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Approximation of Entire Functions of Slow Growth [1]

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Abstract

In the present paper, we study the polynomial approximation of entire functions in Banach spaces ($\mathbf{B}(p,q,\kappa)$ space, Hardy space and Bergman space). The coefficient characterizations of generalized type of entire functions of slow growth have been obtained in terms of the approximation errors.

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1 Introduction

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function and $M(r,f)=\max_{|z|=r}|f(z)|$ be its maximum modulus. The growth of $f(z)$ is measured in terms of its

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order ρ and type τ defined as under

$$\limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \rho, \tag{1}$$

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} = \tau, \tag{2}$$

for $0 < \rho < \infty$. Various workers have given different characterizations for entire functions of fast growth ($\rho = \infty$). M. N. Seremeta [3] defined the generalized order and generalized type with the help of general functions as follows.

Let L^o denote the class of functions h satisfying the following conditions (i) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii)

$$\lim_{x\to\infty} \frac{h\{(1+1/\psi(x))x\}}{h(x)} = 1,$$

for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions h satisfying condition (i) and

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$$

for every $c > 0$, that is, $h(x)$ is slowly increasing.

For the entire function $f(z)$ and functions $\alpha(x) \in \Lambda, \beta(x) \in L^o$, Seremeta [3, Th .1] proved that

$$\rho(\alpha,\ \beta,\ f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r,f)]}{\beta(\log r)} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)}. \tag{3}$$

Further, for $\alpha(x) \in L^o, \beta^{-1}(x) \in L^o, \gamma(x) \in L^o$,

(4)

$$\tau(\alpha,\ \beta,\ f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r,f)]}{\beta[(\gamma(r))^{\rho}]} = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{1/\rho}|a_n|^{-1/n})]^{\rho}\}}.$$

where $0 < \rho < \infty$ is a fixed number.

Above relations were obtained under certain conditions which do not hold if $\alpha = \beta$. To overcome this difficulty, G.P.Kapoor and Nautiyal [2] defined generalized order $\rho(\alpha; f)$ of slow growth with the help of general functions as follows

Let Ω be the class of functions $h(x)$ satisfying (i) and
(iv) there exists a $\delta(x) \in \Omega$ and x_0, K_1 and K_2 such that

$$0 < K_1 \leq \frac{d(h(x))}{d(\delta(\log x))} \leq K_2 < \infty \quad \text{for all } x > x_0.$$

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(v) $$\lim_{x \to \infty} \frac{d(h(x))}{d(\log x)} = K, \quad 0 < K < \infty.$$

Kapoor and Nautiyal [2] showed that classes Ω and $\overline{\Omega}$ are contained in Λ. Further, $\Omega \cap \overline{\Omega} = \phi$ and they defined the generalized order $\rho(\alpha; f)$ for entire functions $f(z)$ of slow growth as

$$\rho(\alpha; f) = \limsup_{r \to \infty} \frac{\alpha(\log M(r,f))}{\alpha(\log r)},$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

Recently Vakarchuk and Zhir [5] considered the approximation of entire functions in Banach spaces. Thus, let $f(z)$ be analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$ and we set

$$M_q(r,f) = \{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^q d\theta\}^{1/q}, \ q > 0.$$

Let H_q denote the Hardy space of functions $f(z)$ satisfying the condition

$$\|f\|_{H_q} = \lim_{r \to 1-0} M_q(r,f) < \infty$$

and let H'_q denote the Bergman space of functions $f(z)$ satisfying the condition

$$\|f\|_{H'_q} = \{\frac{1}{\pi}\int\int_U |f(z)|^q dxdy\}^{1/q} < \infty.$$

For $q = \infty$, let $\|f\|_{H'_\infty} = \|f\|_{H_\infty} = \sup\{|f(z)|, z \in U\}$. Then H_q and H'_q are Banach spaces for $q \geq 1$. Following [5, p.1394], we say that a function $f(z)$ which is analytic in U belongs to the space $\mathbf{B}(p, q, \kappa)$ if

$$\|f\|_{p,q,\kappa} = \{\int_0^1 (1-r)^{\kappa(1/p-1/q)-1} M_q^\kappa(r, f)dr\}^{1/\kappa} < \infty,$$

$0 < p < q \leq \infty,\ 0 < \kappa < \infty$ and

$$\|f\|_{p,q,\infty} = \sup\{(1-r)^{1/p-1/q} M_q(r, f); 0 < r < 1\} < \infty.$$

It is known [1] that $\mathbf{B}(p, q, \kappa)$ is a Banach space for $p > 0$ and $q, \kappa \geq 1$, otherwise it is a Frechet space. Further [4],

$$H_q \subseteq H'_q = \mathbf{B}(q/2, q, q),\ 1 \leq q < \infty. \tag{5}$$

Let X denote one of the Banach spaces defined above and let

$$E_n(f, X) = \inf\{\|f - p\|_X : p \in P_n\}$$

where P_n consists of algebraic polynomials of degree at most n in complex variable z.

Vakarchuk and Zhir [5] obtained the characterizations of generalized order and generalized type of $f(z)$ in terms of the errors $E_n(f, X)$ defined above. These characterizations do not hold good when $\alpha = \beta = \gamma$. i.e. for entire functions of slow growth.In this paper we have tried to fill this gap. We define the generalized type $\tau(\alpha; f)$ of an entire function $f(z)$ having finite generalized order $\rho(\alpha; f)$ as

$$\tau(\alpha; f) \;=\; \limsup_{r \to \infty} \frac{\alpha(\log M(r,f))}{[\alpha(\log r)]^{\rho}}$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

To the best of our knowledge, coefficient characterization for generalized type of slow growth have not been obtained so far. Similarly, the characterization for approximating entire functions in certain Banach spaces by generalized type of slow growth have not been studied so far.

In this paper, we have made an attempt to bridge this gap. First we obtain coefficient characterization for generalized type of slow growth. Next we obtain necessary and sufficient conditions of generalized type of slow growth in certain Banach spaces($\mathbf{B}(p, q, \kappa)$ space, Hardy space and Bergman spaces). We shall assume throughout that function $\alpha \;\in\; \overline{\Omega}$.

Notation:

1. $F[x\;;\;\tau\;,\;\rho] \;=\; \alpha^{-1}\{\overline{\tau}\;[\alpha(x)]^{1/\rho}\}$

   where ρ is fixed number, $0 < \rho < \infty$ and $\overline{\tau} \;=\; \tau + \epsilon$.

2. $E[F[x\;;\;\tau\;,\;\rho]]$ is an integer part of the function F.

2 Main Results

First we prove the following

Theorem 2.1: *Let $\alpha(x) \in \overline{\Omega}$, then the entire function f(z) of generalized order ρ, $1 < \rho < \infty$, is of generalized type τ if and only if*

$$(1) \qquad \tau = \limsup_{R\to\infty} \frac{\alpha(\ln M(R,f))}{[\alpha(\ln R)]^\rho} = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln|a_n|^{-1/n}]\}^{\rho-1}},$$

provided $dF(x\,;\,\tau\,,\,\rho)/d\ln x = O(1)$ as $x \to \infty$ for all τ, $0 < \tau < \infty$.

Proof. Let

$$\limsup_{R\to\infty} \frac{\alpha(\ln M(R,f))}{[\alpha(\ln R)]^\rho} = \tau.$$

We suppose $\tau < \infty$. Then for every $\epsilon > 0$, $\exists\, R(\epsilon) \ni$

$$\frac{\alpha(\ln M(R,f))}{[\alpha(\ln R)]^\rho} \le \tau + \epsilon = \overline{\tau} \quad \forall\, R \ge R(\epsilon).$$

$$\text{(or)} \qquad \ln M(R,f) \le (\alpha^{-1}\{\overline{\tau}[\alpha(\ln R)]^\rho\}).$$

Choose $R = R(n)$ to be the unique root of the equation

$$(2) \qquad n = \frac{\rho}{\ln R} F[\ln R\,;\,\overline{\tau}\,,\,\frac{1}{\rho}].$$

Then

$$(3) \qquad \ln R = \alpha^{-1}[(\frac{1}{\overline{\tau}}\,\alpha(\frac{n}{\rho}))^{1/(\rho-1)}] = F[\frac{n}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\rho-1].$$

By Cauchy's inequality,

$$\begin{aligned} |a_n| &\le R^{-n}M(R;f) \\ &\le \exp\{-n\ln R + (\alpha^{-1}\{\overline{\tau}\,[\alpha(\ln R)]^\rho\})\} \end{aligned}$$

By using (5) and (6) , we get

$$|a_n| \;\le\; \exp\{-nF \;+\frac{n}{\rho}F\}$$

or

$$\frac{\rho}{\rho-1}\ln|a_n|^{-1/n} \;\ge\; \alpha^{-1}\{[(\frac{1}{\overline{\tau}}\;\alpha(\frac{n}{\rho}))^{1/(\rho-1)}]\}$$

or

$$\overline{\tau} \;=\; \tau+\epsilon \;\ge\; \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln|a_n|^{-1/n}]\}^{\rho-1}}.$$

Now proceeding to limits, we obtain

$$\tau \;\ge\; \limsup_{n\to\infty}\;\frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\;\ln|a_n|^{-1/n}]\}^{(\rho-1)}}. \tag{4}$$

Inequality (4) obviously holds when $\tau \;=\; \infty$.

Conversely, let

$$\limsup_{n\to\infty}\;\frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\;\ln|a_n|^{-1/n}]\}^{(\rho-1)}} \;=\; \sigma.$$

Suppose $\sigma \;<\; \infty$. Then for every $\epsilon \;>\; 0$ and for all $n \;\ge\; N(\epsilon)$, we have

$$\frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\;\ln|a_n|^{-1/n}]\}^{(\rho-1)}} \;\le\; \sigma+\epsilon \;=\; \overline{\sigma}$$

$$|a_n| \;\le\; \frac{1}{\exp\{(\rho-1)\;\frac{n}{\rho}\;F[\frac{n}{\rho}\;;\;\frac{1}{\overline{\sigma}}\;,\;\rho-1]\}}. \tag{5}$$

The inequality

$$\sqrt[n]{|a_n|R^n} \;\le\; Re^{-(\frac{\rho-1}{\rho})\;F[\frac{n}{\rho}\;;\;\frac{1}{\overline{\tau}}\;,\;\rho-1]} \;\le\; \frac{1}{2} \tag{6}$$

is fulfilled beginning with some $n = n(R)$. Then

$$(7) \qquad \sum_{n=n(R)+1}^{\infty} |a_n| R^n \leq \sum_{n=n(R)+1}^{\infty} \frac{1}{2^n} \leq 1.$$

We now express $n(R)$ in terms of R. From inequality (6),

$$2R \leq \exp\{(\frac{\rho-1}{\rho})\ F[\frac{n}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]\},$$

we can take $n(R) = E[\rho\ \alpha^{-1}\{\overline{\tau}\ (\alpha(\ln R+\ln 2))^{\rho-1}\}]$. We consider the function $\psi(x) = R^x \exp\{-(\frac{\rho-1}{\rho})\ x\ F[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]\}$. Let

$$(8) \quad \frac{\psi'(x)}{\psi(x)} = \ln R - (\frac{\rho-1}{\rho}) F[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1] - \frac{dF[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]}{d\ln x} = 0.$$

As $x \to \infty$, by the assumption of the theorem, for finite σ $(0 < \sigma < \infty)$, $dF[x\ ;\ \overline{\sigma}\ ,\ \rho-1]\ /\ d\ln x$ is bounded. So there is an $A > 0$ such that for $x \geq x_1$ we have

$$(9) \qquad |\frac{dF[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]}{d\ln x}| \leq A.$$

We can take $A > \ln 2$. It is then obvious that inequalities (6) and (7) hold for $n \geq n_1(R) = E[\rho\ \alpha^{-1}\{\overline{\sigma}\ (\alpha(\ln R+A))^{(\rho-1)}\}]+1$. We let n_0 designate the number $\max(N(\epsilon)\ ,\ E[x_1]+1)$. For $R > R_1(n_0)$ we have $\psi'(n_0)/\psi(n_0) > 0$. From (9)and (8) it follows that $\psi'(n_1(R))/\psi(n_1(R)) < 0$. We hence obtain that if for $R > R_1(n_0)$ we let $x^*(R)$ designate the point where $\psi(x^*(R)) = \max_{n_0 \leq x \leq n_1(R)} \psi(x)$, then

$$n_0 < x^*(R) < n_1(R) \quad \text{and } x^*(R) = \rho\alpha^{-1}\{\overline{\sigma}\ (\alpha(\ln R - a(R)))^{\rho-1}\}.$$

where

$$-A < a(R) = \frac{dF[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]}{d\ln x}|_{x=x^*(R)} < A.$$

Further

$$\max_{n_0<n<n_1(R)} (|a_n|R^n) \;\leq\; \max_{n_0<x<n_1(R)} \psi(x) \;=\; \frac{R^{\rho\alpha^{-1}\{\overline{\sigma}\;(\alpha(\ln R - a(R)))^{\rho-1}\}}}{e^{\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R - a(R)))^{\rho-1}\}\;(\ln R - a(R))}} =$$

$$= \;\exp\{a(R)\rho\alpha^{-1}\{\overline{\sigma}\;(\alpha(\ln R - a(R)))^{\rho-1}\}\}\;\leq$$

$$\leq\; \exp\{A\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R + A))^{\rho-1}\}\}.$$

It is obvious that (for $R > R_1(n_0)$)

$$M(R,f) \;\leq\; \Sigma_{n=0}^{\infty}|a_n|R^n \;=$$

$$= \Sigma_{n=0}^{n_0}|a_n|R^n \;+\; \Sigma_{n=n_0+1}^{n_1(R)}|a_n|R^n \;+\; \Sigma_{n=n_1(R)+1}^{\infty}|a_n|R^n$$

$$\leq\; O(R^{n_0}) \;+\; n_1(R) \max_{n_0<n<n_1(R)} (|a_n|R^n) \;+\; 1$$

$$M(R,f)(1+o(1)) \;\leq\; \exp\{(A\rho + o(1))\;\alpha^{-1}[\overline{\sigma}\;(\alpha(\ln R + A))^{\rho-1}]\}$$

$$\alpha(\ln M(R,f)) \;\leq\; \overline{\sigma}[\alpha(\ln R + A)]^{\rho-1} \;\leq\; \overline{\sigma}\;[\alpha(\ln R + A)]^{\rho}.$$

We then have

$$\frac{\alpha[(A\rho + o(1))^{-1}\ln M(R,f)]}{[\alpha(\ln R + A)]^{\rho}} \;\leq\; \overline{\sigma} \;=\; \sigma + \epsilon.$$

Since $\alpha(x) \;\in \overline{\Omega} \subseteq \Lambda$, now proceeding to limits we obtain

$$\limsup_{R\to\infty} \frac{\alpha(\ln M(R;f))}{[\alpha(\ln R)]^{\rho}} \;\leq\; \sigma. \tag{10}$$

From inequalities (4) and (10), we get the required the result.

We now prove

Theorem 2.2: *Let $\alpha(x) \in \overline{\Omega}$, then a necessary and sufficient condition for an entire function $f(z) \in \boldsymbol{B}(p,q,\kappa)$ to be of generalized type τ having finite generalized order ρ, $1 < \rho < \infty$ is*

$$\tau \ = \ \limsup_{n\to\infty} \ \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ \ln\left(|E_n(\boldsymbol{B}(p,q,\kappa);f)|^{-1/n}\right)\}]^{(\rho-1)}}. \tag{11}$$

Proof. We prove the above result in two steps. First we consider the space $\mathbf{B}(p,q,\kappa)$, $q=2$, $0<p<2$ and $\kappa \geq 1$. Let $f(z) \in \mathbf{B}(p,q,\kappa)$ be of generalized type τ with generalized order ρ. Then from the Theorem 2.1, we have

$$\limsup_{n\to\infty} \ \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln|a_n|^{-1/n}]\}^{\rho-1}} \ = \ \tau. \tag{12}$$

For a given $\epsilon > 0$, and all $n > m = m(\epsilon)$, we have

$$|a_n| \ \leq \ \frac{1}{\exp\{(\rho-1) \ \frac{n}{\rho} \ F[\frac{n}{\rho} \ ; \ \frac{1}{\tau} \ , \ \rho-1]\}}. \tag{13}$$

Let $g_n(f,z) \ = \ \sum_{j=0}^{n} a_j z^j$ be the n^{th} partial sum of the Taylor series of the function $f(z)$. Following [5, p.1396], we get

$$E_n(\mathbf{B}(p,2,\kappa);f) \ \leq \ B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]\{\sum_{j=n+1}^{\infty}|a_j|^2\}^{1/2} \tag{14}$$

where $B(a,b)$ $(a,b>0)$ denotes the beta function. By using (13), we have

$$E_n(\mathbf{B}(p,2,\kappa);f) \ \leq \ \frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{\exp\{(\rho-1) \ \frac{n+1}{\rho} \ F[\frac{n+1}{\rho} \ ; \ \frac{1}{\tau} \ , \ \rho-1]\}} \ \{\sum_{j=n+1}^{\infty}\psi_j^2(\alpha)\}^{1/2}, \tag{15}$$

where

$$\psi_j(\alpha) \ \cong \ \frac{\exp\{\frac{n+1}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}}{\exp\{\frac{j}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{j}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}}.$$

Set

$$\psi(\alpha) \;\cong\; \exp\{-\frac{1}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}.$$

Since $\alpha(x)$ is increasing and $j \geq n+1$, we get

(16)

$$\psi_j(\alpha) \;\leq\; \exp\{\frac{((n+1)-j)}{\rho}\;(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\} \;\leq\; \psi^{j-(n+1)}(\alpha).$$

Since $\psi(\alpha) \;<\; 1$, we get from (15) and (16),

(17)

$$E_n(\mathbf{B}(p,2,\kappa);f) \;\leq\; \frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}[\exp\{\frac{n+1}{\rho}\;(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]}.$$

For $n > m$, (17) yields

$$\tau+\epsilon \;\geq\; \frac{\alpha(\frac{n+1}{\rho})}{\{\alpha(\frac{\rho}{(1+\frac{1}{n})(\rho-1)}\{\ln(|E_n|^{-1/n}) \;+\; \ln(\frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}})^{1/n}\})\}^{(\rho-1)}}$$

Now

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] \;=\; \frac{\Gamma((n+1)\kappa+1)\Gamma(\kappa(1/p-1/2))}{\Gamma((n+1/2+1/p)\kappa+1)}.$$

Hence

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] \;\simeq$$

$$\simeq \frac{e^{-[(n+1)\kappa+1]}[(n+1)\kappa+1]^{(n+1)\kappa+3/2}\Gamma(1/p-1/2)}{e^{[(n+1/2+1/p)\kappa+1]}[(n+1/2+1/p)\kappa+1]^{(n+1/2+1/p)\kappa+3/2}}.$$

Thus

(18) $$\{B[(n+1)\kappa+1;\kappa(1/p-1/2)]\}^{1/(n+1)} \;\cong\; 1.$$

Now proceeding to limits, we obtain

(19) $$\tau \;\geq\; \limsup_{n\to\infty}\; \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}}$$

For reverse inequality, by [5, p.1398], we have

$$|a_{n+1}|B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)] \;\leq\; E_n(\mathbf{B}(p,2,\kappa);f). \tag{20}$$

Then for sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \;\geq$$

$$\frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_{n+1}|^{-1/n}) \;+\ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)])\}\}]^{(\rho-1)}}$$

$$\geq \quad \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_n|^{-1/n}) \;+\ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)])\}\}]^{(\rho-1)}}.$$

By applying limits and from (12), we obtain

$$\limsup_{n\to\infty}\;\frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \;\geq\; \tau. \tag{21}$$

From (19), and (21), we obtain the required relation

$$\limsup_{n\to\infty}\;\frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \;=\; \tau. \tag{22}$$

In the second step, we consider the spaces $\mathbf{B}(p,q,\kappa)$ for $0<p<q, q\neq 2$, and $q,\kappa\geq 1$. Gvaradze [1] showed that, for $p\geq p_1,\; q\leq q_1$, and $\kappa\leq\kappa_1$, if at least one of the inequalities is strict, then the strict inclusion $\mathbf{B}(p,q,\kappa)\subset \mathbf{B}(p_1,q_1,\kappa_1)$ holds and the following relation is true:

$$\|f\|_{p_1,q_1,\kappa_1} \;\leq\; 2^{1/q-1/q_1}[\kappa(1/p-1/q)]^{1/\kappa-1/\kappa_1}\|f\|_{p,q,\kappa}.$$

For any function $f(z)\;\in\;\mathbf{B}(p,q,\kappa)$, the last relation yields

(23)

$$E_n(\mathbf{B}(p_1,q_1,\kappa_1);f) \;\leq\; 2^{1/q-1/q_1}[\kappa(1/p-1/q)]^{1/\kappa-1/\kappa_1}E_n(\mathbf{B}(p,q,\kappa);f).$$

For the general case $\mathbf{B}(p,q,\kappa),\ q\ \neq 2$, we prove the necessity of condition (13).

Let $f(z)\ \in\ \mathbf{B}(p,q,\kappa)$ be an entire transcendental function having finite generalized order $\rho(\alpha;f)$ whose generalized type is defined by (12). Using the relation (13), for $n>m$ we estimate the value of the best polynomial approximation as follows

$$E_n(\mathbf{B}(p,q,\kappa);f)=\|f-g_n(f)\|_{p,q,\kappa}\ \leq\ (\int_0^1(1-r)^{(\kappa(1/p-1/q)-1)}M_q^\kappa dr)^{1/\kappa}.$$

Now

$$|f|^q\ =\ |\sum a_n z^n|^q\ \leq(\sum|a_n r^n|)^q\leq(r^{n+1}\sum_{k=n+1}^{\infty}|a_k|)^q.$$

Hence

$$\begin{aligned} E_n(\mathbf{B}(p,q,\kappa);f)\ &\leq\ B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]\sum_{k=n+1}^{\infty}|a_k| \\ &\leq\ \frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]}{(1-\psi(\alpha))[\exp\{\frac{n+1}{\rho}\ (\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]}. \qquad (24)\end{aligned}$$

For $n>m$, (24) yields

$$\tau+\epsilon\ \geq\ \frac{\alpha(\frac{n+1}{\rho})}{\{\alpha(\frac{\rho}{(1+\frac{1}{n})(\rho-1)}\{\ln(|E_n|^{-1/n})\ +\ \ln(\frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi(\alpha))})^{1/n}\})\}^{(\rho-1)}}$$

Since $\psi(\alpha)\ <\ 1$, and $\alpha\in\overline{\Omega}$, proceeding to limits and using (18), we obtain

$$\tau\ \geq\ \limsup_{n\to\infty}\ \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}}.$$

For the reverse inequality, let $0<p<q<2$ and $\kappa,q\geq 1$. By (23), where $p_1=p,q_1=2$, and $\kappa_1=\kappa$, and the condition (13) is already proved for the space $\mathbf{B}(p,2,\kappa)$, we get

$$\limsup_{n\to\infty}\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(\mathbf{B}(p,q,\kappa);f)|^{-1/n})\}]^{(\rho-1)}}$$

$$\geq \limsup_{n\to\infty}\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(\mathbf{B}(p,2,\kappa);f)|^{-1/n})\}]^{(\rho-1)}} = \tau.$$

Now let $0<p\leq 2<q$. Since we have

$$M_2(r,f) \leq M_q(r,f), \quad 0<r<1,$$

therefore

$$E_n(\mathbf{B}(p,q,\kappa);f) \geq |a_{n+1}|\, B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]. \tag{25}$$

Then for sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \geq$$

$$\frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_{n+1}|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)])\}\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_n|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)])\}\}]^{(\rho-1)}}.$$

By applying limits and from (12), we obtain

$$\limsup_{n\to\infty}\frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \geq \limsup_{n\to\infty}\frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|a_n|^{-1/n})\}]^{(\rho-1)}} = \tau.$$

Now we assume that $2\leq p<q$. Set $q_1=q, \kappa_1=\kappa$, and $0<p_1<2$ in the inequality (23), where p_1 is an arbitrary fixed number. Substituting p_1 for p in (25), we get

$$E_n(\mathbf{B}(p,q,\kappa);f) \geq |a_{n+1}|B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p_1-1/q)]. \tag{26}$$

Using (26) and applying the same analogy as in the previous case $0 < p \leq 2 < q$, for sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \geq$$

$$\frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_{n+1}|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p_1-1/q)])\}\}]^{(\rho-1)}}$$

$$\geq \quad \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_n|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p_1-1/q)])\}\}]^{(\rho-1)}}.$$

By applying limits and using (12), we obtain

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \geq \tau.$$

From relations (19) and (21), and the above inequality, we obtain the required relation (22).

Theorem 2.3: *Assuming that the conditions of Theorem 2.2 are satisfied and $\xi(\alpha)$ is a positive number, a necessary and sufficient condition for a function $f(z) \in H_q$ to be an entire function of generalized type $\xi(\alpha)$ having finite generalized order ρ is that*

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(H_q;f)|^{-1/n})\}]^{\rho-1}} = \xi(\alpha). \tag{27}$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire transcendental function having finite generalized order ρ and generalized type τ . Since

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 \tag{28}$$

$f(z) \in \mathbf{B}(p,q,\kappa)$, where $0 < p < q \leq \infty$ and $q, \kappa \geq 1$. From relation (4), we get

$$E_n(\mathbf{B}(q/2,q,q);f) \leq \varsigma_q E_n(H_q;f), \quad 1 \leq q < \infty. \tag{29}$$

where ς_q is a constant independent of n and f. In the case of Hardy space H_∞,

$$E_n(\mathbf{B}(p,\infty,\infty);f) \leq E_n(H_\infty;f), \quad 1 < p < \infty. \tag{30}$$

Since

$$\begin{aligned}
\xi(\alpha;f) &= \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(H_q;f)|^{-1/n})\}]^{(\rho-1)}} \\
&\geq \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n((\mathbf{B}(q/2,q,q);f)|^{-1/n})\}]^{(\rho-1)}} \\
&\geq \tau, \quad 1 \leq q < \infty. \qquad (31)
\end{aligned}$$

Using estimate (30) we prove inequality (31) in the case $q = \infty$.
For the reverse inequality

$$\xi(\alpha;f) \leq \tau, \tag{32}$$

we use the relation (13), which is valid for $n > m$, and estimate from above, the generalized type τ of an entire transcendental function $f(z)$ having finite generalized order ρ, as follows. We have

$$\begin{aligned}
E_n(H_q;f) &\leq \|f - g_n\|_{H_q} \\
&\leq \sum_{j=n+1}^{\infty} |a_j| \\
&\leq \frac{1}{[\exp\{(\rho-1)\ \frac{n+1}{\rho}\ [\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]} \sum_{j=n+1}^{\infty} \psi_j(\alpha).
\end{aligned}$$

Using (16),

$$\begin{aligned} E_n(H_q;f) &\leq \|f-g_n\|_{H_q} \\ &\leq \frac{1}{(1-\psi(\alpha))\,[\exp\{(\rho-1)\,\frac{n+1}{\rho}\,[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]}. \end{aligned}$$

$$\frac{1}{E_n(H_q;f)} \geq (1-\psi(\alpha))\ \exp\left\{(\rho-1)\frac{(n+1)}{\rho}\left[\alpha^{-1}\left\{\left(\frac{\alpha(\frac{(n+1)}{\rho})}{\overline{\tau}}\right)^{\frac{1}{(\rho-1)}}\right\}\right]\right\}$$

This yields

(33)

$$\tau+\epsilon \geq \frac{\alpha(\frac{n+1}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}[\ln(|E_n(H_q;f)|^{-1/n+1}) + \ln((1-\psi(\alpha))^{-1/n+1})]\}]^{(\rho-1)}}.$$

Since $\psi(\alpha) < 1$ and by applying the properties of the function α, passing to the limit as $n \to \infty$ in (33), we obtain inequality (32). Thus we have finally

(34) $$\xi(\alpha) = \tau.$$

This proves Theorem 2.2.

Remark : *An analog of Theorem 2.3 for the Bergman Spaces follows from* (4) *for* $1 \leq q < \infty$ *and from Theorem 2.2 for* $q = \infty$.

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