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Divisibility of Some Hypergroups Defined from Groups

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Abstract

A hypergroup $(H, \circ)$ is said to be *divisible* if for every $x \in H$ and every positive integer n, there exists an element $y \in H$ such that $x \in (y, \circ)^n$ where $(y, \circ)^n$ denotes the set $y \circ y \circ \ldots \circ y$ (n copies). The following hypergroups defined from groups are known. If G is an abelian group and ρ is the equivalence relation on G defined by $x\rho y \Leftrightarrow x = y$ or $x = y^{-1}$, then $(G/\rho, \circ)$ is a hypergroup where $x\rho \circ y\rho = \{(xy)\rho, (xy^{-1})\rho\}$. Also, if G' is any group and $N \rhd G'$, then $(G', \diamond)$ is a hypergroup where $x \diamond y = Nxy$. In this paper, we show that for a finite abelian group G, $(G/\rho, \circ)$ is divisible if and only if G is of odd order. In addition, if the orders of elements of G' are bounded, then the hypergroup $(G', \diamond)$ is divisible only the case that $N = G'$.

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1 Introduction

For any set X, let $|X|$ denote the cardinality of X. The set of positive integers (natural numbers), the set of integers and the set of rational numbers will be denoted respectively by $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$.

We call a semigroup S a *divisible semigroup* if for every $x \in S$ and every $n \in \mathbb{N}, x = y^n$ for some $y \in S$. It is clearly seen that the group $(\mathbb{Q}, +)$ is a divisible group while the group $(\mathbb{Z}, +)$ and the multiplicative group of positive rational numbers are not divisible. Divisible semigroups have long been studied. See [5], [1], [3], [7] and [6] for examples. Divisible abelian groups have been characterized in terms of $\mathbb{Z}$-injectively. This can be seen in [4], page 195. A. Wasanawichit and the first author have studied the divisibility of some periodic semigroups (that is, semigroups whose elements have finite order) in [7]. Moreover, N. Triphop and A. Wasanawichit introduced some interesting divisible matrix groups in [6].

In this paper, the notion of divisibility is defined extensively. Divisible semihypergroups are defined and the divisibility of some hypergroups defined from groups will be investigated.

Let us recall some hyperstructures which will be used. A *hyperoperation* on a nonempty set H is a mapping $\circ : H \times H \to P^*(H)$ where $P(H)$ is the power set of H and $P^*(H) = P(H) \smallsetminus \{\emptyset\}$, and $(H, \circ)$ is called a *hypergroupoid*. If A and B are nonempty subsets of H, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} (a \circ b).$$

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$ and a semihypergroup $(H, \circ)$ with $H \circ x = x \circ H = H$ for all $x \in H$ is called a *hypergroup*. The concept of commutativity of these hyperstructures is given naturally. A semihypergroup $(H, \circ)$ is called *divisible* if for any $x \in H$ and $n \in \mathbb{N}$, there is an element $y \in H$ such that $x \in (y, \circ)^n$ where $(y, \circ)^n$ denotes the subset $y \circ y \circ \ldots \circ y$ (n copies) of H. It is clear that a total hypergroup, that is, a hypergroup $(H, \circ)$ with $x \circ y = H$ for all $x, y \in H$, is divisible. A semigroup [semihypergroup] is said to be *indivisible* if it is not divisible. The following hypergroups are defined from groups and they can be seen in [2], page 11.

Let G be an abelian group and ρ the equivalence relation on G defined by

$$x \rho y \;\Leftrightarrow\; x = y \text{ or } x = y^{-1}. \tag{1}$$

Then $(G/\rho, \circ)$ is a commutative hypergroup where

$$x\rho \;\circ\; y\rho = \{(xy)\rho, (xy^{-1})\rho\} \text{ for all } x, y \in G. \tag{2}$$

Next, let G be any group and N a normal subgroup of G. Then $(G, \diamond)$ is a hypergroup where

$$x \diamond y = Nxy \text{ for all } x, y \in G. \tag{3}$$

We note that if $N = G$, then $(G, \diamond)$ is a total hypergroup. Also, if $N = \{e\}$ where e is the identity of G, then $x \diamond y = \{xy\}$ for all $x, y \in G$, so $(G, \diamond) = G$.

For more details on hyperstructures, the reader is referred to [2].

Let us recall the following well-known fact which will be referred. If G is a finite abelian group, then $G \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_l}$ for some $k_1, k_2, \ldots, k_l \in \mathbb{N}$ ([4], page 76). Here $\mathbb{Z}_n$ denotes the group under addition of integers modulo n. Note that $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\} = \{\overline{x} \mid x \in \mathbb{Z}\}$.

2 The Hypergroup $(G/\rho, \circ)$

Throughout this section, ρ denotes the equivalence relation on a considering abelian group G defined as in (1) and $\circ$ denotes the hyperoperation on G/ρ defined as in (2). Notice that $e\rho = \{e\}$ where e is the identity of G and $x^{-1}\rho = x\rho = \{x, x^{-1}\}$ for all $x \in G$. Our main interest in this section is to show that for a finite abelian group G, the hypergroup $(G/\rho, \circ)$ is divisible if and only if G is of odd order.

The following two lemmas are needed.

Lema 2.1. *If $x \in G$ and $n \in \mathbb{N}$, then in the hypergroup $(G/\rho, \circ)$,*

$$(x\rho, \circ)^n = \begin{cases} \{e\rho, x^2\rho, \ldots, x^n\rho\} & \text{if } n \text{ is even,} \\ \{x\rho, x^3\rho, \ldots, x^n\rho\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This is clear for $n = 1$. We have from (2) that $(x\rho, \circ)^2 = x\rho \circ x\rho = \{x^2\rho, e\rho\} = \{e\rho, x^2\rho\}$, $(x\rho, \circ)^3 = \{e\rho, x^2\rho\} \circ x\rho = \{x\rho, x^{-1}\rho, x^3\rho, x\rho\} = \{x\rho, x^3\rho\}$ and $(x\rho, \circ)^4 = \{x\rho, x^3\rho\} \circ x\rho = \{x^2\rho, e\rho, x^4\rho, x^2\rho\} = \{e\rho, x^2\rho, x^4\rho\}$.

If $k > 2$ is even and $(x\rho, \circ)^k = \{e\rho, x^2\rho, x^4\rho, \ldots, x^{k-2}\rho, x^k\rho\}$, then

$$\begin{aligned}(x\rho, \circ)^{k+1} &= \{e\rho, x^2\rho, x^4\rho, \ldots, x^{k-2}\rho, x^k\rho\} \circ x\rho \\ &= \{x\rho, x^{-1}\rho, x^3\rho, x\rho, x^5\rho, x^3\rho, \ldots, x^{k-1}\rho, x^{k-3}\rho, x^{k+1}\rho, x^{k-1}\rho\} \\ &= \{x\rho, x^3\rho, x^5\rho, \ldots, x^{k-1}\rho, x^{k+1}\rho\}.\end{aligned}$$

Also, if $k > 1$ is odd and $(x\rho, \circ)^k = \{x\rho, x^3\rho, x^5\rho, \ldots, x^{k-2}\rho, x^k\rho\}$, then

$$\begin{aligned}(x\rho, \circ)^{k+1} &= \{x\rho, x^3\rho, x^5\rho, \ldots, x^{k-2}\rho, x^k\rho\} \circ x\rho \\ &= \{x^2\rho, e\rho, x^4\rho, x^2\rho, x^6\rho, x^4\rho, \ldots, x^{k-1}\rho, x^{k-3}\rho, x^{k+1}\rho, x^{k-1}\rho\} \\ &= \{e\rho, x^2\rho, x^4\rho, \ldots, x^{k-1}\rho, x^{k+1}\rho\}.\end{aligned}$$

Therefore the lemma is proved, as required.

Lema 2.2. *Let $\{k_i \mid i \in I\}$ be a nonempty subset of $\mathbb{N}$. Then the hypergroup $(\prod_{i\in I} \mathbb{Z}_{k_i}/\rho, \circ)$ is divisible if and only if k_i is odd for all $i \in I$. In particular, the hypergroup $(\mathbb{Z}_n/\rho, \circ)$ is divisible if and only if n is odd.*

Proof. Frist assume that the hypergroup $(\prod_{i\in I} \mathbb{Z}_{k_i}/\rho, \circ)$ is divisible. Then there is an element $(\overline{x}_i)_{i\in I}\rho$ of $(\prod_{i\in I} \mathbb{Z}_{k_i}/\rho, \circ)$ such that $(\overline{1})_{i\in I}\rho \in 2((\overline{x}_i)_{i\in I}\rho, \circ)$. Note that we use the notation $n((\overline{x}_i)_{i\in I}\rho, \circ)$ in the hypergroup $(\prod_{i\in I} \mathbb{Z}_{k_i}/\rho, \circ)$ instead of $((\overline{x}_i)_{i\in I}\rho, \circ)^n$. By Lemma 2.1, $2((\overline{x}_i)_{i\in I}\rho, \circ) = \{(\overline{0})_{i\in I}\rho, (2\overline{x}_i)_{i\in I}\rho\}$.

Case 1. $(\overline{1})_{i\in I}\rho = (\overline{0})_{i\in I}\rho$. Then $\overline{1} = \overline{0}$ in $\mathbb{Z}_{k_i}$ for every $i \in I$, so $k_i = 1$ for all $i \in I$.

Case 2. $(\overline{1})_{i\in I}\rho = (2\overline{x_i})_{i\in I}\rho$. Then $(\overline{1})_{i\in I} = (2\overline{x_i})_{i\in I}$ or $(\overline{1})_{i\in I} = (-2\overline{x_i})_{i\in I}$. This implies that

$$k_i | 2x_i - 1 \text{ for all } i \in I \quad \text{or} \quad k_i | 2x_i + 1 \text{ for all } i \in I,$$

and hence k_i is odd for every $i \in I$.

For the converse, assume that each k_i is odd. Then $\frac{k_i + 1}{2} \in \mathbb{N}$ for all $i \in I$. To show that the hypergroup $(\prod_{i\in I} \mathbb{Z}_{k_i}/\rho, \circ)$ is divisible, let $(\overline{x}_i)_{i\in I} \in \prod_{i\in I} \mathbb{Z}_{k_i}$ and $n \in \mathbb{N}$ be given. Note that $k_i\overline{x_i} = \overline{0}$ for every $i \in I$. If n is odd, then $(\overline{x}_i)_{i\in I}\rho \in n((\overline{x_i})_{i\in I}\rho, \circ)$ by Lemma 2.1. If n is even, then by Lemma 2.1,

$$n\big(((\frac{k_i + 1}{2})\overline{x_i})_{i\in I}\rho, \circ\big) = \{(\overline{0})_{i\in I}\rho, \big(2((\frac{k_i + 1}{2})\overline{x_i})_{i\in I}\big)\rho, \ldots, \big(n((\frac{k_i + 1}{2})\overline{x_i})_{i\in I}\big)\rho\},$$

and

$$\big(2((\frac{k_i + 1}{2})\overline{x_i})_{i\in I}\big)\rho = \big((k_i + 1)\overline{x_i}\big)_{i\in I}\rho = (\overline{x_i})_{i\in I}\rho.$$

Therefore the lemma is completely proved.

If $x, y \in G$ and $n \in \mathbb{N}$ are such that $x = y^n$, then by Lemma 2.1, $x\rho = y^n\rho \in (y\rho, \circ)^n$. Hence we have

Proposition 2.1. *If G is a divisible group, then $(G/\rho, \circ)$ is a divisible hypergroup.*

It is natural to ask whether the converse of Proposition 2.3 holds. The following example shows that it is not generally true. From Lemma 2.2, if $n \in \mathbb{N}$ is odd, then the hypergroup $(\mathbb{Z}_n/\rho, \circ)$ is divisible. It is known that a divisible finite abelian group must be trivial ([4], page 198). Moreover, this is true that for any finite group. This fact is given in [7]. Therefore we have that for any odd $n > 1$, the additive group $\mathbb{Z}_n$ is not divisible but the hypergroup $(\mathbb{Z}_n/\rho, \circ)$ is divisible.

Theorem 2.1. *Assume that G is a finite abelian group. Then the hypergroup $(G/\rho, \circ)$ is divisible if and only if G is of odd order.*

Proof. Since G is a finite abelian group, $G \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_l}$ for some $k_1, k_2, \ldots, k_l \in \mathbb{N}$. It then follows that $|G| = k_1 k_2 \ldots k_l$.

First, assume that $(G/\rho, \circ)$ is a divisible hypergroup. Since $G \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_l}$, the hypergroup $\big(\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_l}/\rho, \circ\big)$ is divisible. It then follows from Lemma 2.2, k_i is odd for all $i \in \{1, 2, \ldots, l\}$. Hence $|G| = k_1 k_2 \ldots k_l$ is odd.

Conversely, assume that $|G|$ is odd. Then k_i is odd for every $i \in \{1, 2, \ldots, l\}$ which implies by Lemma 2.2 that the hypergroup $\big(\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_l}/\rho, \circ\big)$ is divisible. But $G \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_l}$, so $(G/\rho, \circ)$ is a divisible hypergroup.

3 The Hypergroup $(G, \diamond)$

In this section, let G be any group, N a normal subgroup of G and $\diamond$ the hyperoperation depending on N defined on G as in (3). Recall that if $N = G$, then $(G, \diamond)$ is a total hypergroup which is divisible. The purpose of this section is to show that if the orders of elements of G are bounded, then the necessary and sufficient condition for $(G, \diamond)$ to be divisible is $N = G$.

First, we prove the following easy fact.

Lema 3.1. *For every $x \in G$ and every $n \in \mathbb{N}, (x, \diamond)^n = Nx^n$.*

Proof. If $x \in G$, by (3), $(x, \diamond)^2 = x \diamond x = Nxx = Nx^2$. Assume that $k \in \mathbb{N}$ and $(x, \diamond)^k = Nx^k$. Then

$$(x, \diamond)^{k+1} = (Nx^k) \diamond x = \bigcup_{t \in Nx^k} t \diamond x = \bigcup_{t \in Nx^k} Ntx = N(Nx^k)x = Nx^{k+1}.$$

For $x, y \in G$ and $n \in \mathbb{N}$, if $x = y^n$, then $x \in Ny^n = (y, \diamond)^n$ by Lemma 3.1. Hence we have

Proposition 3.1. *If G is a divisible group, then $(G, \diamond)$ is a divisible hypergroup.*

If $N = G$, then $(G, \diamond)$ is a divisible hypergroup. This indicates that the converse of Proposition 3.2 is not generally true. However, it is true under the assumption that N is divisible and $N \subseteq C(G)$ where $C(G)$ is the center of G.

Proposition 3.2. *If $(G, \diamond)$ is a divisible hypergroup, N is a divisible group and $N \subseteq C(G)$, then G is a divisible group. In particular, for an abelian group G, if both $(G, \diamond)$ and N are divisible, then so is G.*

Proof. Let $x \in G$ and $n \in \mathbb{N}$. Since $(G, \diamond)$ is a divisible hypergroup, $x \in (y, \diamond)^n$ for some $y \in G$. By Lemma 3.1, $x \in Ny^n$. Thus $x = sy^n$ for some $s \in N$. But N is a divisible group, so $s = t^n$ for some $t \in N$. Thus $x = t^n y^n = (ty)^n$ since $N \subseteq C(G)$. This shows that G is a divisible group, as desired.

Theorem 3.1. *If the orders of elements of G are bounded, then $(G, \diamond)$ is a divisible hypergroup only the case that $N = G$. In particular, if G is finite and $N \subsetneq G$, then $(G, \diamond)$ is indivisible.*

Proof. Assume that the orders of elements of G are bounded by $m \in \mathbb{N}$. This implies that for every $x \in G, x^{m!} = e$ where e is the identity of G.

Suppose that $(G, \diamond)$ is a divisible hypergroup. If $x \in G$, then $x \in (y, \diamond)^{m!}$ for some $y \in G$. But $(y, \diamond)^{m!} = Ny^{m!}$ by Lemma 3.1, so $x \in Ny^{m!} = Ne = N$. Therefore we have that $N = G$.

If G is an infinite cyclic group, then $G \cong (\mathbb{Z}, +)$, so G is indivisible and every nonidentity element of G has infinite order. For this case, the subgroup N of G which makes the hypergroup $(G, \diamond)$ divisible cannot be proper.

Proposition 3.3. *If G is an infinite cyclic group such that $(G, \diamond)$ is a divisible hypergroup, then $N = G$.*

Proof. It suffices to assume that $G = (\mathbb{Z}, +)$. Assume that $(\mathbb{Z}, \diamond)$ is divisible. If $N = \{0\}$, then $(\mathbb{Z}, \diamond) = (\mathbb{Z}, +)$ which is indivisible. This implies that $N \neq \{0\}$. Then $N = m\mathbb{Z}$ for some $m \in \mathbb{N}$. Therefore for $x \in \mathbb{Z}, x \in m(y, \diamond)$ for some $y \in \mathbb{Z}$. But $m(y, \diamond) = N + my$ by Lemma 3.1, so $x \in N + my = m\mathbb{Z} + my = m\mathbb{Z} = N$. Hence $\mathbb{Z} = N$.

Remark 3.6. From Proposition 3.2, Theorem 3.4 and Proposition 3.5, a natural question arises. Are there an indivisible group G and a proper normal subgroup N of G such that $(G, \diamond)$ is a divisible hypergroup? There are such G and N as shown by the following example. Let G be the group $\mathbb{Q} \times \mathbb{Z}$ with usual addition and $N = \{0\} \times \mathbb{Z}$. Then N is a proper subgroup of G. Claim that G is an indivisible group but $(G, \diamond)$ is a divisible hypergroup. Recall that $x \diamond y = N + x + y$ for all $x, y \in G$. Since $(0, 1) \in \mathbb{Q} \times \mathbb{Z}$ and $(0, 1) \neq 2(a, b)$ for all $(a, b) \in \mathbb{Q} \times \mathbb{Z}$, we have that G is indivisible. If $(a, b) \in \mathbb{Q} \times \mathbb{Z}$ and $n \in \mathbb{N}$, then by Lemma 3.1, $n((\frac{a}{n}, b), \diamond) = N + n(\frac{a}{n}, b) = \{0\} \times \mathbb{Z} + (a, nb) = \{a\} \times \mathbb{Z}$ which implies that $(a, b) \in n((\frac{a}{n}, b), \diamond)$. This shows that $(G, \diamond)$ is a divisible hypergroup, as desired.

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