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# $C_g$ asymptotic equivalence for some functional equation of type Voltera[1]

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**Abstract**

In this paper , by using the notion of $\varphi-$ contraction, we study the $C_g$ asymptotic equivalence for the solutions of the equations $x'(t) = A(t)x(t)$ and $y'(t) = A(t)y(t) + f(t, y_t)$,where $f(t, \cdot)$ is a Voltera operator.

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## 1 Introduction

Let $C_g$ be the Banach space of continuous functions defined on $\mathbb{R}_{t_0} = [t_0, \infty), t_0 \geq 0$ which satisfied the condition :

(1) $$|u(t)| = O(g(t)), \quad t \longrightarrow \infty,$$

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where $g$ is a continuous and positive function defined on $R_{t_0}$,$|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.

We define the norm on $C_g$ by relation:

$$|u|_{C_g} = \sup_{t\in\mathbb{R}_{t_0}} \frac{|u(t)|}{g(t)}. \tag{2}$$

We note by $u_t$ the restriction of function $u$ at $[t_0, t]$. For $u_t \in C([t_0, t], \mathbb{R}^n)$ we define

$$\|u_t\| = \sup_{s\in[t_0,t]} |u(s)| \tag{3}$$

On [1] it is presented the following lemma:

**Lemma 1.1.** *Let $g$ be a nondecreasing, positive function defined on $\mathbb{R}_+$ and $x \in C_g$. Then:*

$$|x|_{C_g} = \sup_{t\in\mathbb{R}_+} \frac{\|x_t\|}{g(t)}$$

Next we consider the equations :

$$x' = A(t)x \tag{4}$$

$$y' = A(t)y + f(t, y_t), \tag{5}$$

where for $t \geq t_0$ the application $\psi \longrightarrow f(t, \psi)$ is an application from $C([t_0, t], \mathbb{R}^n)$ to $\mathbb{R}^n$ that satisfies some conditions that assure the existence of the equation (5), conditions that are to be explained below .

**Definition 1.1.** *[1] We say that the equations (4) and (5) are $C_g$-asymptotic equivalence if for all solution $x \in C_g$ of equation (4) corresponding a unique solution $y$ of equation (5) such that :*

$$\lim_{t\longrightarrow\infty} \frac{|x(t) - y(t)|}{g(t)} = 0. \tag{6}$$

Through the following definitions we shall further present the notion of comparison function and $\varphi-$ contraction:

**Definition 1.2.**[2],[3] $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a strict comparison function if* $\varphi$ *satisfies the following:*

*i)* $\varphi$ *is continuous.*

*ii)*$\varphi$ *is monotone increasing.*

*iii)* $\varphi^n(t) \longrightarrow 0$, *for all* $t > 0$.

*iv)* $t$-$\varphi(t) \longrightarrow \infty$,*for* $t \longrightarrow \infty$.

Let $(X, d)$ be an metric space and $f : X \longrightarrow X$ be an operator.

**Definition 1.3.**[2],[3] *The operator* $f$ *is called a strict* $\varphi$*-contraction if:*

*(i)* $\varphi$ *is a strict comparison function.*

*(ii)*$d(f(x), f(y)) \leq \varphi(d(x, y))$, *for all* $x, y \in X$.

We shall make the following hypothesis :

(H) We supose that there exists a comparison function $\varphi$ which satisfies condition

$$\varphi(\lambda \cdot r) \leq \lambda \cdot \varphi(r), \tag{7}$$

for all $r \geq 0$ and $\lambda \geq 1$

An example of such a function is shown in the next figure:

$$\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \varphi(t) = \frac{r}{r+1}$$

On [2] I.A Rus obtains the following result:

**Theorem 1.1.***Let* $(X, d)$ *be a complete metric space and* $f : X \longrightarrow X$ *a* $\varphi$*-contraction.Then* $f$ *, is a Picard operator.*

# 2 Main result

**Theorem 2.1.** *Let $X(t)$ be a fundamental matrix of equation (4). We suppose that :*

*(i) There exists the projectors $P_1$,$P_2$ and a constant $K > 0$ such that*

$$\Big(\int_{t_0}^{t} |X(t)P_1X^{-1}(s)|^q ds + \int_{t}^{\infty} |X(t)P_2X^{-1}(s)|^q ds\Big)^{\frac{1}{q}} \leq K, \tag{8}$$

*for $t \geq t_0,\ q > 1$;*

*(ii) The application $t \longrightarrow f(t, y_t)$ is continuous for all $y \in C_g$.*

*(iii) There exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, a comparison function which satisfies the hypothesis $(H)$ , and $\lambda$ a continuous,nonnegative function defined on $\mathbb{R}_{t_0}$ such that*

$$|f(t, y_t) - f(t, \overline{y_t})| \leq \lambda(t)\varphi(\|y_t - \overline{y_t}\|), \tag{9}$$

*for all $t \geq t_0$,$y \in C_g$*

*(iv)*

$$\left\{\int_{t}^{\infty} (\lambda(s)g(s))^p ds\right\}^{\frac{1}{p}} \in C_g,\ p > 1 \tag{10}$$

$$\left\{\int_{t_0}^{\infty} |f(s, 0)|^p ds\right\}^{\frac{1}{p}} < \infty, \left\{\int_{t_0}^{\infty} (\lambda(s))^p\right\}^{\frac{1}{p}} < \infty,\quad p > 1. \tag{11}$$

*Then for all solution $x \in C_g$ of equation (4) there exists a unique solution $y(t)$ of equation (5).*

*If we replace the condition (10) with*

$$\left\{ \int_t^\infty (\lambda(s)g(s))^p ds \right\}^{\frac{1}{p}} = o(t), \quad t \longrightarrow \infty \ , \tag{12}$$

*then the equations (4)and (5) are $C_g$-asymptotic equivalence.*

**Proof.** The function $g$ being nondecreasing and positive we cant suppose that $g \geq 1$, because $C_g = C_{kg}$ for all $k > 0$.(for more details see [1])

On $C_g$ we define the operator $T$ by relation:

$$T(y)(t) = x(t) + \int_{t_0}^{t} X(t)P_1X^{-1}(s)f(s, y_s)ds - \int_{t}^{\infty} X(t)P_2X^{-1}(s)f(s, y_s)ds$$

Let $x \in C_g$ be a solution for the equation(4).Then $|x(t)| \leq A \cdot g(t)$, for all $t \geq t_0$.We prove that $T(C_g) \subseteq C_g$.Let be $y \in C_g$. Then

$$|T(y)(t)| \leq |x(t)| + \int_{t_0}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, y_s)|ds +$$

$$+ \int_{t}^{\infty} |X(t)P_2X^{-1}(s)| \cdot |f(s, y_s)|ds \leq$$

$$\leq Ag(t) + \int_{t_0}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, y_s) - f(s, 0)|ds +$$

$$+ \int_{t_0}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, 0)|ds +$$

$$+\int_t^\infty |X(t)P_2X^{-1}(s)|\cdot|f(s,y_s)-f(s,0)|ds+$$

$$+\int_t^\infty |X(t)P_2X^{-1}(s)|\cdot|f(s,0)|ds\leq$$

$$\leq Ag(t)+\left\{\int_t^\infty |X(t)P_2X^{-1}(s)|^q ds\right\}^{\frac{1}{q}}\cdot\left\{\int_t^\infty |f(s,0)|^p ds\right\}^{\frac{1}{p}}+$$

$$+\left\{\int_{t_0}^t |X(t)P_1X^{-1}(s)|^q ds\right\}^{\frac{1}{q}}\cdot\left\{\int_{t_0}^t |f(s,0)|^p ds\right\}^{\frac{1}{p}}+$$

$$+\left\{\int_t^\infty |X(t)P_2X^{-1}(s)|^q ds\right\}^{\frac{1}{q}}\cdot\left\{\int_t^\infty (\lambda(s)\varphi(\|y_s\|))^p ds\right\}^{\frac{1}{p}}+$$

$$+\left\{\int_{t_0}^t |X(t)P_1X^{-1}(s)|^q ds\right\}^{\frac{1}{q}}\cdot\left\{\int_{t_0}^t (\lambda(s))\varphi(\|y_s\|)^p ds\right\}^{\frac{1}{p}}\leq$$

$$\leq Ag(t)+K\cdot\left\{\int_{t_0}^t |f(s,0)|^p ds\right\}^{\frac{1}{p}}+$$

$$+K\cdot\left\{\int_t^\infty |f(s,0)|^p ds\right\}^{\frac{1}{p}}+$$

$$+K\cdot\varphi(|y|_{C_g})\cdot\left\{\int_{t_0}^t (\lambda(s)g(s))^p ds\right\}^{\frac{1}{p}}+$$

$$+K\cdot\varphi(|y|_{C_g})\cdot\left\{\int_t^\infty(\lambda(s)g(s))^p ds\right\}^{\frac{1}{p}}\leq$$
$$\leq M\cdot g(t),$$

where

$$M=A+K\cdot\varphi(|y|_{C_g})\cdot\left\{\int_{t_0}^\infty(\lambda(s))^p ds\right\}^{\frac{1}{p}}+$$
$$+K\cdot\varphi(|y|_{C_g})\cdot B_1+2K\cdot\left\{\int_{t_0}^\infty|f(s,0)|^p ds\right\}^{\frac{1}{p}}.$$

Next we consider $y,\overline{y}\in C_g$. We prove that, the operator $T$ is a $\varphi$-contraction.

$$\|T(y)(t)-T(\overline{y})(t)\|\leq\int_{t_0}^t|X(t)P_1X^{-1}(s)|\cdot|f(s,y_s)-f(s,\overline{y}_s)|ds+$$
$$+\int_t^\infty|X(t)P_2X^{-1}(s)|\cdot|f(s,y_s)-f(s,\overline{y}_s)|ds\leq$$
$$\leq K\cdot\left\{\left\{\int_{t_0}^t|f(s,y_s)-f(s,\overline{y}_s)|^p ds\right\}^{\frac{1}{p}}+\left\{\int_t^\infty|f(s,y_s)-f(s,\overline{y}_s)|^p ds\right\}^{\frac{1}{p}}\right\}\leq$$
$$\leq K\cdot\varphi(|y-\overline{y}|_{C_g})\left\{\left\{\int_{t_0}^t(\lambda(s)g(s))^p ds\right\}^{\frac{1}{p}}+\left\{\int_t^\infty(\lambda(s)g(s))^p ds\right\}^{\frac{1}{p}}\right\}\leq$$
$$\leq K\cdot\left\{\left\{\int_{t_0}^\infty(\lambda(s))^p ds\right\}^{\frac{1}{p}}+B_1\right\}\varphi(|y-\overline{y}|_{C_g})\cdot g(t).$$

We choose $t_0 \geq 0$ such that $K \cdot \left\{ \left\{ \int\limits_{t_0}^{\infty} (\lambda(s))^p ds \right\}^{\frac{1}{p}} + B_1 \right\} < 1$

Let be $x$ an solution of equation (4) and $y$ the unique solution of the equation(5) that corresponds to $x$. Then

$$|y(t) - x(t)| \leq \int\limits_{t_0}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, y_s)|ds +$$

$$+ \int\limits_{t}^{\infty} |X(t)P_2X^{-1}(s)| \cdot |f(s, y_s)|ds \leq$$

$$\leq \int\limits_{t_0}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, y_s)|ds + \int\limits_{t}^{\infty} |X(t)P_2X^{-1}(s)| \cdot |f(s, y_s) - f(s, 0)|ds +$$

$$+ \int\limits_{t}^{\infty} |X(t)P_2X^{-1}(s)| \cdot |f(s, 0)|ds = I_1 + I_2.$$

$$I_2 = \int\limits_{t}^{\infty} |X(t)P_2X^{-1}(s)| \cdot |f(s, y_s) - f(s, 0)|ds + \int\limits_{t}^{\infty} |X(t)P_2X^{-1}(s)| \cdot |f(s, 0)|ds$$

$$\leq K\varphi(|y|_{C_g}) \left\{ \int\limits_{t}^{\infty} (\lambda(s)g(s))^p ds \right\}^{\frac{1}{p}} + K \left\{ \int\limits_{t_1}^{\infty} |f(s, 0)|^p ds \right\}^{\frac{1}{p}}.$$

If $t \geq t_1 \geq t_0$ then

$$\left\{ \int\limits_{t}^{\infty} (\lambda(s)g(s))^p ds \right\}^{\frac{1}{p}} < \frac{\epsilon}{6K\varphi(|y|_{C_g})}$$

$$\left\{ \int\limits_{t_1}^{\infty} |f(s, 0)|^p ds \right\}^{\frac{1}{p}} < \frac{\epsilon}{6K}.$$

Then $I_2 \leq \frac{\varepsilon}{3} g(t)$

$$I_1 = \int_{t_0}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, y_s)|ds \leq$$

$$\leq |X(t)P_1| \int_{t_0}^{t_1} |X^{-1}(s)f(s, y_s)|ds + \int_{t_1}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, y_s) - f(s, 0)|ds +$$

$$+ \int_{t_1}^{t} |f(s, 0)|ds$$

$$\leq |X(t)P_1| \int_{t_0}^{t_1} |X^{-1}(s)f(s, y_s)|ds +$$

$$+K \cdot \left\{ \left\{ \int_{t_1}^{t} (\lambda(s)\varphi(\|y_s\|))^p ds \right\}^{\frac{1}{p}} + \left\{ \int_{t_1}^{t} |f(s, 0)|^p \right\}^{\frac{1}{p}} \right\} \leq$$

$$\leq |X(t)P_1| \int_{t_0}^{t_1} |X^{-1}(s)f(s, y_s)|ds +$$

$$+K \cdot \varphi(|y|_{C_g}) \left\{ \int_{t_1}^{t} (\lambda(s)g(s))^p ds \right\}^{\frac{1}{p}} + K \cdot \left\{ \int_{t_1}^{t} |f(s, 0)|^p \right\}^{\frac{1}{p}} \leq$$

$$\leq |X(t)P_1| \int_{t_0}^{t_1} |X^{-1}(s)f(s, y_s)|ds + K \cdot \varphi(|y|_{C_g}) g(t) \left\{ \int_{t_1}^{\infty} (\lambda(s))^p ds \right\}^{\frac{1}{p}} +$$

$$+K \cdot \left\{ \int_{t_1}^{\infty} |f(s, 0)|^p \right\}^{\frac{1}{p}}.$$

Using Lemma 1.1 we obtain that $|X(t)P_1|\int_{t_0}^{t_1}|X^{-1}(s)f(s,y_s)|ds<\frac{\varepsilon}{3}$, for all $t\geq t_2\geq t_1$. Then $I_1<\frac{2\varepsilon}{3}g(t)$

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