+1)} |z|^k \right\}$$

for $0 \leq \mu < 1$, $0 \leq i \leq \lambda - 1$ and $z \in U$. The equalities in (38) and (39) are attained for the function $f(z)$ given by (37).

Proof. We can easily take

$$\frac{\Gamma(p+k-\mu)}{\Gamma(p+1)} z^{\mu} D_z^{\mu} f(z) = z^p - \sum_{n=k}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p-\mu+1)}{\Gamma(p+1)\Gamma(p+n-\mu+1)} (p+n)^i a_{p+n} z^{p+n}.$$

Since the function

$$\phi(n) = \frac{\Gamma(p-\mu+1)\Gamma(p+n)}{\Gamma(p+1)\Gamma(p+n-\mu+1)}; \; n \geq k \geq 2.$$

In decreasing in n, we have

$$0 < \phi(n) \leq \phi(k) = \frac{\Gamma(p-\mu+1)\Gamma(p+k)}{\Gamma(p+1)\Gamma(p+k-\mu+1)}.$$

Further, we note that from [8],

$$\sum_{n=k}^{\infty} (p+n)^{i+1} a_{p+n} \leq \frac{(A-B)\beta(p-\alpha)p^{\lambda}}{[(1-B\beta)k+(A-B)\beta(p-\alpha)]} (p+k)^{-(\lambda-i-1)},$$

$$0 \leq i \leq \lambda - 1,$$

for $f(z) \in T_p^*(\alpha, A, B, k, \beta, \lambda)$.

Then it follows that

$$\left|\frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} z^\mu D_z^\mu (D^i f(z))\right| \geq |z|^p - \phi(k)\,|z|^{p+k} \sum_{n=k}^{\infty} (p+n)^{i+1} a_{p+n} \geq$$

$$\geq |z|^p - \frac{\Gamma(p+k)\Gamma(p-\mu+1)}{\Gamma(p+1)\Gamma(p+k-\mu+1)} \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)k + (A-B)\beta(p-\alpha)]} \cdot$$
$$\cdot (p+k)^{-(\lambda-i-1)}\,|z|^{p+k}$$

and

$$\left|\frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} z^\mu D_z^\mu (D^i f(z))\right| \leq |z|^p + \phi(k)\,|z|^{p+k} \sum_{n=k}^{\infty} (p+n)^{i+1} a_{p+n} \leq$$

$$\leq |z|^p + \frac{\Gamma(p+k)\Gamma(p-\mu+1)}{\Gamma(p+1)\Gamma(p+k-\mu+1)} \frac{(A-B)\beta(p-\alpha)p^\lambda}{[(1-B\beta)k + (A-B)\beta(p-\alpha)]} \cdot$$
$$\cdot (p+k)^{-(\lambda-i-1)}\,|z|^{p+k}$$

which completes the proof of theorem.

Remark. *By taking $i = 0$ and $\lambda = 0$ and letting $\mu \to 1$ in Theorem 9.2 , we have the former results by Aouf and Darwish [4].*

Theorem 9.3. *Let the function f defined by*

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{+np} \qquad ; a_{p+n} \geq 0$$

be in the class $C_p(\alpha, A, B, k, \beta, \lambda)$.

Then

(40) $$\left|D_z^{-\mu}(D^i f(z))\right| \geq |z|^{p+\mu} \left\{ \frac{\Gamma(p+1)}{\Gamma(\mu+p+1)} - \right.$$
$$\left. - \frac{(A-B)\beta(p-\alpha)p^{\lambda+1}\Gamma(p+k+1)}{[(1-B\beta)k + (A-B)\beta(p-\alpha)](p+k)^{\lambda+1-i}\Gamma(\mu+p+k+1)}\,|z|^k \right\}$$

$$\left|D_z^{-\mu}(D^i f(z))\right| \leq |z|^{p+\mu}\left\{\frac{\Gamma(p+1)}{\Gamma(\mu+p+1)}+\right. \tag{41}$$

$$\left.+\frac{(A-B)\beta(p-\alpha)p^{\lambda+1}\Gamma(p+k+1)}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda+1-i}\Gamma(\mu+p+k+1)}|z|^k\right\}$$

for $\mu > 0$, $0 \leq i \leq \lambda$ *and* $z \in U$, *and*

$$\left|D_z^{\mu}(D^i f(z))\right| \geq |z|^{p-\mu}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}-\right. \tag{42}$$

$$\left.-\frac{(A-B)\beta(p-\alpha)p^{\lambda+1}\Gamma(p+k)}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda-i}\Gamma(p+k-\mu+1)}|z|^k\right\}$$

and

$$\left|D_z^{\mu}(D^i f(z))\right| \leq |z|^{p-\mu}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}+\right. \tag{43}$$

$$\left.+\frac{(A-B)\beta(p-\alpha)p^{\lambda+1}\Gamma(p+k)}{[(1-B\beta)k+(A-B)\beta(p-\alpha)](p+k)^{\lambda-i}\Gamma(p+k-\mu+1)}|z|^k\right\}$$

for

$$0 \leq \mu < 1\ ,\ 0 \leq i \leq \lambda - 1 and z \in U.$$

All inequalities in above are attained for the function $f(z)$ *given by (37).*

Remark. *By letting* $\mu \to 0$, *taking* $i = 0$ *and* $\lambda = 0$ *in (40) - (41) and by taking* $i = 0$ *and* $\lambda = 0$ *and letting* $\mu \to 1$*in (42) - (43) of Theorem 9.3, we have the former results by Aouf [6].*

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