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On a class of convergent sequences defined by integrals [1]

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Abstract

The main result shows that if $g : [0,1] \to \mathbb{R}$ is a continuous function such that $\lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{g(x)}{x}$ exists and it is finite, then for any continuous function $f : [0,1] \to \mathbb{R}$

$$\lim_{n\to\infty} n \int_0^1 f(x)g(x^n)dx = f(1)\int_0^1 \frac{g(x)}{x}dx.$$

The order of convergence in the above relation, consequences and some applications are given.

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1 Introduction

There are many important classes of sequences defined by using Riemann integrals. We mention here only two. The first one is called the Riemann-Lebesgue Lemma and it asserts that if $g : [0, +\infty) \to \mathbb{R}$ is a continuous and T-periodic function, then for any continuous function $f : [a, b] \to \mathbb{R}$, $0 \le a < b$, the following relation holds:

$$\lim_{n\to\infty} \int_a^b f(x)g(nx)dx = \frac{1}{T}\int_0^T g(x)dx \int_a^b f(x)dx \tag{1}$$

For the proof we refer to [3] (in special case $a = 0$, $b = T$) and [8]. In the paper [1] we have proved that a similar relation as (1) holds for all continuous and bounded functions $g : [0, +\infty) \to \mathbb{R}$ having finite Cesaro mean. The second one was given in our paper [2] and shows that if $f : [1, +\infty) \to \mathbb{R}$ is a continuous function such that $\lim\limits_{x\to\infty} xf(x)$ exists and it is finite, then

$$\lim_{n\to\infty} n\int_1^a f(x^n)dx = \int_1^\infty \frac{f(x)}{x}dx, \tag{2}$$

for any real number $a > 1$.

In this paper we investigate the class of sequences defined by $n\int_0^1 f(x)g(x^n)dx$, where $f, g : [0, 1] \to \mathbb{R}$ are continuous functions. The main results in [6] are obtained as consequences and some applications are given.

2 The main results

We begin with two preliminary results.

Lemma 1. *Let $g : [0,1] \to \mathbb{R}$ be a continuous function such that $\lim\limits_{\substack{x\to 0\\ x>0}} \frac{g(x)}{x}$ exists and it is finite. Then*

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = \int_0^1 \frac{g(u)}{u} du. \tag{3}$$

Proof. Define the function $h : [0,1] \to \mathbb{R}$,

$$h(x) = \begin{cases} \dfrac{g(x)}{x} & \text{if} \quad x \in (0,1] \\ \lim\limits_{\substack{x\to 0\\ x>0}} & \text{if} \quad x = 0 \end{cases} \tag{4}$$

It is clear that h is continuous and denote

$$H(x) = \int_0^x h(t)dt.$$

We have

$$n \int_0^1 g(x^n)dx = n \int_0^1 x^n h(x^n)dx = xH(x^n)\Big|_0^1 - \int_0^1 H(x^n)dx$$

$$= H(1) - \int_0^1 H(x^n)dx = \int_0^1 \frac{g(x)}{x}dx - \int_0^1 H(x^n)dx.$$

If $0 < a < 1$, then we can write

$$\left| \int_0^1 H(x^n)dx \right| \leq \int_0^1 |H(x^n)|dx = \int_0^a |H(x^n)|dx + \int_a^1 |H(x^n)|dx$$

$$\leq a|H(\alpha_n^n)| + (1-a)M, \tag{5}$$

where $\alpha_n \in [0,a]$ and $M \max\limits_{t\in[0,1]} |H(t)|$.

Consider $\varepsilon > 0$ such that $a > 1 - \dfrac{\varepsilon}{2M}$. Because $\lim\limits_{n\to\infty} |H(\alpha_n^n)| = 0$, it follows that $a|H(\alpha_n^n)| < \dfrac{\varepsilon}{2}$ for all positive integers $n \geq N(\varepsilon)$. From (5) we get

$$\left|\int_0^1 H(x^n)dx\right| \leq \frac{\varepsilon}{2} + (1-a)M < \frac{\varepsilon}{2} + \left(1 - 1 + \frac{\varepsilon}{2M}\right) M = \varepsilon,$$

i.e. $\lim\limits_{n\to\infty} \int_0^1 H(x^n)dx = 0$ and the conclusion follows.

Lemma 2. *Let $g : [0,1] \to \mathbb{R}$ be a continuous function such that $\lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{g(x)}{x}$ exists and it is finite. Then for any function $f : [0,1] \to \mathbb{R}$ of class C^1,*

$$\lim_{n\to\infty} n \int_0^1 f(x)g(x^n)dx = f(1) \int_0^1 \frac{g(x)}{x}dx \tag{6}$$

Proof. Denote $G(x) = \int_0^x \dfrac{g(t)}{t}dt$, $x \in [0,1]$, and note that

$$n \int_0^1 f(x)g(x^n)dx = n \int_0^1 x^n f(x) \frac{g(x^n)}{x^n}dx$$

$$= G(x^n)xf(x)\Big|_0^1 - \int_0^1 (xf'(x) + f(x))G(x^n)dx$$

$$= G(1)f(1) - \int_0^1 (xf'(x) + f(x))G(x^n)dx$$

$$= f(1) \int_0^1 \frac{g(x)}{x}dx - \int_0^1 (xf'(x) + f(x))G(x^n)dx. \tag{7}$$

We will prove that

$$\lim_{n\to\infty} \int_0^1 (xf'(x) + f(x))G(x^n)dx = 0.$$

Indeed, by considering $M = \max_{x\in[0,1]} |xf'(x) + f(x)|$ we have

$$\left|\int_0^1 (xf'(x) + f(x))G(x^n)dx\right| \le \int_0^1 |xf'(x) + f(x)||G(x^n)|dx$$

$$\le M \int_0^1 |G(x^n)|dx.$$

Using that $\lim_{n\to\infty} \int_0^1 |G(x^n)|dx = 0$ (see the proof of Lemma 1) the desired relation (6) follows from (7).

Our main results are the following.

Theorem 1. *Let* $g : [0, 1] \to \mathbb{R}$ *be a continuous function such that* $\lim_{\substack{x\to 0\\ x>0}} \frac{g(x)}{x}$ *exists and it is finite. Then for any continuous function* $f : [0, 1] \to \mathbb{R}$ *the relation (6) holds.*

Proof. According to the well-known Weierstrass approximation theorem, consider $(f_m)_{m\ge 1}$ a sequence of polynomials uniformly convergent to f on the interval $[0, 1]$. Let $\varepsilon > 0$ be a fixed real number. We will show that we can find a positive integer $N(\varepsilon)$ such that for any $n \ge N(\varepsilon)$ and for any $x \in [0, 1]$, we have

$$\left|n\int_0^1 f(x)g(x^n)dx - f(1)\int_0^1 \frac{g(x)}{x}dx\right| < \varepsilon \tag{8}$$

From technical reasons, take $\varepsilon' = \varepsilon/\left(2\int_0^1 \frac{g(x)}{x}dx + 1\right)$ and consider the positive integer $N(\varepsilon)$ with the property that $|f_m(x) - f(x)| < \varepsilon'$ for any $x \in [0, 1]$. Because f and g are bounded it follows that we can assume that $f \ge 0$ and $g \ge 0$. For $m \ge N(\varepsilon)$ we have

$$f_m(x)g(x^n) - \varepsilon' g(x^n) \le f(x)g(x^n) \le f_m(x)g(x^n) + \varepsilon' g(x^n),$$

hence

$$n\int_0^1 f_m(x)g(x^n)dx - \varepsilon' n\int_0^1 g(x^n)dx \le n\int_0^1 f(x)g(x^n)dx$$

$$\le n\int_0^1 f_m(x)g(x^n)dx + \varepsilon' n\int_0^1 g(x^n)dx \tag{9}$$

From Lemma 2 we have

$$\lim_{n\to\infty} n\int_0^1 f_m(x)g(x^n)dx = f_m(1)\int_0^1 \frac{g(x)}{x}dx$$

and

$$\lim_{n\to\infty} n\varepsilon'\int_0^1 g(x^n)dx = \varepsilon'\int_0^1 \frac{g(x)}{x}dx$$

and it follows that for any positive integer $n \ge N'(\varepsilon)$

$$n\int_0^1 f_m(x)g(x^n)dx - \varepsilon' n\int_0^1 g(x^n)dx \ge f_m(1)\int_0^1 \frac{g(x)}{x}dx$$

$$-\varepsilon'\int_0^1 \frac{g(x)}{x}dx - \varepsilon'$$

and

$$n\int_0^1 f_m(x)g(x^n)dx + \varepsilon' n\int_0^1 g(x^n)dx \le f_m(1)\int_0^1 \frac{g(x)}{x}dx$$

$$+\varepsilon'\int_0^1 \frac{g(x)}{x}dx + \varepsilon'$$

But $f(1)-\varepsilon' < f_m(1) < f(1)+\varepsilon'$ imply for all $n \ge N'(\varepsilon)$

$$(f(1)-\varepsilon')\int_0^1 \frac{g(x)}{x}dx - \varepsilon'\left(\int_0^1 \frac{g(x)}{x}dx + 1\right) \le n\int_0^1 f(x)g(x^n)dx$$

$$\le (f(1)+\varepsilon')\int_0^1 \frac{g(x)}{x}dx + \varepsilon'\left(\int_0^1 \frac{g(x)}{x}dx + 1\right).$$

The last relation is equivalent to

$$\left| n \int_0^1 f(x) g(x^n) dx - f(1) \int_0^1 \frac{g(x)}{x} dx \right| < \varepsilon' \left(2 \int_0^1 \frac{g(x)}{x} dx + 1 \right)$$
$$= \varepsilon, \text{ for all } n \geq N'(\varepsilon),$$

and the conclusion follows.

Remarks. 1) Consider the function $h : [0, 1] \to \mathbb{R}$,

$$h(x) = \begin{cases} \dfrac{g(x)}{x} & \text{if } \ x \neq 0 \\ \\ \lim\limits_{\substack{x \to 0 \\ x > 0}} \dfrac{g(x)}{x} & \text{if } \ x = 0. \end{cases}$$

Because $\lim\limits_{\substack{x \to 0 \\ x > 0}} \dfrac{g(x)}{x}$ exists and it is finite, it follows that function h is continuous on $[0, 1]$. Applying the result in Theorem 1 we obtain that for any continuous functions $f, h : [0, 1] \to \mathbb{R}$ the following relation holds:

$$\lim_{n \to \infty} n \int_0^1 x^n f(x) h(x^n) dx = f(1) \int_0^1 h(x) dx \tag{10}$$

Relation (10) was proved in [6] in the case when f is differentiable and f' is continuous on $[0, 1]$.

2) If $u : [0, 1] \to \mathbb{R}$ is a continuous function such that its right derivative at 0 exists and it is finite, then the function $g(x) = u(x) - u(0)$ satisfies the hypotheses in Theorem 1. From (6) it follows

$$\lim_{n \to \infty} \int_0^1 f(x) g(x^n) dx = 0,$$

i.e.

$$\lim_{n \to \infty} \int_0^1 f(x) u(x^n) dx = u(0) \int_0^1 f(x) dx \tag{11}$$

In the paper [6] (see also [3]) is proved that the above relation holds even f, u are only continuous on $[0,1]$.

3) If $f=1$, the constant function on $[0,1]$, from (10) we get the result in paper [7].

The order of convergence in (10) is given in the following result.

Theorem 2. *Let $f:[0,1]\to\mathbb{R}$ be a function of class C^1 and let $h:[0,1]\to\mathbb{R}$ be a continuous function. Then*

$$\lim_{n\to\infty} n\left[f(1)\int_0^1 h(x)dx - n\int_0^1 x^n f(x)h(x^n)dx\right]$$

$$= (f(1)+f'(1))\int_0^1 \frac{H(x)}{x}dx, \tag{12}$$

where $H(x)=\displaystyle\int_0^x h(t)dt$.

Proof. We can write

$$n\int_0^1 x^n f(x)h(x^n)dx = \int_0^1 xf(x)(H(x^n))'dx$$

$$= xf(x)H(x^n)\Big|_0^1 - \int_0^1 (xf(x))'H(x^n)dx.$$

Therefore

$$n\left[f(1)\int_0^1 h(x)dx - n\int_0^1 x^n f(x)h(x^n)dx\right] = n\int_0^1 (xf(x))'H(x^n)dx.$$

Functions $x\mapsto xf(x)$, $x\mapsto H(x)$ satisfy the hypothesis in Theorem 1, hence we have

$$\lim_{n\to\infty}\int_0^1 n(xf(x))'H(x^n)dx = (f(1)+f'(1))\int_0^1 \frac{H(x)}{x}dx$$

and the desired relation follows.

Remarks. 1) Writing $h(x) = \dfrac{g(x)}{x}$ if $x \neq 0$ and $h(0) = \lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{g(x)}{x}$, where $g : [0,1] \to \mathbb{R}$ is a continuous function such that $\lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{g(x)}{x}$ exists and it is finite, from (11) we derive the following relation

$$\lim_{n\to\infty}\left[f(1)\int_0^1 \frac{g(x)}{x}dx - n\int_0^1 f(x)g(x^n)dx\right]$$
$$= (f(1) + f'(1))\int_0^1 \left(\frac{1}{x}\int_0^x \frac{g(t)}{t}dt\right)dx.$$

This is the order of convergence in (6) when f is of class C^1.

2) If $h = 1$, the constant function on $[0,1]$, from (10) we derive Problem 2.83.b) in [3].

3 Some applications

Application 1. 1) If $f : [0,1] \to \mathbb{R}$ is a continuous function, then

$$\lim_{n\to\infty} n\int_0^1 \frac{x^n f(x)}{1+x^{2n}}dx = \frac{\pi}{4}f(1).$$

2) If $f : [0,1] \to \mathbb{R}$ is a function of class C^1, then

$$\lim_{n\to\infty} n\left[\frac{\pi}{4}f(1) - n\int_0^1 \frac{x^n f(x)}{1+x^{2n}}dx\right] = (f(1)+f'(1))\int_0^1 \frac{\operatorname{arctg} x}{x}dx.$$

These results follows from (6) and (13), where

$$g(x) = \frac{x}{1+x^2}, \quad x \in [0,1].$$

If $f(x) = 1$ for all $x \in [0,1]$, then we get Problem 2 of the 12th Form in final Round of Romanian National Olympiad 2006.

Application 2. 1) (Romanian National Olympiad, County round 2001, partial statement) If $a > 0$, then

$$\lim_{n\to\infty} n \int_0^1 \frac{x^n}{a+x^n}dx = \ln\frac{a+1}{a}.$$

2) The following relation holds

$$\lim_{n\to\infty} n\left(\ln\frac{a+1}{a} - n\int_0^1 \frac{x^n}{a+x^n}dx\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{a^n n^2}. \tag{14}$$

Indeed, taking in (6) $f = 1$ and $g(x) = \dfrac{x}{a+x}$ we easily derive the first relation. For the second one we use (13) for the same choosing of functions. The right hand side in (13) becomes

$$\int_0^1 \left(\frac{1}{x}\int_0^x \frac{dt}{a+t}\right)dx = \int_0^1 \frac{\ln(x+a)-\ln a}{x}dx$$

$$= \int_0^1 \frac{1}{x}\ln\left(1+\frac{x}{a}\right)dx = \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{x}{a}\right)^n dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{a^n n^2}.$$

If $a = 1$, from (14) we get the interesting relation

$$\lim_{n\to\infty} n\left(\ln 2 - n\int_0^1 \frac{x^n}{1+x^n}dx\right) = \frac{\pi^2}{12} \tag{15}$$

Application 3. 1) If $f : [0,1] \to \mathbb{R}$ is a continuous function, then

$$\lim_{n\to\infty} n\int_0^1 f(x)\ln(1+x^n)dx = \frac{\pi^2}{12}f(1) \tag{16}$$

2) If $f : [0,1] \to \mathbb{R}$ is a function of class C^1, then

$$\lim_{n\to\infty} n\left[\frac{\pi^2}{12}f(1) - n\int_0^1 f(x)\ln(1+x^n)dx\right] = \frac{3}{4}(f(1)+f'(1))\zeta(3), \tag{17}$$

where ζ is the well-known Riemann's function.

To prove (16) we take in (6), $g(x) = \ln(1+x)$. We have

$$\int_0^1 \frac{g(x)}{x}dx = \int_0^1 \frac{\ln(1+x)}{x}dx = \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}dx$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \zeta(2) - \frac{2}{2^2}\zeta(2) = \frac{1}{2}\zeta(2)\frac{\pi^2}{12}.$$

In order to prove (17) we use relation (13) and observe that in the right hand side we obtain

$$\int_0^1 \left(\frac{1}{x}\int_0^x \frac{\ln(1+t)}{t}dt\right)dx = \int_0^1 \left(\frac{1}{x}\int_0^x \sum_{n=1}^{\infty}\frac{(-1)^{n+1}t^{n-1}}{n}dt\right)dx$$

$$= \int_0^1 \left(\frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n^2}\right)dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3} = \zeta(3) - \frac{2}{2^3}\zeta(3) = \frac{3}{4}\zeta(3).$$

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